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OPTIMAL MATCHING PARAMETERS OF THE INVERSE HILBERT-TYPE INTEGRAL INEQUALITY WITH QUASIHOMOGENEOUS KERNELS AND THEIR APPLICATIONS

ОПТИМАЛЬНІ ПАРАМЕТРИ УЗГОДЖЕННЯ ОБЕРНЕНОЇ ІНТЕГРАЛЬНОЇ НЕРІВНОСТІ ТИПУ ГІЛЬБЕРТА З КВАЗІОДНОРІДНИМИ ЯДРАМИ ТА ЇХ ЗАСТОСУВАННЯ

By using the inverse Hölder inequality and the weight function method, we establish the inverse Hilbert-type integral inequality. In the case of a quasihomogeneous kernel, we obtain the necessary and sufficient conditions for the optimal matching parameters. Finally, their applications in the operator theory are discussed.

За допомогою оберненої нерівності Гельдера та методу вагової функції встановлено обернену інтегральну нерівність типу Гільберта. У випадку квазіоднорідного ядра отримано необхідні та достатні умови для оптимальних параметрів узгодження. Насамкінець обговорено їх застосування в теорії операторів.

1. Introduction. Suppose that $r \neq 0,\ \alpha \in \mathbb{R}$, let $K(x,\ y)$ be nonnegative and measurable. Define

$$L_r^\alpha(0,+\infty) = \left\{ f(x) \geq 0 : ||f||_{r,\alpha} = \left(\int\limits_0^{+\infty} x^\alpha f^r(x) dx \right)^{1/r} < +\infty \right\}.$$

For $\dfrac{1}{p} + \dfrac{1}{q} = 1,\ 0 < p < 1,\ q < 0$, we call

$$A(f,g) \triangleq \int\limits_0^{+\infty} \int\limits_0^{+\infty} K(x,y) f(x) g(y) dx dy \geq M ||f||_{p,\alpha} ||g||_{q,\beta} \tag{1}$$

the inverse Hilbert-type integral inequality, and

$$M_0 = \inf \left\{ \frac{A(f,g)}{||f||_{p,\alpha} ||g||_{q,\beta}} : 0 < ||f||_{p,\alpha} < +\infty,\ 0 < ||g||_{q,\beta} < +\infty \right\}$$

the optimal constant factor of (1).

Let $G(u,\ v)$ be a λ-order nonnegative homogeneous function, $\lambda_1 \lambda_2 \neq 0$, we say $K(x,\ y) = G(x^{\lambda_1},\ y^{\lambda_2})$ is a quasihomogeneous kernel with parameters $\{\lambda,\ \lambda_1,\ \lambda_2\}$. There are some obvious properties of $K(x,\ y)$ as follows: if $t > 0$, then

$$K(tx, y) = t^{\lambda \lambda_1} K(x, t^{-\lambda_1/\lambda_2} y), \quad K(x, ty) = t^{\lambda \lambda_2} K(t^{-\lambda_2/\lambda_1} x, y).$$

In particular,

$$K(t,1) = t^{\lambda \lambda_1} K(1, t^{-\lambda_1/\lambda_2}), \quad K(1,t) = t^{\lambda \lambda_2} K(t^{-\lambda_2/\lambda_1}, 1).$$

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If the matching parameters a and b are introduced, the following inverse Hilbert-type inequalities can be obtained by using the inverse Hölder integral inequality and the weight function method:

$$\int_0^{+\infty}\int_0^{+\infty} K(x,y)f(x)g(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty}\left(\frac{x^a}{y^b}f(x)\right)\left(\frac{y^b}{x^a}g(y)\right)K(x,y)dxdy$$

$$\geq M(\lambda,\lambda_1,\lambda_2,a,b)||f||_{p,\alpha(\lambda,\lambda_1,\lambda_2,a,b)}||g||_{q,\beta(\lambda,\lambda_1,\lambda_2,a,b)}, \tag{2}$$

where the constant factors M,α,β are all related to $\lambda,\lambda_1,\lambda_2,a$ and b.

For any given matching parameters a, b, the constant factor $M(\lambda,\lambda_1,\lambda_2,a,b)$ of (2) is not optimal. When $\frac{1}{p}+\frac{1}{q}=1,\ p>1,\ q>1$, the optimal matching parameters of positive Hilbert-type integral inequality are discussed in [1 – 4], and equivalent conditions of the optimal matching are established. Some inverse Hilbert-type integral inequalities are given in [5 – 9]. In this paper, the inverse Hilbert-type integral inequality with quasihomogeneous kernels is studied in the case of $\frac{1}{p}+\frac{1}{q}=1,\ 0<p<1,\ q<0$, and the necessary and sufficient conditions for optimal matching parameters are obtained.

2. Preliminary lemmas.

Lemma 1 [10]**.** *Assume that* $\Omega_n\subseteq\mathbb{R}^n_+,\ x=(x_1,\ \ldots,\ x_n),\ \frac{1}{p}+\frac{1}{q}=1,\ 0<p<1,\ q<+\infty,$ $f(x)\geq 0,\ g(y)\geq 0,\ \omega(x)\geq 0$. *Then we have an inverse Hölder's integral inequality*

$$\int_{\Omega_n} f(x)g(x)\omega(x)dx \geq \left(\int_{\Omega_n} f^p(x)\omega(x)dx\right)^{1/p}\left(\int_{\Omega_n} g^q(x)\omega(x)dx\right)^{1/q}.$$

Lemma 2. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1,\ 0<p<1,\ q<0,\ \lambda_1\lambda_2>0, a,b\in\mathbb{R},\ K(x,\ y)=G(x^{\lambda_1},y^{\lambda_2})$ *is a quasihomogeneous function with parameters* $\{\lambda,\ \lambda_1,\ \lambda_2\}$. *Denote*

$$W_1(s)=\int_0^{+\infty}K(1,t)t^s dt,\ W_2(s)=\int_0^{+\infty}K(t,1)t^s dt.$$

Then

$$\omega_1(b,p,x)=\int_0^{+\infty}K(x,y)y^{-bp}dy = x^{\lambda_1\left[\lambda-\frac{1}{\lambda_2}(bp-1)\right]}W_1(-bp),$$

$$\omega_2(a,q,y)=\int_0^{+\infty}K(x,y)x^{-aq}dx = y^{\lambda_2\left[\lambda-\frac{1}{\lambda_1}(aq-1)\right]}W_2(-aq).$$

If $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$, *then* $\frac{1}{\lambda_1}W_1(-bp)=\frac{1}{\lambda_2}W_2(-aq)$.

Proof. Since $K(x,\ y)$ is a quasihomogeneous function with parameters $\{\lambda,\ \lambda_1,\ \lambda_2\}$, it follows that

$$\omega_1(b,p,x) = x^{\lambda\lambda_1} \int\limits_0^{+\infty} K(1, x^{-\lambda_1/\lambda_2} y) y^{-bp} dy = x^{\lambda\lambda_1 - \frac{\lambda_1}{\lambda_2}(bp-1)} \int\limits_0^{+\infty} K(1,t) t^{-bp} dt$$

$$= x^{\lambda_1\left[\lambda - \frac{1}{\lambda_2}(bp-1)\right]} W_1(-bp).$$

Similarly, we can prove

$$\omega_2(a,q,y) = y^{\lambda_2\left[\lambda - \frac{1}{\lambda_1}(aq-1)\right]} W_2(-aq).$$

If $\dfrac{1}{\lambda_1} aq + \dfrac{1}{\lambda_2} bp = \lambda + \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2}$, then

$$W_1(-bp) = \int\limits_0^{+\infty} K(t^{-\lambda_2/\lambda_1}, 1) t^{\lambda\lambda_2 - bp} dt = \frac{\lambda_1}{\lambda_2} \int\limits_0^{+\infty} K(u,1) u^{-\frac{\lambda_1}{\lambda_2}(\lambda\lambda_2 - bp) - \frac{\lambda_1}{\lambda_2} - 1} du$$

$$= \frac{\lambda_1}{\lambda_2} \int\limits_0^{+\infty} K(u,1) u^{-aq} du = \frac{\lambda_1}{\lambda_2} W_2(-aq).$$

Hence, $\dfrac{1}{\lambda_1} W_1(-bp) = \dfrac{1}{\lambda_2} W_2(-aq)$.

Lemma 2 is proved.

Lemma 3. *Assume that* $\dfrac{1}{p} + \dfrac{1}{q} = 1,\ 0 < p < 1,\ q < 0,\ \alpha, \beta \in \mathbb{R},\ K(x,\ y) \geq 0,\ f(x) \geq 0, g(y) \geq 0$. *Define* T *by*

$$T(f)(y) = \int\limits_0^{+\infty} K(x,y) f(x) dx.$$

Then (1) *is equivalent to the operator inequality* $||T(f)||_{p,\beta(1-p)} \geq M||f||_{p,\alpha}$.

Proof. If $||T(f)||_{p,\beta(1-p)} \geq M||f||_{p,\alpha}$, then by virtue of Lemma 1 we have

$$\int\limits_0^{+\infty}\int\limits_0^{+\infty} K(x,y) f(x) g(y) dx dy = \int\limits_0^{+\infty} g(y) T(f)(y) dy$$

$$= \int\limits_0^{+\infty} \left(y^{\beta/q} g(y)\right)\left(y^{-\beta/q} T(f)(y)\right) dy$$

$$\geq \left(\int\limits_0^{+\infty} y^{-p\beta/q} (T(f)(y))^p dy\right)^{1/p} \left(\int\limits_0^{+\infty} y^{\beta} g^q(y) dy\right)^{1/q}$$

$$= ||T(f)||_{p,\beta(1-p)}||g||_{q,\beta} \geq M||f||_{p,\alpha}||g||_{q,\beta}.$$

On the contrary, if (1) holds, then

$$||T(f)||^p_{p,\beta(1-p)} = \int_0^{+\infty} y^{\beta(1-p)}(T(f)(y))^p dy$$

$$= \int_0^{+\infty} y^{\beta(1-p)} \left(\int_0^{+\infty} K(x,y)f(x)dx \right)^p dy$$

$$= \int_0^{+\infty} \left(\int_0^{+\infty} K(x,y)f(x)dx \right) \left(y^{-\beta} \int_0^{+\infty} K(x,y)f(x)dx \right)^{p-1} dy.$$

Denote $g(y) = \left(y^{-\beta} \int_0^{+\infty} K(x,y)f(x)dx \right)^{p-1}$. Then we obtain

$$||T(f)||^p_{p,\beta(1-p)} = \int_0^{+\infty}\int_0^{+\infty} K(x,y)f(x)g(y)dxdy$$

$$\geq M||f||_{p,\alpha}||g||_{q,\beta} = M||f||_{p,\alpha} \left(\int_0^{+\infty} y^{\beta} g^q(y)dy \right)^{1/q}$$

$$= M||f||_{p,\alpha} \left(\int_0^{+\infty} y^{\beta(1-p)}(T(f)(y))^p dy \right)^{1/q} = M||f||_{p,\alpha}||T(f)||^{p/q}_{p,\beta(1-p)}.$$

Thus, $||T(f)||_{p,\beta(1-p)} \geq M||f||_{p,\alpha}$.

Lemma 3 is proved.

3. Sufficient and necessary conditions for optimal matching parameters.

Theorem 1. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1,\ 0 < p < 1,\ q < 0,\ \lambda_1\lambda_2 > 0,\ a,\ b \in \mathbb{R},\ K(x,\ y)$ *is a quasihomogeneous nonnegative measurable function with parameters* $\{\lambda,\ \lambda_1,\ \lambda_2\}$, $W_1(-bp) < +\infty$, $W_2(-aq) < +\infty$, *there exists a constant* $\sigma > 0$ *such that at least one of* $W_1(-bp \pm \sigma)$ *and* $W_2(-aq \pm \sigma)$ *converges. Then*:

(i) *Denote*

$$\alpha = \lambda_1 \left[\lambda + \frac{1}{\lambda_2} + p\left(\frac{a}{\lambda_1} - \frac{b}{\lambda_2} \right) \right], \quad \beta = \lambda_2 \left[\lambda + \frac{1}{\lambda_1} + q\left(\frac{b}{\lambda_2} - \frac{a}{\lambda_1} \right) \right],$$

one has

$$A(f,g) = \int_0^{+\infty}\int_0^{+\infty} K(x,y)f(x)g(y)dxdy$$

$$\geq W_1^{1/p}(-bp)W_2^{1/q}(-aq)||f||_{p,\alpha}||g||_{q,\beta}, \tag{3}$$

where $f(x) \in L_p^{\alpha}(0,+\infty),\ g(y) \in L_q^{\beta}(0,+\infty)$.

(ii) *The following three conditions are equivalent*:

(a) *the constant factor* $W_1^{1/p}(-bp)W_2^{1/q}(-aq)$ *of* (3) *is the best*;

(b) $\dfrac{1}{\lambda_1}aq + \dfrac{1}{\lambda_2}bp = \lambda + \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2}$;

(c) $\dfrac{1}{\lambda_1}W_1(-bp) = \dfrac{1}{\lambda_2}W_2(-aq)$.

(iii) *For* $\dfrac{1}{\lambda_1}aq + \dfrac{1}{\lambda_2}bp = \lambda + \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2}$, (3) *becomes*

$$A(f,g) \geq \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}||f||_{p,apq-1}||g||_{q,bpq-1}, \tag{4}$$

where $W_0 = |\lambda_2|W_1(-bp) = |\lambda_1|W_2(-aq)$.

Proof. (i) It follows from Lemmas 1 and 2 that

$$A(f,g) = \int_0^{+\infty}\int_0^{+\infty}\left(\frac{x^a}{y^b}f(x)\right)\left(\frac{y^b}{x^a}g(y)\right)K(x,y)dxdy$$

$$\geq \left(\int_0^{+\infty}\int_0^{+\infty}\frac{x^{ap}}{y^{bp}}f^p(x)K(x,y)dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{y^{bq}}{x^{aq}}g^q(y)K(x,y)dxdy\right)^{1/q}$$

$$= \left(\int_0^{+\infty}x^{ap}f^p(x)\omega_1(b,p,x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{bq}g^q(y)\omega_2(a,q,y)dy\right)^{1/q}$$

$$= W_1^{1/p}(-bp)W_2^{1/q}(-aq)\left(\int_0^{+\infty}x^{ap+\lambda_1\left[\lambda-\frac{1}{\lambda_2}(bp-1)\right]}f^p(x)dx\right)^{1/p}$$

$$\times\left(\int_0^{+\infty}y^{bq+\lambda_2\left[\lambda-\frac{1}{\lambda_1}(aq-1)\right]}g^q(y)dy\right)^{1/q}$$

$$= W_1^{1/p}(-bp)W_2^{1/q}(-aq)||f||_{p,\alpha}||g||_{q,\beta}.$$

Hence, (3) holds.

(ii) (b) $\Rightarrow$ (a) Suppose that $\dfrac{1}{\lambda_1}aq + \dfrac{1}{\lambda_2}bp = \lambda + \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2}$, then $\alpha = apq-1, \beta = bpq-1$. In addition, it follows from Lemma 2 that $\dfrac{1}{\lambda_1}W_1(-bp) = \dfrac{1}{\lambda_2}W_2(-aq)$. Consequently, (3) becomes (4).

If the constant factor in (4) is not optimal, then there exists a constant $M_0 > W_0/(|\lambda_1|^{1/q}|\lambda_2|^{1/p})$ such that $A(f,g) \geq M_0||f||_{p,apq-1}||g||_{q,bpq-1}$.

When $W_1(-bp+\sigma) < +\infty$, for sufficiently small $\varepsilon > 0$, take

$$f(x) = \begin{cases} x^{(-apq-|\lambda_1|\varepsilon)/p}, & x \geq 1, \\ 0, & 0 < x < 1, \end{cases} \qquad g(y) = \begin{cases} y^{(-bpq-|\lambda_2|\varepsilon)/q}, & y \geq 1, \\ 0, & 0 < y < 1, \end{cases}$$

one has

$$\begin{aligned} ||f||_{p,apq-1}||g||_{q,bpq-1} &= \left(\int_1^{+\infty} x^{-1-|\lambda_1|\varepsilon}dx\right)^{1/p}\left(\int_1^{+\infty} y^{-1-|\lambda_2|\varepsilon}dy\right)^{1/q} \\ &= \frac{1}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}\varepsilon}, \\ A(f,g) &= \int_1^{+\infty} x^{-aq-\frac{|\lambda_1|\varepsilon}{p}}\left(\int_1^{+\infty} y^{-bp-\frac{|\lambda_2|\varepsilon}{q}}K(x,y)dy\right)dx \\ &= \int_1^{+\infty} x^{-aq-\frac{|\lambda_1|\varepsilon}{p}+\lambda\lambda_1}\left(\int_1^{+\infty} y^{-bp-\frac{|\lambda_2|\varepsilon}{q}}K(1,x^{-\lambda_1/\lambda_2}y)dy\right)dx \\ &= \int_1^{+\infty} x^{\lambda_1(\lambda+\frac{1}{\lambda_2}-\frac{1}{\lambda_1}aq-\frac{1}{\lambda_2}bp)-|\lambda_1|\varepsilon}\left(\int_{x^{-\lambda_1/\lambda_2}}^{+\infty} t^{-bp-\frac{|\lambda_2|\varepsilon}{q}}K(1,t)dt\right)dx \\ &\leq \int_1^{+\infty} x^{-1-|\lambda_1|\varepsilon}dx\int_0^{+\infty} t^{-bp-\frac{|\lambda_2|\varepsilon}{q}}K(1,t)dt \\ &= \frac{1}{|\lambda_1|\varepsilon}W_1\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right). \end{aligned}$$

Thereupon

$$\frac{1}{|\lambda_1|}W_1\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right) \geq \frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}.$$

Since $q < 0$, then $-\dfrac{|\lambda_2|\varepsilon}{q} > 0$. And since ε is sufficiently small, we have $-\dfrac{|\lambda_2|\varepsilon}{q} < \sigma$, it follows that

$$\begin{aligned} W_1\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right) &= \int_0^1 t^{-bp-\frac{|\lambda_2|\varepsilon}{q}}K(1,t)dt + \int_1^{+\infty} t^{-bp-\frac{|\lambda_2|\varepsilon}{q}}K(1,t)dt \\ &\leq \int_0^1 t^{-bp}K(1,t)dt + \int_1^{+\infty} t^{-bp+\sigma}K(1,t)dt \\ &\leq W_1(-bp) + W_1(-bp+\sigma) < +\infty. \end{aligned}$$

Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0^+} W_1\left(-bp-\frac{|\lambda_2|\varepsilon}{q}\right)=W_1(-bp),$$

consequently,

$$\frac{1}{|\lambda_1|}W_1(-bp)\geq\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}.$$

Thus, $M_0\leq W_0/(|\lambda_1|^{1/q}|\lambda_2|^{1/p})$, which is in contradiction with $M_0>W_0/(|\lambda_1|^{1/q}|\lambda_2|^{1/p})$.

If $W_1(-bp-\sigma)<+\infty$, take, for sufficiently small $\varepsilon>0$,

$$f(x)=\begin{cases}x^{(-apq+|\lambda_1|\varepsilon)/p}, & 0<x\leq 1,\\ 0, & x>1,\end{cases}\qquad g(y)=\begin{cases}y^{(-bpq+|\lambda_2|\varepsilon)/q}, & 0<y\leq 1,\\ 0, & y>1.\end{cases}$$

Then similar results can be obtained

$$\frac{1}{|\lambda_1|}W_1(-bp+\frac{|\lambda_2|\varepsilon}{q})\geq\frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}.$$

It follows from $q<0$ that $\dfrac{|\lambda_2|\varepsilon}{q}<0$. Owing to ε is sufficiently small, then $\dfrac{|\lambda_2|\varepsilon}{q}>-\sigma$ and

$$\begin{aligned}W_1\left(-bp+\frac{|\lambda_2|\varepsilon}{q}\right)&=\int_0^1 t^{-bp+\frac{|\lambda_2|\varepsilon}{q}}K(1,t)dt+\int_1^{+\infty}t^{-bp+\frac{|\lambda_2|\varepsilon}{q}}K(1,t)dt\\&\leq\int_0^1 t^{-bp-\sigma}K(1,t)dt+\int_1^{+\infty}t^{-bp}K(1,t)dt\\&\leq W_1(-bp-\sigma)+W_1(-bp)<+\infty.\end{aligned}$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0^+} W_1\left(-bp+\frac{|\lambda_2|\varepsilon}{q}\right)=W_1(-bp).$$

Thereupon we also get $M_0\leq W_0/(|\lambda_1|^{1/q}|\lambda_2|^{1/p})$, which is in contradiction with $M_0>W_0/(|\lambda_1|^{1/q}|\lambda_2|^{1/p})$.

In conclusion, when $W_1(-bp+\sigma)<+\infty$ or $W_1(-bp-\sigma)<+\infty$, the constant factor in (4) is the best.

Similarly, the case of $W_2(-aq+\sigma)<+\infty$ or $W_1(-aq-\sigma)<+\infty$ can be proved.

(a) $\Rightarrow$ (b) Assume that the constant factor $W_1^{1/p}(-bp)\ W_2^{1/q}(-aq)$ of (3) is the best. Denote

$$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp-\left(\lambda+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)=c,\quad a'=a-\frac{\lambda_1 c}{pq},\quad b'=b-\frac{\lambda_2 c}{pq},$$

then

$$\alpha' = \lambda_1\left[\lambda + \frac{1}{\lambda_2} + p\left(\frac{a'}{\lambda_1} - \frac{b'}{\lambda_2}\right)\right] = \alpha, \quad \beta' = \lambda_2\left[\lambda + \frac{1}{\lambda_1} + q\left(\frac{b'}{\lambda_2} - \frac{a'}{\lambda_1}\right)\right] = \beta$$

and

$$W_2(-aq) = \int_0^{+\infty} K(t,1)t^{-aq}dt = \int_0^{+\infty} t^{\lambda\lambda_1 - aq}K(1, t^{-\lambda_1/\lambda_2})dt$$

$$= \frac{\lambda_2}{\lambda_1}\int_0^{+\infty} u^{-\frac{\lambda_2}{\lambda_1}(\lambda\lambda_1 - aq) - \frac{\lambda_2}{\lambda_1} - 1}K(1,u)du$$

$$= \frac{\lambda_2}{\lambda_1}W_1(-bp + \lambda_2 c).$$

Hence, (3) is equivalent to

$$A(f,g) \geq W_1^{1/p}(-bp)\left(\frac{\lambda_2}{\lambda_1}W_1(-bp+\lambda_2 c)\right)^{1/q}||f||_{p,\alpha'}||g||_{q,\beta'}.$$

Note that $\frac{1}{\lambda_1}a'q + \frac{1}{\lambda_2}b'p = \frac{1}{\lambda_1}q\left(a - \frac{\lambda_1 c}{pq}\right) + \frac{1}{\lambda_2}p\left(b - \frac{\lambda_2 c}{pq}\right) = \frac{1}{\lambda_1}aq + \frac{1}{\lambda_2}bp - c = \lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$. Then $\alpha' = a'pq - 1,\ \beta' = b'pq - 1$. Thus, (3) is further equivalent to

$$A(f,g) \geq W_1^{1/p}(-bp)\left(\frac{\lambda_2}{\lambda_1}W_1(-bp+\lambda_2 c)\right)^{1/q}||f||_{p,a'pq-1}||g||_{q,b'pq-1}. \tag{5}$$

Since the constant factor of (3) is the best, the optimal constant factor of (5) is

$$W_1^{1/p}(-bp)\left(\frac{\lambda_2}{\lambda_1}W_1(-bp+\lambda_2 c)\right)^{1/q}.$$

Note also that $\frac{1}{\lambda_1}a'q + \frac{1}{\lambda_2}b'p = \lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ and $W_1(-b'p - \frac{\lambda_2 c}{q}) = \frac{\lambda_1}{\lambda_2}W_2(-bp) < +\infty$. According to the proof of (b) $\Rightarrow$ (a), the optimal constant factor of (5) is

$$\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\big(|\lambda_2|W_1(-b'p)\big) = \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1\left(-bp + \frac{\lambda_2 c}{q}\right).$$

Therefore,

$$W_1\left(-bp + \frac{\lambda_2 c}{q}\right) = W_1^{1/p}(-bp)W_1^{1/q}(-bp+\lambda_2 c). \tag{6}$$

Since $\frac{1}{p} + \frac{1}{q} = 1,\ 0 < p < 1,\ q < 0$, applying the reverse Hölder inequality, we deduce

$$W_1\left(-bp + \frac{\lambda_2 c}{q}\right) = \int_0^{+\infty} 1 \cdot t^{\lambda_2 c/q}K(1,t)t^{-bp}dt$$

$$\geq \left(\int_0^{+\infty} 1^p \cdot K(1,t)t^{-bp}dt \right)^{1/p} \left(\int_0^{+\infty} t^{\lambda_2 c} K(1,t)t^{-bp}dt \right)^{1/q}$$

$$= W_1^{1/p}(-bp)W_1^{1/q}(-bp+\lambda_2 c). \tag{7}$$

It follows from (6) that (7) takes the equal sign. Then according to the condition with the equal sign of the reverse Hölder inequality holds, we get $t^{\lambda_2 c} = \text{constant}$. Thus $c = 0$, that is, $\frac{1}{\lambda_1}aq + \frac{1}{\lambda_2}bp = \lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

(b) $\Rightarrow$ (c) It can be obtained by using Lemma 2.

(c) $\Rightarrow$ (b) Assuming $\frac{1}{\lambda_1}W_1(-bp) = \frac{1}{\lambda_2}W_2(-aq)$, denote still $\frac{1}{\lambda_1}aq + \frac{1}{\lambda_2}bp - \left(\lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) = c$, then

$$W_1(-bp) = \frac{\lambda_1}{\lambda_2}W_2(-aq) = \frac{\lambda_1}{\lambda_2}\int_0^{+\infty} t^{\lambda\lambda_1 - aq}K(1, t^{-\lambda_1/\lambda_2})dt$$

$$= \int_0^{+\infty} t^{\lambda_2\left[\frac{1}{\lambda_1}aq - \left(\lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right]}K(1,t)dt = \int_0^{+\infty} t^{\lambda_2 c}K(1,t)t^{-bp}dt.$$

Set $\frac{1}{r} + \frac{1}{s} = 1,\ 0 < r < 1,\ s < 0$, we have

$$W_1(-bp) = \int_0^{+\infty} 1 \cdot t^{\lambda_2 c}K(1,t)t^{-bp}dt$$

$$\geq W_1^{1/r}(-bp)\left(\int_0^{+\infty} t^{\lambda_2 cs}K(1,t)t^{-bp}dt\right)^{1/s}.$$

Thus,

$$W_1(-bp) \geq \int_0^{+\infty} t^{\lambda_2 cs}K(1,t)t^{-bp}dt.$$

If $\lambda_2 c > 0$, then $\lambda_2 cs < 0$. Therefore,

$$W_1(-bp) \geq \int_0^{1/2} t^{\lambda_2 cs}K(1,t)t^{-bp}dt \geq \left(\frac{1}{2}\right)^{\lambda_2 cs} \int_0^{1/2} K(1,t)t^{-bp}dt.$$

We have $W_1(-bp) = +\infty$ as $s \to -\infty$, which contradicts $W_1(-bp) < +\infty$.

If $\lambda_2 c < 0$, then $\lambda_2 cs > 0$ and

$$W_1(-bp) \geq \int_2^{+\infty} t^{\lambda_2 cs} K(1,t) t^{-bp} dt \geq 2^{\lambda_2 cs} \int_2^{+\infty} K(1,t) t^{-bp} dt.$$

Similarly, $W_1(-bp) = +\infty$ as $s \to -\infty$, which contradicts $W_1(-bp) < +\infty$.

In conclusion, we get $\lambda_2 c = 0$, hence $c = 0$, that is, $\frac{1}{\lambda_1} aq + \frac{1}{\lambda_2} bp = \lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

(iii) It can be obtained by following the proof of (b) $\Rightarrow$ (a).

Theorem 1 is proved.

4. Applications in operator theory. According to Lemma 3 and Theorem 1, the following theorem concerning integral operators can be obtained.

Theorem 2. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1,\ 0 < p < 1,\ q < 0,\ \lambda_1\lambda_2 > 0,\ a,\ b \in \mathbb{R},\ K(x,\ y)$ *is a quasihomogeneous nonnegative measurable function with parameters* $\{\lambda,\ \lambda_1,\ \lambda_2\}$, $W_1(-bp) < +\infty$, $W_2(-aq) < +\infty$, *there exist a constant* $\sigma > 0$ *such that at least one of* $W_1(-bp \pm \sigma)$ *and* $W_2(-aq \pm \sigma)$ *converges. Then*:

(i) *Denote*

$$\alpha = \lambda_1 \left[\lambda + \frac{1}{\lambda_2} + p\left(\frac{a}{\lambda_1} - \frac{b}{\lambda_2} \right) \right], \quad \beta = \lambda_2 \left[\lambda + \frac{1}{\lambda_1} + q\left(\frac{b}{\lambda_2} - \frac{a}{\lambda_1} \right) \right],$$

then

$$||T(f)||_{p,\beta(1-p)} \geq W_1^{1/p}(-bp) W_2^{1/q}(-aq) ||f||_{p,\alpha}, \tag{8}$$

where $f(x) \in L_p^{\alpha}(0,+\infty)$.

(ii) *If* $T(f)(y) \in L_p^{\beta(1-p)}(0,+\infty)$, *then* $f(x) \in L_p^{\alpha}(0,+\infty)$ *if and only if*

$$||T||^* \triangleq \inf \left\{ \frac{||T(f)||_{p,\beta(1-p)}}{||f||_{p,\alpha}} : T(f)(y) \in L_p^{\beta(1-p)}(0,+\infty), ||f||_{p,\alpha} > 0 \right\}$$
$$= W_1^{1/p}(-bp) W_2^{1/q}(-aq).$$

(iii) *For* $\frac{1}{\lambda_1} aq + \frac{1}{\lambda_2} bp = \lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$, *(8) becomes*

$$||T(f)||_{p,(bpq-1)(1-p)} \geq \frac{W_0}{|\lambda_1|^{1/q} |\lambda_2|^{1/p}} ||f||_{p,apq-1},$$

where $W_0 = |\lambda_1| W_2(-aq) = |\lambda_2| W_1(-bp)$.

***Example* 1.** Let $\frac{1}{p} + \frac{1}{q} = 1,\ 0 < p < 1,\ q < 0,\ \lambda > 0,\ \lambda_1 > 0,\ \lambda_2 > 0,\ aq < 1,\ bp < 1,$ $\frac{1}{\lambda_1} aq + \frac{1}{\lambda_2} bp = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \lambda$. Define the integral operator T by

$$T(f)(y) = \int_0^{+\infty} \frac{1}{(x^{\lambda_1} + y^{\lambda_2})^{\lambda}} f(x) dx. \tag{9}$$

Then

$$||T(f)||_{p,(bpq-1)(1-p)} \geq \frac{B\left(\frac{1}{\lambda_1}(1-aq), \frac{1}{\lambda_2}(1-bp)\right)}{\lambda_1^{1/q}\lambda_2^{1/p}} ||f||_{p,apq-1}$$

and

$$||T||^* = \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} B\left(\frac{1}{\lambda_1}(1-aq), \frac{1}{\lambda_2}(1-bp)\right).$$

Proof. (i) Given $K(x,y) = 1/(x^{\lambda_1} + y^{\lambda_2})^\lambda$, then $K(x,\, y)$ is a quasihomogeneous function with parameters $\{-\lambda,\, \lambda_1,\, \lambda_2\}$. Since

$$W_1(-bp) = \int\limits_0^{+\infty} \frac{1}{(1+t^{\lambda_2})^\lambda} t^{-bp} dt = \frac{1}{\lambda_2} \int\limits_0^{+\infty} \frac{1}{(1+u)^\lambda} u^{\frac{1}{\lambda_2}(1-bp)-1} du$$

$$= \frac{1}{\lambda_2} B\left(\frac{1}{\lambda_2}(1-bp), \lambda - \frac{1}{\lambda_2}(1-bp)\right) = \frac{1}{\lambda_2} B\left(\frac{1}{\lambda_1}(1-aq), \frac{1}{\lambda_2}(1-bp)\right),$$

then $W_0 = |\lambda_2| W_1(-bp) = B\left(\frac{1}{\lambda_1}(1-aq), \frac{1}{\lambda_2}(1-bp)\right)$.

Note that $1 - aq > 0, \lambda_i > 0,\ i = 1, 2$, there exists $\sigma > 0$ such that $\frac{1}{\lambda_1}(1-aq) - \frac{\sigma}{\lambda_2} > 0$. Thus,

$$W_1(-bp+\sigma) = \int\limits_0^{+\infty} \frac{1}{(1+t^{\lambda_2})^\lambda} t^{-bp+\sigma} dt$$

$$= \frac{1}{\lambda_2} \int\limits_0^{+\infty} \frac{1}{(1+u)^\lambda} u^{\frac{1}{\lambda_2}(1-bp+\sigma-1)} du$$

$$= \frac{1}{\lambda_2} B\left(\frac{1}{\lambda_2}(1-bp+\sigma), \lambda - \frac{1}{\lambda_2}(1-bp+\sigma)\right)$$

$$= \frac{1}{\lambda_2} B\left(\frac{1}{\lambda_2}(1-bp+\sigma), \frac{1}{\lambda_1}(1-aq) - \frac{\sigma}{\lambda_2}\right) < +\infty.$$

The example holds according to Theorem 2.

Take $b = \frac{1}{pq}$ in Example 1, then $bp = \frac{1}{q} < 0 < 1,\ bpq - 1 = 0$. We obtain the following example.

***Example* 2.** Let $\frac{1}{p} + \frac{1}{q} = 1,\ 0 < p < 1,\ q < 0,\ \lambda > 0,\ \lambda_1 > 0, \lambda_2 > 0,\ aq < 1,\ \frac{1}{\lambda_1}(1-aq) + \frac{1}{\lambda_2 p} = \lambda$. Then the integral operator T defined by (9) satisfies

$$||T(f)||_p \geq \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} B\left(\frac{1}{\lambda_1}(1-aq), \frac{1}{\lambda_2 p}\right) ||f||_{p,apq-1}$$

and

$$||T||^* = \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} B\left(\frac{1}{\lambda_1}(1 - aq), \frac{1}{\lambda_2 p}\right).$$

Remark. Under the conditions of Example 2, the necessary condition of $T(f)(y) \in L_p(0, +\infty)$ is $f(x) \in L_p^{apq-1}(0, +\infty)$. Thus, we can infer $T(f)(y) \notin L_p(0, +\infty)$ from $f(x) \notin L_p^{apq-1}(0, +\infty)$.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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