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**T. Panigrahi, E. Pattnayak** (Institute of Mathematics and Applications, Andharua, Bhubaneswar, Odisha, India),
**R. M. El-Ashwah**[1] (Department of Mathematics, Faculty of Science, Damietta University, New Damietta, Egypt)

# THIRD HANKEL DETERMINANT FOR THE CLASS OF ANALYTIC FUNCTIONS DEFINED BY MATHIEU-TYPE SERIES RELATED TO A PETAL-SHAPED DOMAIN

# ТРЕТІЙ ДЕТЕРМІНАНТ ГАНКЕЛЯ ДЛЯ КЛАСУ АНАЛІТИЧНИХ ФУНКЦІЙ, ВИЗНАЧЕНИХ РЯДАМИ ТИПУ МАТЬЄ І ПОВ'ЯЗАНИХ З ОБЛАСТЮ ПЕЛЮСТКОВОЇ ФОРМИ

We introduce a new subclass of analytic functions based on the Mathieu-type series related to a petal-shaped domain. We investigate the bounds of the initial coefficient estimates, the Fekete – Szegö inequality, and the Hankel determinant of order two and three.

Введено новий підклас аналітичних функцій на основі рядів типу Матьє, що пов'язані з областю пелюсткової форми. Досліджено межі початкових оцінок коефіцієнтів, нерівність Фекете – Сего та визначник Ганкеля другого та третього порядку.

**1. Introduction and motivation.** Denote by $\mathcal{B}$, the family of all functions $\chi$ of the form

$$\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \alpha_n \zeta^n \tag{1.1}$$

which are holomorphic in the open unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Let $\mathcal{S}$ represent the subclass of $\mathcal{B}$ which are univalent in $\mathbb{D}$. One of the important subclass of the class $\mathcal{B}$ is the family of starlike function. A function $\chi \in \mathcal{B}$ is said to be starlike if $\Re\left\{\dfrac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right\} > 0,\ \zeta \in \mathbb{D}$. We denote such class by $\mathcal{S}^*$. In term of subordination, Ma and Minda (see [12]) define the class $\mathcal{S}^*$ as

$$\mathcal{S}^*(\phi) = \left\{\chi \in \mathcal{B} \colon \frac{\zeta\chi'(\zeta)}{\chi(\zeta)} \prec \phi(\zeta)\right\},$$

where $\phi(\zeta) = \dfrac{1+\zeta}{1-\zeta}$ and $\prec$ denote the subordination between two analytic functions.

In recent time, several subfamilies of the normalized analytic function class $\mathcal{B}$ were studied as a special cases of the function $\phi$ (see [6, 7, 15, 19, 21, 24]).

Further, Kumar and Arora (see [11]) considered the class $\mathcal{S}^*(\phi)$ where $\phi(\zeta) = 1 + \sin h^{-1}\zeta$. The function $\phi(\zeta)$ is a multivalued function and has the branch cuts about the line segments $(-i\infty, -i) \cup (i, i\infty)$, on the imaginary axis, and hence it is holomorphic in $\mathbb{D}$. Geometrically, the function $\phi(\zeta)$ maps the unit disk $\mathbb{D}$ onto a petal-shaped region $\Omega_\phi$ given below:

$$\Omega_\phi = \{\omega \in \mathbb{C} \colon |\sin h(\omega - 1)| < 1\}.$$

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[1] Corresponding author, e-mails: r_elashwah@yahoo.com, relashwah@du.edu.eg.

One of the interesting area of research in the geometric function theory is the study of coefficient problem and the Hankel determinant is one of the medium for the estimating it. For the function $\chi \in \mathcal{B}$ of the form (1.1), Pommerenke [18] introduced the Hankel determinant $H_{q,n}(\chi)$ as

$$H_{q,n}(\chi) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+q-1} \\ \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n+q-1} & \alpha_{n+q} & \cdots & \alpha_{n+2q-2} \end{vmatrix}, \quad n, q \in \mathbb{N} := \{1, 2, 3, \ldots\}, \quad \alpha_1 = 1.$$

In particular, for different values of $q$ and $n$, we obtain the Hankel determinant of various orders:

For $n = 1$ and $q = 2$,

$$H_{2,1}(\chi) = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{vmatrix} = \alpha_3 - \alpha_2^2$$

is popularly known as Fekete – Szegö functional. For $n = q = 2$, we have

$$H_{2,2}(\chi) = \begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = \alpha_2\alpha_4 - \alpha_3^2$$

is well-known as the second Hankel determinant. In recent time, many authors have contributed their results in form of research papers for finding the upper bounds of $|H_{2,2}(\chi)|$ for various subclasses of analytic function and their results are listed in literature. For $n = 1$ and $q = 3$,

$$H_{3,1}(\chi) = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_3 & \alpha_4 & \alpha_5 \end{vmatrix}$$

is known as the third Hankel determinant. Babalola [3] obtained the upper bound of $|H_{3,1}(\chi)|$ for the family of $\mathcal{S}^*$, $\mathcal{C}$ and the class $\mathcal{R}$. Later, many researchers extended their ideas for finding the upper bounds of $|H_{3,1}(\chi)|$ for various subclasses of holomorphic functions which stood as a base for research in the field of geometric function theory. For recent exposition works on the third Hankel determinant, see [16, 20, 22, 25].

In our investigation we use the following infinite series named after E. L. Mathieu (see [14]) as

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{n^2 + r^2}, \quad r > 0.$$

Emersleben (see [9]) represent the series $S(r)$ in a closed integral form as

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{t \sin(rt)}{e^t - 1} dt.$$

Tomovski [23] has defined the Mathieu-type power series as

$$S(r,\zeta)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}\zeta^n, \quad r>0, \quad z\in\mathbb{D}.$$

Basically this series was defined for function of real variables. It was redefined by Bansal and Sokół [4] for function of complex variables.

A good amount of literature available on the study of Mathieu's series, its generalization and its inequalities. For recent expository work on the Mathieu-type series, see [1, 4–6, 8, 13, 17, 23].

Clearly, $S(r,\zeta)\notin\mathcal{B}$. We have to normalize the series as follows:

$$S_1(r,\zeta)=\frac{(r^2+1)^2}{2}S(r,\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{n(r^2+1)^2}{(n^2+r^2)^2}\zeta^n.$$

For functions $\chi\in\mathcal{S}$ given by (1.1) and $\tau\in\mathcal{S}$ defined by

$$\tau(\zeta)=\zeta+\sum_{n=2}^{\infty}\beta_n\zeta^n, \quad \zeta\in\mathbb{D},$$

we define the Hadamard product or convolution of $\chi$ and $\tau$ by

$$(\chi*\tau)(\zeta)=\zeta+\sum_{n=2}^{\infty}\alpha_n\beta_n\zeta^n, \quad \zeta\in\mathbb{D}.$$

Making use of convolution operator, we define a new linear operator $\mathcal{R}(n,r)\colon\mathcal{B}\longrightarrow\mathcal{B}$ as

$$\mathcal{R}(n,r)\chi=S_1(r,\zeta)*\chi(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{n(r^2+1)^2}{(n^2+r^2)^2}\alpha_n\zeta^n, \quad \zeta\in\mathbb{D}. \tag{1.2}$$

Motivated essentially by aforementioned works, we introduce the subclass of $\mathcal{B}$ as follows:

**Definition 1.1.** *A function* $\chi$ *given by* (1.1) *is in the class* $\mathcal{S}^{*,r}_{\sin h}$ *if and only if*

$$\frac{\zeta(R(n,r)\chi)'(\zeta)}{R(n,r)\chi(\zeta)}\prec 1+\sin h^{-1}\zeta, \quad \zeta\in\mathbb{D}. \tag{1.3}$$

Equivalently, we can write the above subordination condition (1.3) as follows:

$$\left|\sin h\left(\frac{\zeta(R(n,r)\chi)'(\zeta)}{R(n,r)\chi(\zeta)}-1\right)\right|<1, \quad \zeta\in\mathbb{D}.$$

**2. Preliminaries.** Let $\mathcal{P}$ denote the set of all functions $p$ that are analytic in $\mathbb{D}$ with $\operatorname{Re}(p(\zeta))>0$ and has the following form:

$$p(\zeta)=1+\sum_{n=1}^{\infty}c_n\zeta^n, \quad \zeta\in\mathbb{D}. \tag{2.1}$$

We need the following lemmas for deriving our results.

**Lemma 2.1** [10, 12]. *Let* $p \in \mathcal{P}$ *be of the form* (2.1). *Then, for any complex parameter* $\nu$,

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}. \tag{2.2}$$

*Specially, if the number* $\nu$ *is a real parameter, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2, & \nu \leq 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu \geq 1. \end{cases} \tag{2.3}$$

**Lemma 2.2** [18]. *If* $p \in \mathcal{P}$ *of the form* (2.1), *then*

$$|c_n| \leq 2 \quad \forall n \geq 1, \tag{2.4}$$

$$|c_{n+k} - \mu c_n c_k| \leq \begin{cases} 2, & 0 \leq \mu \leq 1, \\ 2|2\mu - 1|, & \textit{otherwise}, \end{cases}$$

$$|c_m c_n - c_k c_l| < 4 \quad \textit{for} \quad m + n = l + k,$$

$$|c_{n+2k} - \mu c_n c_k^2| \leq 2(1 + 2\mu) \quad \textit{for} \quad \mu \in \mathbb{R},$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

**Lemma 2.3** [2]. *If* $p \in \mathcal{P}$ *and has the series of the form* (2.1), *then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \leq 2|J| + 2|K - 2J| + 2|J - K + L|, \tag{2.5}$$

*where* $J$, $K$ *and* $L$ *are real numbers.*

**3. Coefficient estimates.** The following theorem gives the bounds of the initial coefficient estimates for the class $\mathcal{S}^{*,r}_{\sin h}$.

**Theorem 3.1.** *Let the function* $\chi \in \mathcal{S}^{*,r}_{\sin h}$. *Then*

$$|\alpha_2| \leq \frac{(4 + r^2)^2}{2(r^2 + 1)^2},$$

$$|\alpha_3| \leq \frac{(9 + r^2)^2}{6(r^2 + 1)^2},$$

$$|\alpha_4| \leq \frac{13(16 + r^2)^2}{144(r^2 + 1)^2},$$

*and*

$$|\alpha_5| \leq \frac{13(25 + r^2)^2}{144(r^2 + 1)^2}.$$

***Proof.*** Since $\chi \in \mathcal{S}^{*,r}_{\sin h}$, then by Definition 1.1 there exists an analytic function $\omega(\zeta)$ satisfying the condition of Schwarz's lemma such that

$$\frac{\zeta(\mathcal{R}(n,r)\chi)'(\zeta)}{\mathcal{R}(n,r)\chi(\zeta)} = 1 + \sin h^{-1}\omega(\zeta), \quad \zeta \in \mathbb{D}. \tag{3.1}$$

As $p \in \mathcal{P}$, it can be written in terms of the Schwarz function $\omega$ by

$$p(\zeta) = \frac{1+\omega(\zeta)}{1-\omega(\zeta)} = 1 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \ldots,$$

which implies

$$\begin{aligned}\omega(\zeta) = \frac{p(\zeta)-1}{p(\zeta)+1} &= \frac{c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \ldots}{2 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \ldots} \\ &= \frac{c_1}{2}\zeta + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\zeta^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)\zeta^3 + \ldots.\end{aligned}$$

From (1.2), it is easy to see that

$$\begin{aligned}\frac{\zeta(\mathcal{R}(n,r)\chi)'(\zeta)}{\mathcal{R}(n,r)\chi(\zeta)} &= 1 + \frac{2(r^2+1)^2}{(4+r^2)^2}\alpha_2\zeta + \left[\frac{6(r^2+1)^2}{(9+r^2)^2}\alpha_3 - \frac{4(r^2+1)^4}{(4+r^2)^4}\alpha_2^2\right]\zeta^2 \\ &+ \left[\frac{12(r^2+1)^2}{(16+r^2)^2}\alpha_4 - \frac{18(r^2+1)^4}{(4+r^2)^2(9+r^2)^2}\alpha_2\alpha_3 + \frac{8(r^2+1)^6}{(4+r^2)^6}\alpha_2^3\right]\zeta^3 \\ &+ \left[\frac{20(r^2+1)^2}{(25+r^2)^2}\alpha_5 - \frac{18(r^2+1)^4}{(9+r^2)^4}\alpha_3^2 - \frac{32(r^2+1)^4}{(4+r^2)^2(16+r^2)^2}\alpha_2\alpha_4 \right. \\ &\left. - \frac{16(r^2+1)^8}{(4+r^2)^8}\alpha_2^4 + \frac{48(r^2+1)^6}{(4+r^2)^4(9+r^2)^2}\alpha_2^2\alpha_3\right]\zeta^4 + \ldots.\end{aligned} \tag{3.2}$$

On the other hand, the right-hand side of (3.1) after simplification becomes

$$\begin{aligned}1 + \sin h^{-1}(\omega(\zeta)) &= 1 + \frac{c_1}{2}\zeta + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\zeta^2 + \left(\frac{c_3}{2} + \frac{5}{48}c_1^3 - \frac{c_1c_2}{2}\right)\zeta^3 \\ &+ \left(\frac{c_4}{2} - \frac{c_2^2}{4} - \frac{c_1^4}{32} + \frac{5}{16}c_1^2c_2 - \frac{c_1c_3}{2}\right)\zeta^4 + \ldots.\end{aligned} \tag{3.3}$$

Making use of (3.2) and (3.3) in (3.1) and comparing the coefficients of various powers of $\zeta$, we get

$$\alpha_2 = \frac{(4+r^2)^2}{4(r^2+1)^2}c_1, \tag{3.4}$$

$$\alpha_3 = \frac{(9+r^2)^2}{12(r^2+1)^2}c_2, \tag{3.5}$$

$$\alpha_4 = \frac{(16+r^2)^2}{12(r^2+1)^2}\left[\frac{c_3}{2} - \frac{c_1^3}{48} - \frac{1}{8}c_1c_2\right], \tag{3.6}$$

and

$$\alpha_5 = \frac{(25+r^2)^2}{20(r^2+1)^2}\left[\frac{c_4}{2} - \frac{c_2^2}{8} - \frac{c_1c_3}{6} + \frac{5c_1^4}{288} - \frac{c_1^2c_2}{48}\right]. \tag{3.7}$$

Applications of Lemma 2.2 in (3.4) and (3.5) and Lemma 2.3 in (3.6) give the desire estimate for $\alpha_2$, $\alpha_3$, and $\alpha_4$.

Rearranging the terms in (3.7), we obtain

$$|\alpha_5| = \frac{(25+r^2)^2}{20(r^2+1)^2}\left[c_1\left(\frac{5}{288}c_1^3 - \frac{1}{48}c_1c_2 - \frac{1}{6}c_3\right) + \frac{1}{2}\left(c_4 - \frac{c_2^2}{4}\right)\right].$$

Applications of triangle inequality and followed by Lemmas 2.2 and 2.3 gives the bound for $\alpha_5$.

Theorem 3.1 is proved.

Next theorem gives the bound for the Fekete–Szegö functional $(\alpha_3 - \mu\alpha_2^2)$ when $\mu$ is complex.

**Theorem 3.2.** *Let* $\chi \in \mathcal{S}^{*,r}_{\sin h}$. *Then, for any complex parameter* $\mu$, *we have*

$$|\alpha_3 - \mu\alpha_2^2| \le \frac{(9+r^2)^2}{6(r^2+1)^2}\max\left\{1,\ \left|\frac{2(r^2+1)^2(9+r^2)^2 - 3(4+r^2)^4\mu}{2(r^2+1)^2(9+r^2)^2}\right|\right\}.$$

***Proof.*** From the relations (3.4) and (3.5) one yields

$$\left|\alpha_3 - \mu\alpha_2^2\right| = \left|\frac{(9+r^2)^2}{12(r^2+1)^2}c_2 - \mu\frac{(4+r^2)^4}{16(r^2+1)^4}c_1^2\right| = \frac{(9+r^2)^2}{12(r^2+1)^2}|c_2 - \nu c_1^2|, \tag{3.8}$$

where

$$\nu = \frac{3(4+r^2)^4}{4(r^2+1)^2(9+r^2)^2}\mu.$$

By virtue of (2.2) gives the required estimate as stated in the theorem.

Theorem 3.2 is proved.

Letting $\mu = 1$ in the above theorem yields the following corollary.

***Corollary* 3.1.** *If* $\chi \in \mathcal{S}^{*,r}_{\sin h}$, *then*

$$|\alpha_3 - \alpha_2^2| \le \frac{(9+r^2)^2}{6(r^2+1)^2}.$$

The Fekete–Szegö inequality for real parameter $\mu$ is given in the following theorem.

**Theorem 3.3.** *Let* $\chi \in \mathcal{S}^{*,r}_{\sin h}$. *Then, for any real parameter* $\mu$, *we have*

$$\left|\alpha_3 - \mu\alpha_2^2\right| \le \begin{cases} \dfrac{[2(r^2+1)^2(9+r^2)^2 - 3(4+r^2)^4\mu]}{12(r^2+1)^4}, & \mu \le 0, \\ \dfrac{(9+r^2)^2}{6(r^2+1)^2}, & 0 \le \mu \le \dfrac{4(r^2+1)^2(9+r^2)^2}{3(4+r^2)^4}, \\ -\dfrac{[2(r^2+1)^2(9+r^2)^2 - 3(4+r^2)^4\mu]}{12(r^2+1)^4}, & \mu \ge \dfrac{4(r^2+1)^2(9+r^2)^2}{3(4+r^2)^4}. \end{cases} \tag{3.9}$$

***Proof.*** An application of (2.3) to relation (3.8) yields the required estimate as stated in (3.9).

Theorem 3.3 is proved.

**Theorem 3.4.** *Let the function* $\chi$ *given by* (1.1) *is in the class* $\mathcal{S}^{*,r}_{\sin h}$. *Then*

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| \le \frac{13(4+r^2)(16+r^2)^2 + 8(9+r^2)^4}{288(r^2+1)^4}.$$

***Proof.*** From relations (3.4), (3.5) and (3.6), we get

$$\alpha_2\alpha_4 - \alpha_3^2 = \frac{(4+r^2)^2(16+r^2)^2}{48(r^2+1)^4}c_1\left[\frac{c_3}{2} - \frac{c_1^3}{48} - \frac{1}{8}c_1c_2\right] - \frac{(9+r^2)^4}{144(r^2+1)^4}c_2^2.$$

Application of triangle inequality and followed by (2.4), Lemma 2.3 give the bounds of $[\alpha_2\alpha_4 - \alpha_3^2]$.

Theorem 3.4 is proved.

**Theorem 3.5.** *Let the function* $\chi$ *given by* (1.2) *is in the class* $\mathcal{S}^{*,r}_{\sin h}$. *Then*

$$|\alpha_2\alpha_3 - \alpha_4| \le \frac{13(16+r^2)(r^2+1)^2 + 12(4+r^2)^2(9+r^2)^2}{144(r^2+1)^4}.$$

***Proof.*** By using (3.4), (3.5), and (3.6), we have

$$\alpha_2\alpha_3 - \alpha_4 = \frac{(16+r^2)^2}{576(r^2+1)^2}c_1^3 + \frac{2(4+r^2)^2(9+r^2)^2 + (16+r^2)^2(r^2+1)^2}{96(r^2+1)^4}c_1c_2 - \frac{(16+r^2)^2}{24(r^2+1)^2}c_3. \tag{3.10}$$

Taking modulus on both sides of (3.10) and then applying Lemma 2.3 to the resulting relation we get the required estimate.

Theorem 3.5 is proved.

**Theorem 3.6.** *If the function* $\chi$ *given by* (1.2) *is in the class* $\mathcal{S}^{*,r}_{\sin h}$, *then*

$$|H_3(1)| \le \frac{312(4+r^2)^2(16+r^2)^2(9+r^2)^2 + 96(9+r^2)^6}{20736(r^2+1)^6} + \frac{169(16+r^2)^4(r^2+1)^2 + 312(25+r^2)^2(9+r^2)^2(r^2+1)^2}{20736(r^2+1)^6}.$$

***Proof.*** From the definition of the Hankel determinant, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{3.11}$$

Application of triangle inequality to the relation (3.11) gives

$$|H_3(1)| \le |a_3||a_2a_4 - {a_3}^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|.$$

Making use of Theorems 3.1, 3.4, 3.5 and Corollary 3.1 yield the desired estimate as stated.

Theorem 3.6 is proved.

**Concluding remark.** In this paper, by making use of Mathieu's type series and subordination, we introduced a subclass of analytic function associated with petal-shaped domain and studied the coefficient estimates, the Fekete–Szegö inequalities and the Hankel determinant. Several other properties of the class like Krushal inequality and Zalcman conjecture can also be investigated.

**Conflict of interest.** The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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