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LYAPUNOV-TYPE INEQUALITIES FOR A NONLINEAR SYSTEM INCLUDING OPERATORS

НЕРІВНОСТІ ТИПУ ЛЯПУНОВА ДЛЯ ДЕЯКОЇ НЕЛІНІЙНОЇ СИСТЕМИ З ОПЕРАТОРАМИ

We obtain new Lyapunov-type inequalities for a nonlinear system including p-relativistic operator and q-prescribed curvature operator under the Dirichlet or antiperiodic boundary condition.

Отримано нові нерівності типу Ляпунова для нелінійної системи, що включає p-релятивістський оператор і q-оператор заданої кривини у випадку Діріхле або антиперіодичної граничної умови.

1. Introduction. In this paper, we get new Lyapunov-type inequalities for the following nonlinear system:

$$-\big(\phi_{p_m}(x_m')\big)' = h_m(t)\Phi_{\alpha_{mm}}(x_m)\prod_{\substack{i=1\\ i\neq m}}^{n}\frac{|x_i|^{\alpha_{mi}}}{\left(1+|x_i|^{\alpha_{mi}}\right)^{\frac{\alpha_{mi}-1}{\alpha_{mi}}}}, \quad m=1,2,\ldots,n, \quad n\in\mathbb{N}, \tag{1.1}$$

including p-relativistic operator

$$\phi_p(v)=\frac{|v|^{p-2}v}{\left(1-|v|^p\right)^{\frac{p-1}{p}}}, \quad v\in(-1,1) \quad \text{and} \quad p>1,$$

and q-prescribed curvature operator

$$\Phi_q(u)=\frac{|u|^{q-2}u}{\left(1+|u|^q\right)^{\frac{q-1}{q}}}, \quad u\in\mathbb{R} \quad \text{and} \quad q>1.$$

Here, the function h_m satisfies $h_m\not\equiv 0$ in any compact subinterval of $[t_1,t_2]$ and

$$h_m^+\in\mathcal{A}\stackrel{\triangle}{=}\left\{h_m^+\in L^1_{\text{loc}}(t_1,t_2):\ \int\limits_{t_1}^{t_2} h_m^+(\tau)\prod_{i=1}^{n}\left(\frac{[(\tau-t_1)(t_2-\tau)]^{p_i-1}}{(\tau-t_1)^{p_i-1}+(t_2-\tau)^{p_i-1}}\right)^{\frac{\alpha_{mi}}{p_i}}d\tau<\infty\right\},$$

where

$$h_m^+(t)=\max\{0,h_m(t)\}, \tag{1.2}$$

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$1 < p_m < \infty$, and α_{mi} is a nonnegative constant, $m,\ i = 1, 2, \ldots, n$. Assume that $(x_1, x_2, \ldots, x_n)$ is a real solution (not identically zero) of the nonlinear boundary-value problem (1.1) satisfying the Dirichlet boundary condition

$$x_m(t_1) = 0 = x_m(t_2) \tag{1.3}$$

or the antiperiodic boundary condition

$$x_m^{(k)}(t_1) + x_m^{(k)}(t_2) = 0, \tag{1.4}$$

where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and x_m is not identically zero on $[t_1, t_2]$, $k = 0, 1$, $m = 1, 2, \ldots, n$. We say that $(x_1, x_2, \ldots, x_n)$ is a solution of nonlinear system (1.1), if $x_m \in C^1[t_1, t_2]$, $\|x'_m\|_\infty < 1$, and $\phi_{p_m}(x'_m(\cdot))$, $m = 1, 2, \ldots, n$, is absolutely continuous in any compact subinterval of (t_1, t_2), and x_m satisfies the nonlinear system (1.1) and the Dirichlet boundary condition (1.3) (or the antiperiodic boundary condition (1.4)).

Note that the inverse of p-relativistic operator ϕ_p is $\Phi_q(u) = \phi_p^{-1}(u)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Moreover, whereas ϕ_p is a singular function at ± 1, Φ_q is a bounded regular function ($|\Phi_q| < 1$). If we take $p = q = 2$, they turn out to be the relativistic and prescribed curvature operator, which have been attracting much attention in differential geometry and partial differential equation [12, 21]. It has many important applications in physics, biology and other interdisciplines. In this paper, we mainly consider the Lyapunov-type inequalities for the nonlinear system (1.1) including the p-relativistic operator ϕ_p and q-prescribed curvature operator Φ_q under the Dirichlet boundary condition (1.3) (or the antiperiodic boundary condition (1.4)). The reader can refer to [1 – 3, 13 – 15] for the results including these operators. To the best of our knowledge, this is the first paper using the p-relativistic operator ϕ_p and q-prescribed curvature operator Φ_q to study the system (1.1).

In 2019, Yang et al. [22] proved that if

$$h_1 \in \mathcal{A}_1 = \left\{ h_1 \in L^1_{\text{loc}}((t_1, t_2), [0, \infty)) : \int_{t_1}^{t_2} h_1(\tau)(\tau - t_1)(t_2 - \tau)\, d\tau < \infty \right\}$$

and nonlinear problem

$$-\left(\frac{x'_1}{\sqrt{1 - |x'_1|^2}} \right)' = h_1(t) x_1$$

with the Dirichlet boundary condition (1.3) with $n = 1$ has a positive solution, then the Lyapunov-type inequality

$$t_2 - t_1 < \int_{t_1}^{t_2} (\tau - t_1)(t_2 - \tau) h_1(\tau)\, d\tau \tag{1.5}$$

holds.

If we take $n = 1$ and $\alpha_{11} = p_1 = 2$ in the nonlinear system (1.1), then it reduces to the following single equation:

$$-\left(\frac{x_1'}{\sqrt{1-|x_1'|^2}}\right)' = h_1(t)\frac{x_1}{\sqrt{1+|x_1|^2}}. \tag{1.6}$$

Recently, Aktaş [1, 2] has considered the Lyapunov-type inequalities for Hamiltonian-type system

$$\begin{aligned} x_1' &= h_2(t)\,x_1 + \frac{|y_1|^{p-2}y_1}{(1+|y_1|^p)^{\frac{p-1}{p}}}, \\ y_1' &= -h_1(t)\frac{|x_1|^{q-2}x_1}{(1+|x_1|^q)^{\frac{q-1}{q}}} - h_2(t)y_1, \end{aligned} \tag{1.7}$$

under the Dirichlet boundary condition and $(n+1)$st order nonlinear differential equations

$$-\left(\frac{\left|x_1^{(n)}\right|^{p-2}x_1^{(n)}}{\left(1-\left|x_1^{(n)}\right|^p\right)^{\frac{p-1}{p}}}\right)' = \sum_{i=0}^{n-1} h_{i+1}(t)\frac{\left|x_1^{(i)}\right|^{q-2}x_1^{(i)}}{\left(1+\left|x_1^{(i)}\right|^q\right)^{\frac{q-1}{q}}} \tag{1.8}$$

under the antiperiodic boundary conditions. Note that if we take $p = q = 2$ and $h_2(t) = 0$ in the system (1.7) and $p = q = 2$ and $n = 1$ in Eq. (1.8), then they reduce to Eq. (1.6).

Our motivation comes from the recent papers of Aktaş [1, 2], Tiryaki et al. [18], and Yang et al. [22]. In this paper, on Lyapunov-type inequalities, we prove new results for nonlinear system (1.1) with (1.3) (or (1.4)) including p-relativistic operator and q-prescribed curvature operator. For some of the most recent works on Lyapunov-type inequalities for the systems under various types of boundary conditions, the reader is referred to [4–11, 17–19].

2. Main results. Now, we give one of the main results.

Theorem 2.1. *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system*

$$e_m\left(1-\frac{\alpha_{mm}}{p_m}\right) - \sum_{\substack{i=1\\ i\neq m}}^{n}\frac{\alpha_{im}}{p_m}e_i = 0, \tag{2.1}$$

where $0 < \sum_{i=1}^{n} e_i^2$ *and* $0 \le e_m,\ m = 1, 2, \ldots, n$. *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.3) *and* $h_m^+ \in \mathcal{A},\ m = 1, 2, \ldots, n$, *then*

$$1 < \prod_{m=1}^{n}\left[\int_{t_1}^{t_2} h_m^+(\tau)\prod_{i=1}^{n}\left(\frac{[(\tau-t_1)(t_2-\tau)]^{p_i-1}}{(\tau-t_1)^{p_i-1}+(t_2-\tau)^{p_i-1}}\right)^{\frac{\alpha_{mi}}{p_i}} d\tau\right]^{e_m} \tag{2.2}$$

holds, where $h_m^+(t)$ *is given in* (1.2).

Proof. Let $x_m(t_1) = 0 = x_m(t_2),\ m = 1, 2, \ldots, n$ where $n \in \mathbb{N},\ t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $x_m,\ m = 1, 2, \ldots, n,$ is nonzero on $[t_1, t_2]$. From $x_m(t_1) = 0$ and Hölder's inequality, we get

$$|x_m(t)| \leq \int_{t_1}^{t} \left|x'_m(\tau)\right| d\tau \leq (t - t_1)^{\frac{p_m - 1}{p_m}} \left(\int_{t_1}^{t} \left|x'_m(\tau)\right|^{p_m} d\tau \right)^{\frac{1}{p_m}}$$

for $m = 1, 2, \ldots, n$ and $t \in [t_1, t_2]$. Then we have

$$|x_m(t)|^{p_m} (t - t_1)^{1 - p_m} \leq \int_{t_1}^{t} \left|x'_m(\tau)\right|^{p_m} d\tau \tag{2.3}$$

for $m = 1, 2, \ldots, n$ and $t \in [t_1, t_2]$. On the other hand, from $x_m(t_2) = 0$ and Hölder's inequality, we obtain

$$|x_m(t)|^{p_m} (t_2 - t)^{1 - p_m} \leq \int_{t}^{t_2} \left|x'_m(\tau)\right|^{p_m} d\tau \tag{2.4}$$

for $m = 1, 2, \ldots, n$ and $t \in [t_1, t_2]$. Adding (2.3) and (2.4), we have

$$|x_m(t)|^{p_m} \leq \frac{[(t - t_1)(t_2 - t)]^{p_m - 1}}{(t - t_1)^{p_m - 1} + (t_2 - t)^{p_m - 1}} \int_{t_1}^{t_2} \left|x'_m(\tau)\right|^{p_m} d\tau \tag{2.5}$$

for $m = 1, 2, \ldots, n$ and $t \in [t_1, t_2]$. Note that $x'_m(t) \neq 0,\ m = 1, 2, \ldots, n$ for all $t \in (t_1, t_2)$. If $x'_m(t) \equiv 0,\ m = 1, 2, \ldots, n,\ t \in (t_1, t_2)$, then, from (2.5), we obtain that $x_m(t) \equiv 0,\ m = 1, 2, \ldots, n,\ t \in (t_1, t_2)$, which contradicts (1.3). Then $x'_m(t) \neq 0,\ m = 1, 2, \ldots, n,$ for all $t \in (t_1, t_2)$ holds. Therefore, we get

$$|x_m(t)|^{p_m} < B_m C_m(t), \tag{2.6}$$

where

$$B_m = \int_{t_1}^{t_2} \left|x'_m(\tau)\right|^{p_m} d\tau \quad \text{and} \quad C_m(t) = \frac{[(t - t_1)(t_2 - t)]^{p_m - 1}}{(t - t_1)^{p_m - 1} + (t_2 - t)^{p_m - 1}}$$

for $m = 1, 2, \ldots, n$ and $t \in [t_1, t_2]$. If we take the $\dfrac{\alpha_{im}}{p_m}$th power of (2.6), we have

$$|x_m(t)|^{\alpha_{im}} < B_m^{\frac{\alpha_{im}}{p_m}} C_m^{\frac{\alpha_{im}}{p_m}}(t) \tag{2.7}$$

for $m, i = 1, 2, \ldots, n$ and $t \in [t_1, t_2]$. Multiplying of inequality (2.7) with $i = m$ by $h_m^+(t) \prod_{\substack{i=1 \\ i \neq m}}^{n} |x_i(t)|^{\alpha_{mi}}$ and integrating from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} |x_i(\tau)|^{\alpha_{mi}} d\tau < B_m^{\frac{\alpha_{mm}}{p_m}} \int_{t_1}^{t_2} h_m^+(\tau) C_m^{\frac{\alpha_{mm}}{p_m}}(\tau) \prod_{\substack{i=1 \\ i \neq m}}^{n} |x_i(\tau)|^{\alpha_{mi}} d\tau \tag{2.8}$$

for $m = 1, 2, \ldots, n$.

On the other hand, multiplying the mth equation of nonlinear system (1.1) by x_m and integrating from t_1 to t_2, we get

$$\begin{aligned} B_m &= \int_{t_1}^{t_2} \left|x'_m(\tau)\right|^{p_m} d\tau \leq \int_{t_1}^{t_2} \frac{|x'_m(\tau)|^{p_m}}{\left(1 - |x'_m(\tau)|^{p_m}\right)^{\frac{p_m-1}{p_m}}} d\tau \\ &= \int_{t_1}^{t_2} h_m(\tau) \prod_{i=1}^{n} \frac{|x_i(\tau)|^{\alpha_{mi}}}{\left(1 + |x_i(\tau)|^{\alpha_{mi}}\right)^{\frac{\alpha_{mi}-1}{\alpha_{mi}}}} d\tau \\ &\leq \int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} |x_i(\tau)|^{\alpha_{mi}} d\tau \end{aligned} \tag{2.9}$$

for $m = 1, 2, \ldots, n$. By using (2.9) in (2.8), we have

$$B_m < B_m^{\frac{\alpha_{mm}}{p_m}} \int_{t_1}^{t_2} h_m^+(\tau) C_m^{\frac{\alpha_{mm}}{p_m}}(\tau) \prod_{\substack{i=1 \\ i \neq m}}^{n} |x_i(\tau)|^{\alpha_{mi}} d\tau \tag{2.10}$$

for $m = 1, 2, \ldots, n$. Thus, by using (2.7) in inequality (2.10), we obtain

$$B_m^{1-\frac{\alpha_{mm}}{p_m}} < \int_{t_1}^{t_2} h_m^+(\tau) C_m^{\frac{\alpha_{mm}}{p_m}}(\tau) \prod_{\substack{i=1 \\ i \neq m}}^{n} B_i^{\frac{\alpha_{mi}}{p_i}} C_i^{\frac{\alpha_{mi}}{p_i}}(\tau)\, d\tau$$

and so

$$B_m^{1-\frac{\alpha_{mm}}{p_m}} < \prod_{\substack{i=1 \\ i \neq m}}^{n} B_i^{\frac{\alpha_{mi}}{p_i}} \int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} C_i^{\frac{\alpha_{mi}}{p_i}}(\tau)\, d\tau \tag{2.11}$$

for $m = 1, 2, \ldots, n$. Raising the both sides of inequality (2.11) to the power e_m for each $m = 1, 2, \ldots, n$, respectively, and multiplying the resulting inequalities side by side, we get

$$\prod_{m=1}^{n} B_m^{e_m\left(1-\frac{\alpha_{mm}}{p_m}\right)} < \prod_{m=1}^{n} \left(\prod_{\substack{i=1 \\ i \neq m}}^{n} B_i^{\frac{\alpha_{mi}}{p_i}} \right)^{e_m} \prod_{m=1}^{n} \left(\int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} C_i^{\frac{\alpha_{mi}}{p_i}}(\tau)\, d\tau \right)^{e_m}$$

and, hence,

$$\prod_{m=1}^{n} B_m^{e_m\left(1-\frac{\alpha_{mm}}{p_m}\right)} < \prod_{m=1}^{n} B_m^{\sum\limits_{\substack{i=1 \\ i \neq m}}^{n} \frac{\alpha_{im}}{p_m} e_i} \prod_{m=1}^{n} \left(\int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} C_i^{\frac{\alpha_{mi}}{p_i}}(\tau)\, d\tau \right)^{e_m}.$$

Next, we prove that $B_m > 0$ for $m = 1, 2, \ldots, n$. If $B_m > 0,\ m = 1, 2, \ldots, n$, is not true, then there exists $m_0 \in \{1, 2, \ldots, n\}$ such that $B_{m_0} = \int_{t_1}^{t_2} \left|x'_{m_0}(\tau)\right|^{p_{m_0}} d\tau = 0$. It follows that

$$x'_{m_0}(t) \equiv 0, \quad t \in [t_1, t_2]. \tag{2.12}$$

Combining (2.5) with (2.12), we obtain that $x_{m_0}(t) \equiv 0,\ t_1 \le t \le t_2$, which contradicts (1.3). Therefore, $B_m > 0,\ m = 1, 2, \ldots, n$, holds. Thus, we have

$$\prod_{m=1}^{n} B_m^{\theta_m} < \prod_{m=1}^{n} \left(\int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} C_i^{\frac{\alpha_{mi}}{p_i}}(\tau)\, d\tau \right)^{e_m}, \tag{2.13}$$

where

$$\theta_m = e_m \left(1 - \frac{\alpha_{mm}}{p_m} \right) - \sum_{\substack{i=1 \\ i \ne m}}^{n} \frac{\alpha_{im}}{p_m} e_i$$

for $m = 1, 2, \ldots, n$. From the assumption, the linear system (2.1) has a solution (not identically zero) $(e_1, e_2, \ldots, e_n)$ such that $\theta_m = 0,\ m = 1, 2, \ldots, n$, where $0 \le e_m,\ m = 1, 2, \ldots, n$, and at least one $0 < e_j,\ j \in \{1, 2, \ldots, n\}$. If we choose one of the solutions $(e_1, e_2, \ldots, e_n)$, then, from (2.13), we obtain inequality (2.2).

Theorem 2.1 is proved.

***Remark* 2.1.** If we take $n = 1$ and $\alpha_{11} = p_1 = 2$ in inequality (2.2) with $h_1(t) \ge 0$, then it reduces to inequality (1.5).

It is obvious that the function $M(t) = \dfrac{[(t - t_1)(t_2 - t)]^{p_m - 1}}{(t - t_1)^{p_m - 1} + (t_2 - t)^{p_m - 1}}$ takes its absolute maximum value at $\dfrac{t_1 + t_2}{2}$, i.e.,

$$M(t) \le \max_{t_1 \le t \le t_2} M(t) = M\left(\frac{t_1 + t_2}{2} \right) = \frac{(t_2 - t_1)^{p_m - 1}}{2^{p_m}}$$

for $m = 1, 2, \ldots, n$. Thus, we obtain the following result.

***Corollary* 2.1.** *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1). *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution (not identically zero) on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.3) *and*

$$h_m^+ \in \mathcal{A}_2 \overset{\triangle}{=} \left\{ h_m^+ \in L^1_{\text{loc}}(t_1, t_2) : \int_{t_1}^{t_2} h_m^+(\tau)\, d\tau < \infty \right\}$$

for $m = 1, 2, \ldots, n$, *then*

$$1 < \prod_{m=1}^{n} \left[\prod_{i=1}^{n} \left(\frac{(t_2 - t_1)^{p_i - 1}}{2^{p_i}} \right)^{\frac{\alpha_{mi}}{p_i}} \int_{t_1}^{t_2} h_m^+(\tau)\, d\tau \right]^{e_m} \tag{2.14}$$

holds, where $h_m^+(t)$ *is given in* (1.2).

***Remark* 2.2.** If we take $n = 1$ and $\alpha_{11} = p_1 = 2$ in inequality (2.14) with $h_1(t) \geq 0$, then we have the classical Lyapunov inequality

$$\frac{4}{t_2 - t_1} < \int\limits_{t_1}^{t_2} h_1(\tau)\, d\tau. \tag{2.15}$$

Note that if the condition

$$\sum_{m=1}^{n} \frac{\alpha_{im}}{p_m} = 1, \quad i = 1, 2, \ldots, n, \tag{2.16}$$

holds, then, from Theorem 2.1, we get the following result.

***Corollary* 2.2.** *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.3) *under condition* (2.16) *and* $h_m^+ \in \mathcal{A}$, $m = 1, 2, \ldots, n$, *then*

$$1 < \prod_{m=1}^{n} \left[\int\limits_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} \left(\frac{[(\tau - t_1)(t_2 - \tau)]^{p_i - 1}}{(\tau - t_1)^{p_i - 1} + (t_2 - \tau)^{p_i - 1}} \right)^{\frac{\alpha_{mi}}{p_i}} d\tau \right]^{\sum\limits_{\substack{j=1\\ j \neq m}}^{n} \frac{\alpha_{jm}}{p_m}} \tag{2.17}$$

holds, where $h_m^+(t)$ *is given in* (1.2).

Proof. Proceeding as in the proof of Theorem 2.1, we obtain inequality (2.13), where

$$\theta_m = e_m \left(1 - \frac{\alpha_{mm}}{p_m} \right) - \sum_{\substack{i=1\\ i \neq m}}^{n} \frac{\alpha_{im}}{p_m} e_i$$

for $m = 1, 2, \ldots, n$. From Theorem 2.1, the linear system (2.1) has a solution (not identically zero) $(e_1, e_2, \ldots, e_n)$ such that $\theta_m = 0$, $m = 1, 2, \ldots, n$, where $0 \leq e_m$, $m = 1, 2, \ldots, n$ and at least one $0 < e_j$, $j \in \{1, 2, \ldots, n\}$. If we use condition (2.16) in the linear system (2.1), then we have the equality

$$\sum_{\substack{i=1\\ i \neq m}}^{n} e_m \frac{\alpha_{mi}}{p_i} = \sum_{\substack{i=1\\ i \neq m}}^{n} \frac{\alpha_{im}}{p_m} e_i \tag{2.18}$$

for $m = 1, 2, \ldots, n$. If we perform elementary row operations on the coefficient matrix of the homogeneous system (2.18), we obtain a matrix having a zero row. So, the homogeneous system (2.18) has infinitely many solutions (not identically zero) $(e_1, e_2, \ldots, e_n)$. Hence, we may take $e_m = \dfrac{\alpha_{jm}}{p_m}$, $j \neq m$, j, $m = 1, 2, \ldots, n$, and, from inequality (2.13), we get

$$1 < \prod_{m=1}^{n} \left(\int\limits_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} C_i^{\frac{\alpha_{mi}}{p_i}}(\tau)\, d\tau \right)^{\frac{\alpha_{jm}}{p_m}}, \quad j \neq m, \quad j = 1, 2, \ldots, n.$$

Multiplying the resulting inequalities for $j \neq m$, $j = 1, 2, \ldots, n$, we have inequality (2.17).

Corollary 2.2 is proved.

Applying Hölder's inequality to inequality (2.17), we obtain the following result.

***Corollary* 2.3.** *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.3) *under condition* (2.16) *and* $h_m^+ \in \mathcal{A}$, $m = 1, 2, \ldots, n$, *then*

$$1 < \prod_{m=1}^{n} \prod_{i=1}^{n} \left(\int_{t_1}^{t_2} h_m^+(\tau) \frac{[(\tau - t_1)(t_2 - \tau)]^{p_i - 1}}{(\tau - t_1)^{p_i - 1} + (t_2 - \tau)^{p_i - 1}} \, d\tau \right)^{\frac{\alpha_{mi}}{p_i} \sum\limits_{\substack{j=1 \\ j \neq m}}^{n} \frac{\alpha_{jm}}{p_m}}$$

holds, where $h_m^+(t)$ *is given in* (1.2).

Now, if we take the condition

$$\sum_{i=1}^{n} \frac{\alpha_{im}}{p_m} = 1, \quad m = 1, 2, \ldots, n, \tag{2.19}$$

instead of condition (2.16), then we get the following result from Theorem 2.1. The proof is almost the same as the proof of Corollary 2.2 with $e_m = 1$, $m = 1, 2, \ldots, n$, and hence is omitted.

***Corollary* 2.4.** *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.3) *under condition* (2.19) *and* $h_m^+ \in \mathcal{A}$, $m = 1, 2, \ldots, n$, *then*

$$1 < \prod_{m=1}^{n} \int_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} \left(\frac{[(\tau - t_1)(t_2 - \tau)]^{p_i - 1}}{(\tau - t_1)^{p_i - 1} + (t_2 - \tau)^{p_i - 1}} \right)^{\frac{\alpha_{mi}}{p_i}} d\tau$$

holds, where $h_m^+(t)$ *is given in* (1.2).

Now, we state a lemma which we will use in the proofs of results obtained for the problem (1.1) under the antiperiodic boundary condition (1.4). The proof of the following lemma proceeds along the lines of that of Lemmas in [8, Lemma 2.1] and [20, Lemma 3.1], and hence is omitted.

Lemma 2.1. *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *of the problem* (1.1) *under the antiperiodic boundary condition* (1.4), *then we have*

$$|x_m(t)| \leq \frac{(t_2 - t_1)^{\frac{p_m - 1}{p_m}}}{2} \left(\int_{t_1}^{t_2} \left| x_m'(\tau) \right|^{p_m} d\tau \right)^{\frac{1}{p_m}} \tag{2.20}$$

for $m = 1, 2, \ldots, n$.

The following theorem is another main result of this paper for the problem (1.1) under the antiperiodic boundary condition (1.4).

Theorem 2.2. *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1). *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.4) *and* $h_m^+ \in \mathcal{A}_2$, $m = 1, 2, \ldots, n$, *then*

$$1 < \prod_{m=1}^{n} \left[\prod_{i=1}^{n} \left(\frac{(t_2 - t_1)^{p_i - 1}}{2^{p_i}} \right)^{\frac{\alpha_{mi}}{p_i}} \int_{t_1}^{t_2} h_m^+(\tau) \, d\tau \right]^{e_m} \tag{2.21}$$

holds, where $h_m^+(t)$ *is given in* (1.2).

Proof. Let $x_m^{(k)}(t_1) + x_m^{(k)}(t_2) = 0,\ k = 0, 1,\ m = 1, 2, \ldots, n,$ where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $x_m,\ m = 1, 2, \ldots, n,$ are nonzero on $[t_1, t_2]$. By using inequality (2.20), we get

$$|x_m(t)|^{p_m} \le \frac{(t_2 - t_1)^{p_m - 1}}{2^{p_m}} \int\limits_{t_1}^{t_2} \left|x_m'(\tau)\right|^{p_m} d\tau$$

for $m = 1, 2, \ldots, n,\ x_m'(t) \ne 0,\ m = 1, 2, \ldots, n,$ for all $t \in (t_1, t_2)$, as shown in the proof of Theorem 2.1. Therefore, it is easy to see that we have

$$|x_m(t)|^{p_m} < \frac{(t_2 - t_1)^{p_m - 1}}{2^{p_m}} \int\limits_{t_1}^{t_2} \frac{|x_m'(\tau)|^{p_m}}{\left(1 - |x_m'(\tau)|^{p_m}\right)^{\frac{p_m - 1}{p_m}}}\, d\tau$$

for $m = 1, 2, \ldots, n$. The proof is completed by proceeding the same as the proof of Theorem 2.1 and hence is omitted.

***Remark* 2.3.** When $n = 1$ and $\alpha_{11} = p_1 = 2$, if $x_1(t)$ is a solution (not identically zero) on $[t_1, t_2]$ for the problem (1.6) with (1.4) and $h_1^+ \in \mathcal{A}_2$, then from inequality (2.21), we have the classical Lyapunov inequality (2.15) [16].

The proofs of the following results obtained under the antiperiodic boundary condition (1.4) are almost the same as the proof of the results obtained under the Dirichlet boundary condition (1.3), respectively, and hence are omitted.

***Corollary* 2.5.** *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.4) *under condition* (2.16) *and* $h_m^+ \in \mathcal{A}_2,\ m = 1, 2, \ldots, n,$ *then*

$$1 < \prod_{m=1}^{n} \left[\prod_{i=1}^{n} \left(\frac{(t_2 - t_1)^{p_i - 1}}{2^{p_i}} \right)^{\frac{\alpha_{mi}}{p_i}} \int\limits_{t_1}^{t_2} h_m^+(\tau)\, d\tau \right]^{\sum\limits_{\substack{j=1 \\ j \ne m}}^{n} \frac{\alpha_{jm}}{p_m}}$$

holds, where $h_m^+(t)$ *is given in* (1.2).

***Corollary* 2.6.** *If* $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a solution* (*not identically zero*) *on* $[t_1, t_2]$ *for the problem* (1.1) *with* (1.4) *under condition* (2.19) *and* $h_m^+ \in \mathcal{A}_2,\ m = 1, 2, \ldots, n,$ *then*

$$1 < \prod_{m=1}^{n} \prod_{i=1}^{n} \left(\frac{(t_2 - t_1)^{p_i - 1}}{2^{p_i}} \right)^{\frac{\alpha_{mi}}{p_i}} \int\limits_{t_1}^{t_2} h_m^+(\tau)\, d\tau$$

holds, where $h_m^+(t)$ *is given in* (1.2).

Now, we give sufficient conditions for the nonexistence of nontrivial solutions of the boundary-value problem (1.1) with (1.3) or (1.4) as immediate consequences of Theorems 2.1 and 2.2.

***Corollary* 2.7.** (a) *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1) *and*

$$1 \ge \prod_{m=1}^{n} \left[\int\limits_{t_1}^{t_2} h_m^+(\tau) \prod_{i=1}^{n} \left(\frac{[(\tau - t_1)(t_2 - \tau)]^{p_i - 1}}{(\tau - t_1)^{p_i - 1} + (t_2 - \tau)^{p_i - 1}} \right)^{\frac{\alpha_{mi}}{p_i}} d\tau \right]^{e_m}, \tag{2.22}$$

where $h_m^+(t)$ *is given in* (1.2). *The problem* (1.1) *with* (1.3) *has no nontrivial solution.*
(b) *Assume that there exists a solution (not identically zero)* $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1) *and*

$$1 \geq \prod_{m=1}^{n} \left[\prod_{i=1}^{n} \left(\frac{(t_2 - t_1)^{p_i - 1}}{2^{p_i}} \right)^{\frac{\alpha_{mi}}{p_i}} \int_{t_1}^{t_2} h_m^+(\tau)\, d\tau \right]^{e_m}, \tag{2.23}$$

where $h_m^+(t)$ *is given in* (1.2). *The problem* (1.1) *with* (1.4) *has no nontrivial solution.*

Next, the following result gives a sufficient condition for the uniqueness of the solution of the nonhomogeneous boundary-value problem

$$- \left(\phi_{p_m}(x_m')\right)' = h_m(t)\Phi_{\alpha_{mm}}(x_m)$$

$$\times \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{|x_i|^{\alpha_{mi}}}{\left(1 + |x_i|^{\alpha_{mi}}\right)^{\frac{\alpha_{mi}-1}{\alpha_{mi}}}} + f_m(t), \quad m = 1, 2, \ldots, n, \quad n \in \mathbb{N}, \quad t \in (t_1, t_2), \tag{2.24}$$

$$x_m(t_1) = 0 = x_m(t_2) \quad \text{or} \quad x_m^{(k)}(t_1) + x_m^{(k)}(t_2) = 0, \quad k = 0, 1, \quad m = 1, 2, \ldots, n,$$

where $p_m > 1,\ \alpha_{mi},\ m, i = 1, 2, \ldots, n$, is a nonnegative constant, $h_k,\ f_k \in C([a, b], \mathbb{R}),\ k = 1, 2, \ldots, n,\ n \in \mathbb{N}$, as an application of Lyapunov-type inequalities (2.2) and (2.21).

Theorem 2.3. *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1). *If* (2.22) *or* (2.23) *holds, then nonhomogeneous boundary problem* (2.24) *has a unique solution.*

Proof. To prove the uniqueness, it is sufficient to show that the homogeneous boundary-value problem (2.24) has only trivial solution. Assume on the contrary that $x(t) \not\equiv 0$ is a solution of the homogeneous boundary-value problem (2.24). Then, by using Lyapunov-type inequality (2.2) or (2.21), we have

$$1 < \prod_{m=1}^{n} \left[\int_{t_1}^{t_2} |h_m(\tau)| \prod_{i=1}^{n} \left(\frac{[(\tau - t_1)(t_2 - \tau)]^{p_i - 1}}{(\tau - t_1)^{p_i - 1} + (t_2 - \tau)^{p_i - 1}} \right)^{\frac{\alpha_{mi}}{p_i}} d\tau \right]^{e_m}$$

or

$$1 < \prod_{m=1}^{n} \left[\prod_{i=1}^{n} \left(\frac{(t_2 - t_1)^{p_i - 1}}{2^{p_i}} \right)^{\frac{\alpha_{mi}}{p_i}} \int_{t_1}^{t_2} |h_m(\tau)|\, d\tau \right]^{e_m},$$

which gives contradiction to (2.22) or (2.23), respectively. Therefore, the homogeneous boundary-value problem (2.24) has only trivial solution. Because of the theory of boundary-value problems, the nonhomogeneous boundary problem (2.24) has a unique solution.

Theorem 2.3 is proved.

As an example of Theorem 2.3, we consider the following nonhomogeneous boundary-value problem:

$$\left(\frac{x'}{\sqrt{1-|x'|^2}}\right)' + \frac{x}{4\sqrt{1+|x|^2}} = -\frac{1}{\cos^2 t} + \frac{\cos t}{4\sqrt{1+\cos^2 t}}, \quad t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

$$x\left(\frac{\pi}{2}\right) = 0 = x\left(\frac{3\pi}{2}\right) \quad \text{or} \quad x\left(\frac{\pi}{2}\right) + x\left(\frac{3\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) + x'\left(\frac{3\pi}{2}\right) = 0. \tag{2.25}$$

Since the condition (2.22) or (2.23) is satisfied, the problem (2.25) has a unique solution. One such solution of the problem (2.25) is $x(t) = \cos t$. Note that $|x'|$ is bounded with 1 and the homogeneous boundary-value problem corresponding to the problem (2.25) has only trivial solution.

In the following part, we apply the obtained Lyapunov-type inequalities (2.2) and (2.21) to the eigenvalue problems associated with the nonlinear boundary-value problem

$$-\left(\phi_{p_m}(x_m')\right)' = \lambda h_m(t)\Phi_{\alpha_{mm}}(x_m)\prod_{\substack{i=1\\ i\neq m}}^{n} \frac{|x_i|^{\alpha_{mi}}}{\left(1+|x_i|^{\alpha_{mi}}\right)^{\frac{\alpha_{mi}-1}{\alpha_{mi}}}}, \quad m = 1, 2, \ldots, n, \quad n \in \mathbb{N},$$

$$x_m(t_1) = 0 = x_m(t_2) \quad \text{or} \quad x_m^{(k)}(t_1) + x_m^{(k)}(t_2) = 0, \quad k = 0, 1, \quad m = 1, 2, \ldots, n, \tag{2.26}$$

where $p_m > 1$, α_{mi}, $m, i = 1, 2, \ldots, n$, is a nonnegative constant, $h_k \in C([a,b], \mathbb{R})$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$ is an eigenvalue parameter. As direct consequences of Theorems 2.1 and 2.2, we obtain the following result.

Theorem 2.4. (a) *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1) *and* λ *is an eigenvalue of the boundary-value problem* (2.26). *Then*

$$\frac{1}{\prod_{m=1}^{n}\left[\int_{t_1}^{t_2} |h_m(\tau)| \prod_{i=1}^{n}\left(\frac{[(\tau-t_1)(t_2-\tau)]^{p_i-1}}{(\tau-t_1)^{p_i-1}+(t_2-\tau)^{p_i-1}}\right)^{\frac{\alpha_{mi}}{p_i}} d\tau\right]^{e_m}} < |\lambda|.$$

(b) *Assume that there exists a solution* (*not identically zero*) $(e_1, e_2, \ldots, e_n)$ *of the linear system* (2.1) *and* λ *is an eigenvalue of the boundary-value problem* (2.26). *Then*

$$\frac{1}{\prod_{m=1}^{n}\left[\prod_{i=1}^{n}\left(\frac{(t_2-t_1)^{p_i-1}}{2^{p_i}}\right)^{\frac{\alpha_{mi}}{p_i}} \int\limits_{t_1}^{t_2} |h_m(\tau)|\, d\tau\right]^{e_m}} < |\lambda|.$$

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References

1. M. F. Aktaş, *On Lyapunov-type inequalities for nonlinear Hamiltonian-type problems*, Filomat, **36**, № 10, 3423 – 3432 (2022).
2. M. F. Aktaş, *On Lyapunov-type inequalities for $(n+1)$st order nonlinear differential equations with the antiperiodic boundary conditions*, Turk. J. Math., **45**, 2614 – 2622 (2021).
3. M. F. Aktaş, *Periodic solutions of relativistic Liénard-type equations*, Electron. J. Qual. Theory Different. Equat., **38**, 1 – 12 (2020).
4. M. F. Aktaş, D. Çakmak, A. Tiryaki, *Lyapunov-type inequality for quasilinear systems with anti-periodic boundary conditions*, J. Math. Inequal., **8**, 313 – 320 (2014).
5. M. F. Aktaş, *Lyapunov-type inequalities for n-dimensional quasilinear systems*, Electron. J. Different. Equat., **67**, 1 – 8 (2013).
6. M. F. Aktaş, D. Çakmak, A. Tiryaki, *A note on Tang and He's paper*, Appl. Math. and Comput., **218**, 4867 – 4871 (2012).
7. D. Çakmak, *On Lyapunov-type inequality for a class of quasilinear systems*, Electron. J. Qual. Theory Different. Equat., **9**, 1 – 10 (2014).
8. D. Çakmak, *Lyapunov-type inequalities for two classes of nonlinear systems with anti-periodic boundary conditions*, Appl. Math. and Comput., **223**, 237 – 242 (2013).
9. D. Çakmak, *On Lyapunov-type inequality for a class of nonlinear systems*, Math. Inequal. Appl., **16**, 101 – 108 (2013).
10. D. Çakmak, A. Tiryaki, *On Lyapunov-type inequality for quasilinear systems*, Appl. Math. and Comput., **216**, 3584 – 3591 (2010).
11. D. Çakmak, A. Tiryaki, *Lyapunov-type inequality for a class of Dirichlet quasilinear systems involving the $(p_1, p_2, \ldots, p_n)$-Laplacian*, J. Math. Anal. and Appl., **369**, 76 – 81 (2010).
12. D. Gilbarg, N.S. Trundinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, New York (1983).
13. P. Jebelean, J. Mawhin, C. Serban, *Multiple critical orbits to partial periodic perturbations of the p-relativistic operator*, Appl. Math. Lett., **104**, Articl 106220 (2020).
14. P. Jebelean, J. Mawhin, C. Serban, *A vector p-Laplacian type approach to multiple periodic solutions for the p-relativistic operator*, Commun. Contemp. Math., **19**,1 – 16 (2017).
15. P. Jebelean, C. Serban, *Boundary-value problems for discontinuous perturbations of singular ϕ-Laplacian operator*, J. Math. Anal. and Appl., **431**, 662 – 681 (2015).
16. A. M. Liapunov, *Probléme général de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse, **2**, 203 – 407 (1907).
17. I. Sim, Y. H. Lee, *Lyapunov inequalities for one-dimensional p-Laplacian problems with a singular weight function*, J. Inequal. and Appl., **2010**, Article ID 865096 (2010).
18. A. Tiryaki, D. Çakmak, M. F. Aktaş, *Lyapunov-type inequalities for two classes of Dirichlet quasilinear systems*, Math. Inequal. Appl., **17**, 843 – 863 (2014).
19. A. Tiryaki, D. Çakmak, M. F. Aktaş, *Lyapunov-type inequalities for a certain class of nonlinear systems*, Comput. Math. Appl., **64**, 1804 – 1811 (2012).
20. Y. Wang, *Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions*, Appl. Math. Lett., **25**, 2375 – 2380 (2012).
21. X. J. Wang, *Interior gradient estimates for mean curvature equations*, Math. Z., **228**, 73 – 81 (1998).
22. R. Yang, I. Sim, Y. H. Lee, *Lyapunov-type inequalities for one-dimensional Minkowski-curvature problems*, Appl. Math. Lett., **91**, 188 – 193 (2019).

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