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Bülent Nafi Örnek[1] (Department of Computer Engineering, Amasya University, Turkey)

A REFINEMENT OF SCHWARZ'S LEMMA AT THE BOUNDARY

УТОЧНЕННЯ ЛЕМИ ШВАРЦА НА МЕЖІ

We investigate a boundary version of the Schwarz lemma for analytic functions. In addition, an analytic function satisfying the equality case is found by deducing inequalities related to the modulus of the derivative of analytic functions at a boundary point of the unit disk. Some coefficients used in the Taylor expansion of the function are considered in these inequalities. In the last theorem, by analyzing the Taylor expansion about two points, we obtain the modulus of the derivative of the function at point 1.

Досліджено граничну версію леми Шварца для аналітичних функцій. Крім того, аналітичну функцію, що задовольняє випадок рівності, знайдено шляхом отримання нерівностей, які пов'язані з модулем похідної аналітичних функцій у певній граничній точці одиничного круга. В цих нерівностях розглядаються деякі коефіцієнти, що використовуються в розкладі Тейлора даної функції. В останній теоремі, враховуючи розклад Тейлора в околі двох точок, отримано модуль похідної функції в точці 1.

1. Introduction. We have the following well-known result, also referred to Schwarz's lemma, regarding the values of an analytic function in a disc.

Let f be an analytic function in the unit disc $U = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. For any point z in the unit disc U, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ [4, p. 329]. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2, 9, 11]). A complex valued activation function which is obtained by using the Schwarz lemma is proposed in this study and a complex-valued extreme learning classifier is utilized to analyze its classification performance [11]. Use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and multinotch filter design in signal processing [9].

Let $g(z) = b + c_1 z + c_2 z^2 + \ldots$ be analytic in U, $b = g(0)$ and $|g(z)| < 1$ for $z \in U$. Then

$$|g'(0)| \leq 1 - |b|^2.$$

Obviously, this inequality is sharp. This expression will be used to prove Theorem 3.

Now, at a unit disc boundary point, our main interest is with analytic functions that map the unit disc into itself. According to the Schwarz lemma, if an analytic function f of the unit disc into itself has $f(0) = 0$ and extends continuously to the boundary point c with $|c| = 1$, $|f(c)| = 1$, and $f'(c)$ exists, then $|f'(c)| \geq 1$. The following lemma, known as the boundary Schwarz lemma, is found in [7, 13].

Lemma 1. *If $f(z)$ extends continuously to some boundary point c with $|c| = 1$, $|f(z)| < 1$ for $|z| < 1$ and $f(0) = 0$, and if $|f(c)| = 1$ and $f'(c)$ exists, then*

$$|f'(c)| \geq \frac{2}{1 + |f'(0)|} \tag{1.1}$$

[1] E-mails: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr.

and

$$|f'(c)| \geq 1. \tag{1.2}$$

In addition, the equality in (1.2) *holds if and only if* $f(z) = ze^{i\theta}$, *where* θ *is a real number. Also, the equality in* (1.1) *holds if and only if* f *is of the form* $f(z) = -z\dfrac{\tau - z}{1 - \overline{\tau} z}$ $\forall z \in U$ *for some constant* $\tau \in (-1, 0]$.

In proving our main results, we shall need the following lemma due to Julia – Wolff [12].

Lemma 2 (Julia – Wolff lemma)**.** *Let* f *be an analytic function in* U, $f(0) = 0$ *and* $f(U) \subset U$. *If, in addition, the function* f *has an angular limit* $f(c)$ *at* $c \in \partial U$, $|f(c)| = 1$, *then the angular derivative* $f'(c)$ *exists and* $1 \leq |f'(c)| \leq \infty$.

The geometric theory of functions greatly benefits from the inequality (1.2) and its generalizations, which are still hot subjects in the mathematics literature [5 – 8, 10, 13].

2. Main results. In this section, the derivative of the function at point 1 is evaluated from below. Some of the coefficients in the Taylor expansion of the function are used in this evaluation. In addition, in the last theorem, Taylor's expansion around two points will be used to obtain more general inequalities.

Theorem 1. *Let* $g(z) = b + c_1 z + c_2 z^2 + \ldots$ *be analytic in* U, $b = g(0)$, $\alpha = \arg \overline{b}$ *and* $|g(z)| < 1$ *for* $z \in U$. *Assume that, for* $1 \in \partial U$, g *has an angular limit* $g(1)$ *at* 1, $g(1) = -e^{-i\alpha}$. *Then we have the inequality*

$$|g'(1)| \geq \frac{1 + |b|}{1 - |b|}. \tag{2.1}$$

The equality in (2.1) *occurs for the function*

$$g(z) = e^{-i\alpha} \frac{|b| - z}{1 - z|b|}.$$

Proof. Consider the function

$$\varphi(z) = \frac{e^{i\alpha} g(z) - |b|}{e^{i\alpha} g(z)|b| - 1}.$$

$\varphi(z)$ is an analytic function in the unit disc U, $\varphi(0) = 0$, $|\varphi(z)| < 1$ for $|z| < 1$. Also, we have $|\varphi(1)| = 1$ for $1 \in \partial U$. From (1.2), we obtain

$$1 \leq |\varphi'(1)| = \left| \frac{e^{i\alpha} g'(1)\left(|b|^2 - 1\right)}{\left(e^{i\alpha} g(1)|b| - 1\right)^2} \right| = \left| \frac{e^{i\alpha} g'(1)\left(|b|^2 - 1\right)}{\left(e^{i\alpha}(-e^{-i\alpha})|b| - 1\right)^2} \right|$$

$$= \left| \frac{e^{i\alpha} g'(1)\left(|b|^2 - 1\right)}{\left(-|b| - 1\right)^2} \right| = \frac{|g'(1)|(1 - |b|)}{|b| + 1}$$

and

$$|g'(1)| \geq \frac{1 + |b|}{1 - |b|}.$$

Now, we shall show that inequality (2.1) is sharp. Let

$$g(z) = e^{-i\alpha} \frac{|b| - z}{1 - z|b|}.$$

Then we take

$$g'(z) = e^{-i\alpha}\frac{-(1 - z|b|) + |b|(|b| - z)}{(1 - z|b|)^2},$$

$$g'(1) = e^{-i\alpha}\frac{-(1 - |b|) + |b|(|b| - 1)}{(1 - |b|)^2}$$

and

$$|g'(1)| = \frac{1 + |b|}{1 - |b|}.$$

Theorem 1 is proved.

By considering c_1, the first coefficient in the expansion of the function $g(z)$, inequality (2.1) can be strengthened as shown in the following.

Theorem 2. *Under the assumptions of Theorem* 1, *we have*

$$|g'(1)| \geq \frac{2(1 + |b|)^2}{1 - |b|^2 + |c_1|}. \tag{2.2}$$

Inequality (2.2) *is sharp with extremal function*

$$g(z) = e^{-i\alpha}\frac{|b| - z}{1 - z|b|}.$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 1. Using inequality (1.1) for the function $\varphi(z)$, we obtain

$$\frac{2}{1 + |\varphi'(0)|} \leq |\varphi'(1)| = \frac{|g'(1)|(1 - |b|)}{|b| + 1}.$$

Since

$$|\varphi'(0)| = \frac{|g'(0)|}{1 - |b|^2} = \frac{|c_1|}{1 - |b|^2},$$

we have

$$\frac{2}{1 + \dfrac{|c_1|}{1 - |b|^2}} \leq \frac{|g'(1)|(1 - |b|)}{|b| + 1},$$

$$\frac{2\left(1 - |b|^2\right)}{1 - |b|^2 + |c_1|} \leq \frac{|g'(1)|(1 - |b|)}{|b| + 1}$$

and

$$|g'(1)| \geq \frac{2(1 + |b|)^2}{1 - |b|^2 + |c_1|}.$$

To show that inequality (2.2) is sharp, take an analytic function

$$g(z) = e^{-i\alpha}\frac{|b| - z}{1 - z|b|}.$$

Then

$$|g'(1)| = \frac{1 + |b|}{1 - |b|}.$$

On the other hand, we have

$$b + c_1 z + c_2 z^2 + \ldots = e^{-i\alpha} \frac{|b| - z}{1 - z|b|}.$$

If we take the derivative of both sides of the last equation, we obtain

$$c_1 + 2c_2 z + \ldots = e^{-i\alpha} \frac{-1 + |b|^2}{(1 - z|b|)^2}.$$

Passing to limit as z tends to 0 in the last equality, we get

$$c_1 = -e^{-i\alpha}\left(1 - |b|^2\right)$$

and

$$|c_1| = 1 - |b|^2.$$

Therefore, we have

$$\frac{2(1 + |b|)^2}{1 - |b|^2 + |c_1|} = \frac{2(1 + |b|)^2}{1 - |b|^2 + 1 - |b|^2} = \frac{1 + |b|}{1 - |b|}.$$

Theorem 2 is proved.

In the following theorem, inequality (2.2) is strengthened by adding term c_2 of the function $g(z)$.

Theorem 3. *Under the assumptions of Theorem* 1, *we have*

$$|g'(1)| \geq \frac{1 + |b|}{1 - |b|} \left(1 + \frac{2\left(1 - |b|^2 - |c_1|\right)^2}{\left(1 - |b|^2\right)^2 - |c_1|^2 + \left|c_2\left(1 - |b|^2\right) + e^{i\alpha}|b|c_1^2\right|} \right). \tag{2.3}$$

Inequality (2.3) *is sharp for the function given by*

$$g(z) = e^{-i\alpha} \frac{|b| - z^2}{1 - z^2|b|}.$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 1 and $B(z) = z$. By the maximum principle, for each $z \in U$, we get $|\varphi(z)| \leq |B(z)|$. So,

$$h(z) = \frac{\varphi(z)}{B(z)} = \frac{e^{i\alpha} g(z) - |b|}{z(e^{i\alpha} g(z)|b| - 1)}$$

is an analytic function in U and $|h(z)| < 1$ for $|z| < 1$.

In particular, we obtain

$$|h(0)| = \frac{|c_1|}{1 - |b|^2} \leq 1$$

and

$$|h'(0)| = \frac{\left|c_2\left(1 - |b|^2\right) + e^{i\alpha}|b|c_1^2\right|}{\left(1 - |b|^2\right)^2}.$$

The function

$$\Phi(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

is analytic in the unit disc U, $|\Phi(z)|<1$ for $|z|<1$, $\Phi(0)=0$ and $|\Phi(1)|=1$ for $1\in\partial U$.

From (1.1), we have

$$\frac{2}{1+|\Phi'(0)|}\le\left|\Phi'(1)\right|=\frac{1-|h(0)|^2}{|1-\overline{h(0)}h(1)|^2}\left|h'(1)\right|$$
$$\le\frac{1+|h(0)|}{1-|h(0)|}|h'(1)|=\frac{1+|h(0)|}{1-|h(0)|}\big\{|\varphi'(1)|-1\big\}.$$

Since

$$\Phi'(z)=\frac{1-|h(0)|^2}{\big(1-\overline{h(0)}h(z)\big)^2}h'(z),$$

$$|\Phi'(0)|=\frac{|h'(0)|}{1-|h(0)|^2}$$
$$=\frac{\dfrac{\left|c_2\big(1-|b|^2\big)+e^{i\alpha}|b|c_1^2\right|}{(1-|b|^2)^2}}{1-\left(\dfrac{|c_1|}{1-|b|^2}\right)^2}=\frac{\left|c_2\big(1-|b|^2\big)+e^{i\alpha}|b|c_1^2\right|}{\big(1-|b|^2\big)^2-|c_1|^2},$$

we get

$$\frac{2}{1+\dfrac{\left|c_2\big(1-|b|^2\big)+e^{i\alpha}|b|c_1^2\right|}{\big(1-|b|^2\big)^2-|c_1|^2}}\le\frac{1-|b|^2+|c_1|}{1-|b|^2-|c_1|}\left\{\frac{|g'(1)|(1-|b|)}{|b|+1}-1\right\},$$

$$\frac{2\Big(\big(1-|b|^2\big)^2-|c_1|^2\Big)}{\big(1-|b|^2\big)^2-|c_1|^2+\left|c_2\big(1-|b|^2\big)+e^{i\alpha}|b|c_1^2\right|}\frac{1-|b|^2-|c_1|}{1-|b|^2+|c_1|}\le\frac{|g'(1)|(1-|b|)}{|b|+1}-1$$

and

$$\frac{2\big(1-|b|^2-|c_1|\big)^2}{\big(1-|b|^2\big)^2-|c_1|^2+\left|c_2(1-|b|^2)+e^{i\alpha}|b|c_1^2\right|}+1\le\frac{|g'(1)|(1-|b|)}{|b|+1}.$$

Therefore, we obtain

$$|g'(1)|\ge\frac{1+|b|}{1-|b|}\left(1+\frac{2\big(1-|b|^2-|c_1|\big)^2}{\big(1-|b|^2\big)^2-|c_1|^2+\left|c_2\big(1-|b|^2\big)+e^{i\alpha}|b|c_1^2\right|}\right).$$

Now, we shall show that inequality (2.3) is sharp. Consider the function

$$g(z)=e^{-i\alpha}\frac{|b|-z^2}{1-z^2|b|}.$$

Then we have

$$|g'(1)| = 2\frac{1+|b|}{1-|b|}.$$

On the other hand, we obtain

$$b + c_1 z + c_2 z^2 + c_3 z^3 + \ldots = e^{-i\alpha}\frac{|b| - z^2}{1 - z^2|b|}.$$

If we take the derivative of both sides of the last equation, we get

$$c_1 + 2c_2 z + 3c_3 z^2 + \ldots = e^{-i\alpha}\frac{-2z\left(1-|b|^2\right)}{\left(1 - z^2|b|\right)^2}.$$

Passing to limit in the last equality yields $c_1 = 0$. Similarly, using straightforward calculations, we take $c_2 = -e^{-i\alpha}\left(1-|b|^2\right)$. Therefore, we obtain

$$\frac{1+|b|}{1-|b|}\left(1 + \frac{2\left(1-|b|^2-|c_1|\right)^2}{\left(1-|b|^2\right)^2 - |c_1|^2 + \left|c_2(1-|b|^2) + e^{i\alpha}|b|c_1^2\right|}\right)$$

$$= \frac{1+|b|}{1-|b|}\left(1 + \frac{2\left(1-|b|^2\right)^2}{\left(1-|b|^2\right)^2 + \left|-e^{-i\alpha}\left(1-|b|^2\right)(1-|b|^2)\right|}\right)$$

$$= \frac{1+|b|}{1-|b|}\left(1 + \frac{2\left(1-|b|^2\right)^2}{\left(1-|b|^2\right)^2 + (1-|b|^2)^2}\right) = 2\frac{1+|b|}{1-|b|}.$$

Theorem 3 is proved.

In this theorem, considering the Taylor expansion around two points, the modulus of the derivative of the function at point 1 is obtained. Based on the results of the work presented in [1], the following result is obtained.

Theorem 4. *Let $g(z) = b + c_1 z + c_2 z^2 + \ldots$ be analytic in U, $b = g(0)$, $g(\varsigma_0) = g(0)$, $0 < |\varsigma_0| < 1$, $\alpha = \arg \overline{b}$ and $|g(z)| < 1$ for $z \in U$. Assume that, for $1 \in \partial U$, g has an angular limit $g(1)$ at 1, $g(1) = -e^{-i\alpha}$. Then we have the inequality*

$$|g'(1)| \geq \frac{1+|b|}{1-|b|}\Bigg(1 + \frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2} + \frac{\left(1-|b|^2\right)|\varsigma_0| - |g'(0)|}{\left(1-|b|^2\right)|\varsigma_0| + |g'(0)|}$$

$$\times\left[1 + \frac{\left(1-|b|^2\right)^2|\varsigma_0|^2 + |g'(\varsigma_0)|\left(1-|\varsigma_0|^2\right)|g'(0)| - \left(1-|b|^2\right)|g'(\varsigma_0)|\left(1-|\varsigma_0|^2\right) - \left(1-|b|^2\right)|f'(0)|}{\left(1-|b|^2\right)^2|\varsigma_0|^2 + |g'(\varsigma_0)|(1-|\varsigma_0|^2)|g'(0)| + \left(1-|b|^2\right)|g'(\varsigma_0)|(1-|\varsigma_0|^2) + (1-|b|^2)|f'(0)|}\frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2}\right]\Bigg). \tag{2.4}$$

Inequality (2.4) *is sharp, with equality for each possible value of $|g'(0)|$ and $|g'(\varsigma_0)|$.*

Proof. Let

$$u(z) = \frac{z - \varsigma_0}{1 - \overline{\varsigma_0} z}.$$

Also, let $r : U \to U$ be an analytic function and $\varsigma_0 \in U$ such that

$$\left|\frac{r(z)-r(\varsigma_0)}{1-\overline{r(\varsigma_0)}r(z)}\right| \le \left|\frac{z-\varsigma_0}{1-\overline{\varsigma_0}z}\right| = |u(z)|$$

and

$$|r(z)| \le \frac{|r(\varsigma_0)|+|u(z)|}{1+|r(\varsigma_0)||u(z)|} \tag{2.5}$$

by the Schwarz–Pick lemma [4]. For an analytic function $v\colon U \to U$ and $0<|\varsigma_0|<1$, if we take into account the function

$$r(z) = \frac{v(z)-v(0)}{z\big(1-\overline{v(0)}v(z)\big)}$$

in (2.5), then we get

$$\left|\frac{v(z)-v(0)}{z\big(1-\overline{v(0)}v(z)\big)}\right| \le \frac{\left|\dfrac{v(\varsigma_0)-v(0)}{\varsigma_0\Big(1-\overline{v(0)}v(\varsigma_0)\Big)}\right|+|u(z)|}{1+\left|\dfrac{v(\varsigma_0)-v(0)}{\varsigma_0\big(1-\overline{v(0)}v(\varsigma_0)\big)}\right||u(z)|}$$

and

$$|v(z)| \le \frac{|v(0)|+|z|\dfrac{|C|+|u(z)|}{1+|C||u(z)|}}{1+|v(0)||z|\dfrac{|C|+|u(z)|}{1+|C||u(z)|}}, \tag{2.6}$$

where

$$m = \frac{v(\varsigma_0)-v(0)}{\varsigma_0\big(1-\overline{v(0)}v(\varsigma_0)\big)}.$$

If we take

$$v(z) = \frac{\varphi(z)}{z\left(\dfrac{z-\varsigma_0}{1-\overline{\varsigma_0}z}\right)},$$

then

$$v(0) = \frac{\varphi'(0)}{-\varsigma_0}, \qquad v(\varsigma_0) = \frac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}$$

and

$$m = \frac{\dfrac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}+\dfrac{\varphi'(0)}{\varsigma_0}}{\varsigma_0\left(1+\dfrac{\overline{\varphi'(0)}}{\overline{\varsigma_0}}\dfrac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}\right)},$$

where $|m| \leq 1$. Let $|v(0)| = \beta$ and

$$l = \frac{\left|\frac{\varphi'(\varsigma_0)\left(1-|\varsigma_0|^2\right)}{\varsigma_0}\right| + \left|\frac{\varphi'(0)}{\varsigma_0}\right|}{|\varsigma_0|\left(1+\left|\frac{\varphi'(\varsigma_0)\left(1-|\varsigma_0|^2\right)}{\varsigma_0}\right|\left|\frac{\varphi'(0)}{\varsigma_0}\right|\right)}.$$

By (2.6), we get

$$|\varphi(z)| \leq |z||u(z)|\frac{\beta + |z|\frac{l+|u(z)|}{1+l|u(z)|}}{1+\beta|z|\frac{l+|u(z)|}{1+l|u(z)|}}$$

and

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1+\beta|z|\frac{l+|u(z)|}{1+l|u(z)|} - \beta|z||u(z)| - |u(z)||z|^2\frac{l+|u(z)|}{1+l|u(z)|}}{(1-|z|)\left(1+\beta|z|\frac{l+|u(z)|}{1+l|u(z)|}\right)}.$$

Let $\xi(z) = 1+\beta|z|\frac{l+|u(z)|}{1+l|u(z)|}$ and $\eta(z) = 1+l|u(z)|$. Taking into account the functions $\xi(z)$ and $\eta(z)$ in the previous inequality, we have

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1}{\xi(z)\eta(z)}\left\{\frac{1-|z|^2|u(z)|^2}{1-|z|} + l|u(z)|\frac{1-|z|^2}{1-|z|} + \beta|z|l\frac{1-|u(z)|^2}{1-|z|}\right\}. \qquad (2.7)$$

Since

$$\lim_{z\to 1}\xi(z) = \lim_{z\to 1}\left(1+\beta|z|\frac{l+|u(z)|}{1+l|u(z)|}\right) = 1+\beta,$$

$$\lim_{z\to 1}\eta(z) = \lim_{z\to 1}(1+l|u(z)|) = 1+M,$$

$$\lim_{z\to 1}\frac{1-|z|^i\left|\frac{z-\varsigma_0}{1-\overline{\varsigma_0}z}\right|^j}{1-|z|} = i + j\frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2}$$

for nonnegative integers i and j and

$$1-|u(z)|^2 = 1-\left|\frac{z-\varsigma_0}{1-\overline{\varsigma_0}z}\right|^2 = \frac{\left(1-|\varsigma_0|^2\right)\left(1-|z|^2\right)}{|1-\overline{\varsigma_0}z|^2},$$

passing to the angular limit in (2.7) gives

$$|\varphi'(1)| \geq 1 + \frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2} + \frac{1-\beta}{1+\beta}\left[1+\frac{1-l}{1+l}\frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2}\right].$$

In addition, since

$$\frac{1-\beta}{1+\beta}=\frac{1-|v(0)|}{1+|v(0)|}=\frac{1-\left|\dfrac{\varphi'(0)}{\varsigma_0}\right|}{1+\left|\dfrac{\varphi'(0)}{\varsigma_0}\right|}=\frac{|\varsigma_0|-|\varphi'(0)|}{|\varsigma_0|+|\varphi'(0)|}=\frac{\big(1-|b|^2\big)|\varsigma_0|-|g'(0)|}{\big(1-|b|^2\big)|\varsigma_0|+|g'(0)|},$$

$$\frac{1-l}{1+l}=\frac{1-\dfrac{\left|\dfrac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}\right|+\left|\dfrac{\varphi'(0)}{\varsigma_0}\right|}{|\varsigma_0|\left(1+\left|\dfrac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}\right|\left|\dfrac{\varphi'(0)}{\varsigma_0}\right|\right)}}{1+\dfrac{\left|\dfrac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}\right|+\left|\dfrac{\varphi'(0)}{\varsigma_0}\right|}{|\varsigma_0|\left(1+\left|\dfrac{\varphi'(\varsigma_0)\big(1-|\varsigma_0|^2\big)}{\varsigma_0}\right|\left|\dfrac{\varphi'(0)}{\varsigma_0}\right|\right)}},$$

$$\frac{1-l}{1+l}=\frac{|\varsigma_0|^2+|\varphi'(\varsigma_0)|\big(1-|\varsigma_0|^2\big)|\varphi'(0)|-|\varphi'(\varsigma_0)|\big(1-|\varsigma_0|^2\big)-|\varphi'(0)|}{|\varsigma_0|^2+|\varphi'(\varsigma_0)|(1-|\varsigma_0|^2)|\varphi'(0)|+|\varphi'(\varsigma_0)|(1-|\varsigma_0|^2)+|\varphi'(0)|}$$

and

$$\frac{1-l}{1+l}=\frac{\big(1-|b|^2\big)^2|\varsigma_0|^2+|g'(\varsigma_0)|\big(1-|\varsigma_0|^2\big)|g'(0)|-\big(1-|b|^2\big)|g'(\varsigma_0)|\big(1-|\varsigma_0|^2\big)-\big(1-|b|^2\big)|f'(0)|}{\big(1-|b|^2\big)^2|\varsigma_0|^2+|g'(\varsigma_0)|(1-|\varsigma_0|^2)|g'(0)|+\big(1-|b|^2\big)|g'(\varsigma_0)|(1-|\varsigma_0|^2)+(1-|b|^2)|f'(0)|},$$

we obtain

$$|\varphi'(1)|\geq 1+\frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2}+\frac{\big(1-|b|^2\big)|\varsigma_0|-|g'(0)|}{\big(1-|b|^2\big)|\varsigma_0|+|g'(0)|}$$

$$\times\left[1+\frac{\big(1-|b|^2\big)^2|\varsigma_0|^2+|g'(\varsigma_0)|\big(1-|\varsigma_0|^2\big)|g'(0)|-\big(1-|b|^2\big)|g'(\varsigma_0)|\big(1-|\varsigma_0|^2\big)-\big(1-|b|^2\big)|f'(0)|}{\big(1-|b|^2\big)^2|\varsigma_0|^2+|g'(\varsigma_0)|(1-|\varsigma_0|^2)|g'(0)|+\big(1-|b|^2\big)|g'(\varsigma_0)|(1-|\varsigma_0|^2)+(1-|b|^2)|f'(0)|}\,\frac{1-|\varsigma_0|^2}{|1-\varsigma_0|^2}\right].$$

From definition of $\varphi(z)$, we have

$$|\varphi'(1)|=\frac{|g'(1)|(1-|b|)}{|b|+1}.$$

We thus take inequality (2.4). Let us choose arbitrary real numbers z_0, x and y such that $0<x=|\varphi'(0)|<|\varsigma_0|^2$, $0<y=|\varphi'(\varsigma_0)|<\dfrac{|\varsigma_0|^2}{(1-|\varsigma_0|^2)^2}$ to show that inequality (2.4) is sharp. Let

$$\varphi(z)=z\left(\frac{z-\varsigma_0}{1-\overline{\varsigma_0}z}\right)\frac{-\dfrac{x}{\varsigma_0}+z\dfrac{\sigma+\dfrac{z-\varsigma_0}{1-\overline{\varsigma_0}z}}{1+\sigma\dfrac{z-\varsigma_0}{1-\overline{\varsigma_0}z}}}{1-\dfrac{x}{\varsigma_0}z\dfrac{\sigma+\dfrac{z-\varsigma_0}{1-\overline{\varsigma_0}z}}{1+\sigma\dfrac{z-\varsigma_0}{1-\overline{\varsigma_0}z}}},\tag{2.8}$$

where

$$\sigma = \frac{1}{\varsigma_0^2} \frac{y\big(1-|\varsigma_0|^2\big)+x}{1+xy\dfrac{1-|\varsigma_0|^2}{\varsigma_0^2}}.$$

From (2.8), with the simple calculations we obtain

$$\varphi'(0)=x, \qquad \varphi'(\varsigma_0)=y$$

and

$$\varphi'(1)=1+\frac{1-\varsigma_0^2}{(1-\varsigma_0)^2}+\frac{\varsigma_0+x}{\varsigma_0-x}\left(1+\frac{1-\varsigma_0^2}{(1-\varsigma_0)^2}\frac{\varsigma_0^2+xy\big(1-|\varsigma_0|^2\big)-y\big(1-|\varsigma_0|^2\big)-x}{\varsigma_0^2+xy(1-|\varsigma_0|^2)+y(1-|\varsigma_0|^2)+x}\right).$$

Choosing suitable signs of the numbers z_0, x and y, we conclude from the last equality that inequality (2.4) is sharp.

Theorem 4 is proved.

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