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TWO-POINT BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH GENERALIZED PIECEWISE-CONSTANT ARGUMENT

ДВОТОЧКОВІ КРАЙОВІ ЗАДАЧІ ДЛЯ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З УЗАГАЛЬНЕНИМ КУСКОВО-СТАЛИМ АРГУМЕНТОМ

We consider a two-point boundary-value problem for a system of differential equations with generalized piecewise-constant argument. To solve the problem, we propose to use a constructive method based on the Dzhumabaev parametrization method and a new approach to the concept of general solution. The interval is partitioned with regard for the singularities of the argument. The values of the solution at the interior points of the partition are regarded as additional parameters, and the differential equation is transformed into a system of Cauchy problems with parameters on subintervals of the partition. By using the solutions of these problems, we obtain a new general solution of the differential equation with piecewise-constant argument and establish its properties. The new general solution, boundary conditions, and the conditions of continuity of the solution at the interior points of the partition are used to construct a linear system of algebraic equations for the introduced parameters. The coefficients and the right-hand side of the system are found as a result of the solution of Cauchy problems for linear ordinary differential equations on the subintervals of the partition. It is shown that the solvability of the boundary-value problem is equivalent to the solvability of the constructed system. We propose algorithms of the parametrization method for solving the analyzed boundary-value problem and establish necessary and sufficient conditions for the well-posedness of this problem.

Розглянуто двоточкову крайову задачу для системи диференціальних рівнянь із узагальненим кусково-сталим аргументом. Для її розв'язання запропоновано конструктивний метод, що базується на методі параметризації Джумабаєва та новому підході до поняття загального розв'язку. Інтервал розбито з урахуванням особливостей аргументу. Значення розв'язку у внутрішніх точках розбиття розглянуто як додаткові параметри, а диференціальне рівняння перетворено на систему задач Коші з параметрами, заданими на підінтервалах розбиття. За допомогою розв'язків цих задач отримано новий загальний розв'язок диференціального рівняння з кусково-сталим аргументом і встановлено його властивості. Новий загальний розв'язок, граничні умови та умови неперервності розв'язку у внутрішніх точках розбиття використовують для побудови лінійної системи алгебраїчних рівнянь щодо введених параметрів. Коефіцієнти та праву частину системи знайдено шляхом розв'язування задач Коші для лінійних звичайних диференціальних рівнянь на підінтервалах розбиття. Показано, що розв'язність крайової задачі еквівалентна розв'язності побудованої системи. Запропоновано алгоритми методу параметризації для розв'язування досліджуваної крайової задачі та встановлено необхідні й достатні умови її коректної постановки.

1. Introduction. Mathematical modeling of processes with discontinuity effects has necessitated the need to develop the theory of differential equations with discontinuities. An important class of such equations is comprised of differential equations with piecewise constant argument (DEPCA). The study of DEPCA was initiated by Busenberg, Cooke, Shah, and Wiener in [1 – 3]. The questions of the existence and uniqueness of solutions to DEPCA, their oscillations and stability, integral manifolds and periodic solutions have been extensively discussed by many authors [4 – 9]. DEPCA has been used in constructing various models in biology, mechanics and electronics.

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The founders of the theory of DEPCA suggested that the method of studying these equations is based on their reduction to discrete systems. This means that only the values of solutions at integer moments (or multiples of integers) were of interest. Moreover, for nondeviating arguments, systems are required to be linear with respect to the values of solutions. Such restrictions narrow the scope of theoretical research as well as of application problems that can be modeled via these equations.

When modeling with DEPCA, the deviation of the argument, taken as the greatest integer function, is always constant and equal to one. But this approach can contradict real phenomena. The generalization of DEPCA has been undertaken by Akhmet [10 – 13]. In his works the greatest integer function as deviating argument was replaced by an arbitrary piecewise constant function. Thus, differential equations with piecewise constant argument of generalized type (DEPCAG) are more suitable for modeling and solving various application problems, including areas of neural networks, discontinuous dynamical systems, hybrid systems, etc. To date, the theory of DEPCAG on the entire axis has been developed and their applications have been implemented. The results have been extended to periodic impulse systems of DEPCAG [14 – 21].

Along with the study of various properties of DEPCA, a number of authors investigated the questions of solvability and construction of solutions to boundary-value problems for these equations on a finite interval [22 – 34]. For DEPCAG, however, the questions of solvability of boundary-value problems on a finite interval still remain open. This issue can be resolved by developing constructive methods.

The Dzhumabaev parametrization method [35] is a constructive method for studying and solving boundary-value problems for differential equations. The method was originally applied to linear boundary-value problems for ordinary differential equations to establish coefficient criteria for their unique solvability and develop algorithms for finding their solutions. The parametrization method was further extended to loaded differential equations and integro-differential equations to establish criteria for the solvability and well-posedness of boundary-value problems. Based on the method, a new approach to the concept of general solution was proposed [36 – 40], that made it possible to develop new algorithms for finding approximate and numerical solutions to boundary-value problems.

The Dzhumabaev parametrization method has been successfully applied to boundary-value problems for impulsive differential equations, delay differential equations, and integro-differential equations with parameters [41 – 45]. The algorithms for finding their solutions has been constructed, and criteria for their unique solvability have been obtained in terms of input data. In the present paper, we apply the parametrization method to a boundary-value problem for a system of DEPCAG.

On $[0,T]$, we consider the following two-point boundary-value problem for a system of DEPCAG:

$$\frac{dx}{dt} = A(t)x + A_0(t)x(\gamma(t)) + f(t), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{1.1}$$

$$Bx(0) + Cx(T) = d, \quad d \in \mathbb{R}^n. \tag{1.2}$$

Here, $x(t) = \mathrm{col}(x_1(t), x_2(t), \ldots, x_n(t))$ is the unknown function, $(n \times n)$ matrices $A(t)$, $A_0(t)$ and n-vector $f(t)$ are continuous on $[0,T]$; $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$; $\theta_j \le \zeta_j \le \theta_{j+1}$ for all $j = 0, 1, \ldots, N-1$; $0 = \theta_0 < \theta_1 < \ldots < \theta_{N-1} < \theta_N = T$; B and C are constant $(n \times n)$ matrices, $\|x\| = \max_{i=\overline{1,n}} |x_i|$.

A function $x^*(t) : [0,T] \to \mathbb{R}^n$ is a solution to problem (1.1), (1.2) if:

(i) $x^*(t)$ is continuous on $[0,T]$;

(ii) $x^*(t)$ is differentiable on $[0,T]$ with the possible exception of the points θ_j, $j = \overline{0, N-1}$, at which the one-sided derivatives exist;

(iii) $x^*(t)$ satisfies the system of equations (1.1) on each interval (θ_j, θ_{j+1}), $j = \overline{0, N-1}$; at the points θ_j, $j = \overline{0, N-1}$, system (1.1) is satisfied by the right-hand derivative of $x^*(t)$;

(iv) $x^*(t)$ satisfies the boundary condition (1.2) at $t = 0$ and $t = T$.

The aim of this paper is to develop a constructive method for investigation and solving the problem (1.1), (1.2), as well as to construct algorithms for finding approximate solutions to the problem. To this end, we use the parametrization method [36] and a new approach to the concept of the general solution. This approach to the general solution was originally introduced for linear Fredholm integro-differential equations [38] and then applied to linear loaded differential equations and families of such equations [38, 39] and to ordinary differential equations [40].

In Section 2, we present the scheme of the parametrization method and the definition of the new general solution to differential equations with generalized piecewise constant argument (1.1). The interval $[0, T]$ is partitioned into N subintervals, and the values of the desired function at the points $t = \zeta_j$ of each subinterval $[\theta_j, \theta_{j+1})$, $j = 0, 1, \ldots, N-1$, are considered as additional parameters. The definition of the Δ_N-general solution to equation (1.1) is introduced. The Δ_N-general solution, denoted by $x(\Delta_N, t, \lambda)$, contains an arbitrary vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^{nN}$. The application of the parametrization method to system (1.1) gives rise to auxiliary Cauchy problems for differential equations with parameters [40]. We establish some properties of the Δ_N-general solution to DEPCAG.

In Section 3, we use the constructed general solution $x(\Delta_N, t, \lambda)$ to establish the solvability criteria for the problem under consideration and propose an algorithm for solving the problem. The application of the new concept of the general solution allows us to reduce the solvability of the problem (1.1), (1.2) to that of the system of linear algebraic equations

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^{nN}. \tag{1.3}$$

The system (1.3) is constructed by using the Δ_N-general solution to DEPCAG (1.1), the boundary condition (1.2), and the condition for the continuity of the solution at the interior points of the partition Δ_N. We propose a method for solving the problem (1.1), (1.2), that is based on solving system (1.3).

Section 4 is devoted to constructing algorithms of the parametrization method for solving two-point boundary-value problems for DEPCAG. We establish the conditions for the unique solvability and well-posedness of the problem (1.1), (1.2) in terms of the matrix $Q_\nu(\Delta_N)$ constructed through input data.

2. Scheme of the method and the concept of new general solution to system (1.1). Let Δ_N denote the partition of the interval $[0, T)$ by points $t = \theta_r, r = \overline{1, N-1}$, $\theta_0 = 0$, $\theta_N = T$: $[0, T) = \bigcup\limits_{r=1}^{N} [\theta_{r-1}, \theta_r)$.

We define the following spaces:

$C([0, T], \mathbb{R}^n)$ is the space of continuous functions $x\colon [0, T] \to \mathbb{R}^n$ with the norm

$$\|x\|_1 = \max_{t \in [0,T]} \|x(t)\| = \max_{t \in [0,T]} \max_{i = \overline{1,n}} |x_i(t)|;$$

$C([0, T], \Delta_N, \mathbb{R}^{nN})$ is the space of systems of functions $x[t] = \bigl(x_1(t), x_2(t), \ldots, x_N(t)\bigr)$, where $x_r\colon [\theta_{r-1}, \theta_r) \to \mathbb{R}^n$ are continuous functions that have finite left-hand limits $\lim_{t \to \theta_r - 0} x_r(t)$ for all $r = \overline{1, N}$, with the norm

$$\bigl\|x[\cdot]\bigr\|_2 = \max_{r = \overline{1,N}} \sup_{t \in [\theta_{r-1}, \theta_r)} |x_r(t)|.$$

Let $x_r(t)$ denote the restriction of the function $x(t)$ to the rth subinterval $[\theta_{r-1}, \theta_r)$, i.e., $x_r(t) = x(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$.

Then the system $x[t] = \big(x_1(t), x_2(t), \ldots, x_N(t)\big)$ belongs to the space $C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$, and its elements $x_r(t)$, $r = \overline{1, N}$, satisfy the following system of DEPCAG:

$$\frac{dx_r}{dt} = A(t)x_r(t) + A_0(t)x_r(\zeta_{r-1}) + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}. \tag{2.1}$$

Here, we take into account that $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$.

We introduce parameters $\lambda_r = x_r(\zeta_{r-1})$, $r = \overline{1, N}$. By using the substitution $z_r(t) = x_r(t) - \lambda_r$ on each subinterval $[\theta_{r-1}, \theta_r)$, we get the system of differential equations with parameters

$$\frac{dz_r}{dt} = A(t)(z_r(t) + \lambda_r) + A_0(t)\lambda_r + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}, \tag{2.2}$$

subject to the initial conditions

$$z_r(\zeta_{r-1}) = 0, \quad r = \overline{1, N}. \tag{2.3}$$

Thus we obtain the Cauchy problems (2.2), (2.3) for systems of ordinary differential equations with parameters on the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$. For fixed $\lambda_r \in \mathbb{R}^n$ and r, the Cauchy problem (2.2), (2.3) has a unique solution $z_r(t, \lambda_r)$, and the system $z[t, \lambda] = \big(z_1(t, \lambda_1), \ldots, z_N(t, \lambda_N)\big)$ belongs to $C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$. We note that the obtained Cauchy problems differ from the Cauchy problems in [39]. The initial values (2.3) are assigned at the points $\zeta_{r-1} \in [\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$. If $\zeta_{r-1} = \theta_{r-1}$ then conditions (2.3) become usual initial conditions at the left endpoints of the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, as in [39]. If $\theta_{r-1} < \zeta_{r-1} < \theta_r$, then conditions (2.3) are specified at the interior points of the partition subintervals. In the case $\zeta_{r-1} = \theta_r$, we get the initial conditions at the right endpoints.

A system of functions $z[t, \lambda]$ is called a solution to the Cauchy problems with parameters (2.2), (2.3). If the system of functions $\widetilde{x}[t] = \big(\widetilde{x}_1(t), \widetilde{x}_2(t), \ldots, \widetilde{x}_N(t)\big)$ belongs to $C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$, and the functions $\widetilde{x}_r(t)$, $r = \overline{1, N}$, satisfy equations (2.1), then the system of functions $z[t, \widetilde{\lambda}] = \big(z_1(t, \widetilde{\lambda}_1), z_2(t, \widetilde{\lambda}_2), \ldots, z_N(t, \widetilde{\lambda}_N)\big)$ with elements $z_r(t, \widetilde{\lambda}_r) = \widetilde{x}_r(t) - \widetilde{\lambda}_r$, $\widetilde{\lambda}_r = \widetilde{x}_r(\zeta_{r-1})$, $r = \overline{1, N}$, is a solution to the Cauchy problem with parameters (2.2), (2.3) with $\lambda_r = \widetilde{\lambda}_r$. Conversely, if a system of functions $z[t, \lambda^*] = \big(z_1(t, \lambda_1^*), z_2(t, \lambda_2^*), \ldots, z_N(t, \lambda_N^*)\big)$ is a solution to problem (2.2), (2.3) with $\lambda_r = \lambda_r^*$, $r = \overline{1, N}$, then system of functions $x^*[t] = \big(x_1^*(t), x_2^*(t), \ldots, x_N^*(t)\big)$ with $x_r^*(t) = \lambda_r^* + z_r(t, \lambda_r^*)$, $r = \overline{1, N}$, belongs to $C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ and the functions $x_r^*(t)$, $r = \overline{1, N}$, satisfy equations (2.1).

We now use the new definition of the general solution proposed in [38] to introduce the new general solution to DEPCAG (1.1).

Definition 1. *Let* $z[t, \lambda] = \big(z_1(t, \lambda_1), z_2(t, \lambda_2), \ldots, z_N(t, \lambda_N)\big)$ *be a solution of the Cauchy problem* (2.2), (2.3) *with parameters* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^{nN}$. *Then the function* $x(\Delta_N, t, \lambda)$, *defined by the equalities* $x(\Delta_N, t, \lambda) = \lambda_r + z_r(t, \lambda_r)$ *for* $t \in [\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$, *and* $x(\Delta_N, T, \lambda) = \lambda_N + \lim_{t \to T-0} z_N(t, \lambda_N)$, *is called the* Δ_N*-general solution to the system of DEPCAG* (1.1).

It follows from Definition 1 that the Δ_N-general solution depends on N arbitrary vectors $\lambda_r \in \mathbb{R}^n$ and satisfies the system of DEPCAG (1.1) for all $t \in (0, T) \backslash \big\{\theta_p, p = \overline{1, N-1}\big\}$.

Let $X_r(t)$ be a fundamental matrix of the system of ordinary differential equations

$$\frac{dz_r}{dt} = A(t)z_r(t), \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N}.$$

Hence, the solutions to the Cauchy problems with parameters (2.2), (2.3) can be represented as

$$z_r(t) = X_r(t) \int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau) + A_0(\tau)]d\tau\lambda_r$$

$$+ X_r(t) \int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)f(\tau)d\tau, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N}.$$

We consider the Cauchy problems on the partition subintervals

$$\frac{dx}{dt} = A(t)x + P(t), \quad x(\zeta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N}, \tag{2.4}$$

where $P(t)$ is a square matrix of order n or an n-dimensional vector, continuous on $[0, T]$, $\theta_{r-1} \leq \zeta_{r-1} \leq \theta_r$ for all $r = 1, 2, \ldots, N$. Let $A_r(P, t)$ denote the unique solution of the Cauchy problem (2.4) on each rth subinterval. It follows from the unique solvability of the Cauchy problem for linear ordinary differential equations that

$$A_r(P, t) = X_r(t) \int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N}.$$

We can now represent the Δ_N-general solution of system of DEPCAG (1.1) in the form

$$x(\Delta_N, t, \lambda) = \lambda_r + A_p(A + A_0, t)\lambda_r + A_p(f, t), \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N-1}, \tag{2.5}$$

$$x(\Delta_N, t, \lambda) = \lambda_N + A_N(A + A_0, t)\lambda_N + A_N(f, t), \quad t \in [\theta_{N-1}, \theta_N]. \tag{2.6}$$

The following statement justifies the fact that the function $x(\Delta_N, t, \lambda)$ can be considered as the general solution of system (1.1).

Theorem 1. *Let* $\widetilde{x}(t)$ *be a pointwise continuous on* $[0, T]$ *function with possible discontinuity points* $t = \theta_p$, $p = \overline{1, N-1}$, *and let* $x(\Delta_N, t, \lambda)$ *be the* Δ_N*-general solution of the system of DEPCAG* (1.1). *Suppose that the function* $\widetilde{x}(t)$ *has a continuous derivative and satisfies equation* (1.1) *for all* $t \in (0, T) \backslash \{\theta_p, p = \overline{1, N-1}\}$. *Then there exists a unique* $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N) \in \mathbb{R}^{nN}$ *such that the equality* $x(\Delta_N, t, \widetilde{\lambda}) = \widetilde{x}(t)$ *holds for all* $t \in [0, T]$.

We omit the proof which is quite straightforward.

Lemma 1. *Let* $x^*(t)$ *be a solution to a solution to the system of DEPCAG* (1.1), *and let* $x(\Delta_N, t, \lambda)$ *be the* Δ_N*-general solution to equation* (1.1). *Then there exists a unique* $\lambda^* = (\lambda_1^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ *such that* $x(\Delta_N, t, \lambda^*) = x^*(t)$ *for all* $t \in [0, T]$.

If $x(t)$ is a solution to system (1.1) and $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$ is the system of functions composed of its restrictions to the subintervals $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$, then the following equalities hold:

$$\lim_{t \to \theta_p - 0} x_p(t) = x_{p+1}(\theta_p), \quad p = \overline{1, N-1}. \tag{2.7}$$

These equations express the conditions for the continuity of the solution to system (1.1) at the interior points of the partition Δ_N.

Theorem 2. *Let a system of functions* $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$ *belong to the space* $C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$. *Suppose that the functions* $x_r(t)$, $r = \overline{1,N}$, *satisfy the systems of equations* (2.1) *and the continuity conditions* (2.7). *Then the function* $x^*(t)$, *defined by the equalities*

$$x^*(t) = x_r(t) \quad \textit{for} \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1,N}, \qquad \textit{and} \qquad x^*(T) = \lim_{t \to T-0} x_N(t),$$

is continuous on $[0,T]$, *continuously differentiable on* $(0,T)$ *and satisfies the system of DEPCAG* (1.1).

Proof. The equations (2.7) and the fact that $x[t] = \big(x_1(t), x_2(t), \ldots, x_N(t)\big)$ belongs to the space $C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ ensure the continuity of the function $x^*(t)$ on the interval $[0,T]$. Since the functions $x_r(t)$, $r = \overline{1,N}$, satisfy systems (2.1), the function $x^*(t)$ is continuously differentiable and satisfies system (1.1) for all $t \in [0,T] \backslash \big\{\theta_p, p = \overline{1, N-1}\big\}$. The existence of the one-sided derivative of the function $x^*(t)$ at the points $t = \theta_p$, $p = \overline{1,N-1}$, follows from the equalities

$$\lim_{t \to \theta_p - 0} \dot{x}^*(t) = A(\theta_p)x^*(\theta_p) + A_0(\theta_p)x^*(\zeta_{p-1}) + f(\theta_p), \quad p = \overline{1, N-1},$$

$$\lim_{t \to \theta_p + 0} \dot{x}^*(t) = A(\theta_p)x^*(\theta_p) + A_0(\theta_p)x^*(\zeta_p) + f(\theta_p), \quad p = \overline{1, N-1}.$$

Hence the function $x^*(t)$ satisfies (1.1) at the interior points of the partition Δ_N.

Theorem 2 is proved.

3. Solvability of problem (1.1), (1.2). The introduction of the Δ_N-general solution allows one to reduce the solvability of the boundary-value problem under consideration to that of a system of linear algebraic equations in arbitrary vectors $\lambda_r \in \mathbb{R}^n$, $r = \overline{1,N}$.

By substituting the expressions (2.5), (2.6) of the Δ_N-general solution into the boundary condition (1.2) and the continuity conditions (2.7), we obtain the system of linear algebraic equations

$$B\lambda_1 + BA_1(A + A_0, \theta_0)\lambda_1 + C\lambda_N + CA_N(A + A_0, T)\lambda_N = d - BA_1(f, \theta_0) - CA_N(f, T), \tag{3.1}$$

$$\lambda_p + A_p(A, \theta_p)\lambda_p - \lambda_{p+1} - A_{p+1}(A + A_0, \theta_p)\lambda_{p+1} = -A_p(f, \theta_p) + A_{p+1}(f, \theta_p), \quad p = \overline{1, N-1}. \tag{3.2}$$

Let $Q_*(\Delta_N)$ denote the square matrix of order nN composed of coefficients of λ_r, $r = \overline{1,N}$. We can now rewrite system (3.1), (3.2) in the form

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \quad \lambda \in R^{nN}, \tag{3.3}$$

where

$$F_*(\Delta_N) = \big(-d + BA_1(f, \theta_0) + CA_N(f, T), A_1(f, \theta_1) \\ - A_2(f, \theta_1), A_2(f, \theta_2) - A_3(f, \theta_2), \ldots, A_{N-1}(f, \theta_{N-1}) - A_N(f, \theta_{N-1})\big) \in \mathbb{R}^{nN}.$$

Theorems 1 and 2 imply that for any partition Δ_N the following statement holds true.

Lemma 2. *If* $x^*(t)$ *is a solution to problem* (1.1), (1.2), *and* $\lambda_r^* = x^*(\zeta_{r-1})$, $r = \overline{1,N}$, *then the vector* $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ *is a solution to system* (3.3).

Conversely, if $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N) \in \mathbb{R}^{nN}$ *is a solution to* (3.3) *and* $z[t, \tilde{\lambda}] = (z_1(t, \tilde{\lambda}_1), \ldots, z_N(t, \tilde{\lambda}_N))$ *is the solution to the Cauchy problem* (2.2), (2.3) *with* $\tilde{\lambda} \in \mathbb{R}^{nN}$, *then the function* $\tilde{x}(t)$ *defined by the equalities*

$$\tilde{x}(t) = \tilde{\lambda}_r + z_r(t, \tilde{\lambda}_r), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}, \qquad \textit{and} \qquad \tilde{x}(T) = \tilde{\lambda}_N + \lim_{t \to T-0} z_N(t, \tilde{\lambda}_N),$$

is the solution to problem (1.1), (1.2).

Definition 2. *The boundary-value problem* (1.1), (1.2) *is called uniquely solvable if it has a unique solution for any pair* $(f(t), d)$ *with* $f(t) \in C([0, T], \mathbb{R}^n)$ *and* $d \in \mathbb{R}^n$.

Lemma 2 and well-known theorems of linear algebra imply the following statements.

Theorem 3. *The boundary-value problem* (1.1), (1.2) *has a solution if and only if the vector* $F_*(\Delta_N)$ *is orthogonal to the kernel of the transposed matrix* $(Q_*(\Delta_N))'$, *i.e., the equality*

$$(F_*(\Delta_N), \eta) = 0$$

holds for all $\eta \in Ker(Q_*(\Delta_N))'$, *where* $(\cdot, \cdot)$ *is the dot product in* $\mathbb{R}^{nN}$.

Theorem 4. *The boundary-value problem* (1.1), (1.2) *is uniquely solvable if and only if the matrix* $Q_*(\Delta_N)$ *is invertible.*

Based on the results obtained in Section 2, we propose ***Algorithm A*** for solving the linear boundary-value problem (1.1), (1.2).

Step **1.** On the partition subintervals $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$, solve the Cauchy problems

$$\frac{dz}{dt} = A(t)z + A(t) + A_0(t), \quad z(\zeta_{r-1}) = 0,$$

and

$$\frac{dz}{dt} = A(t)z + f(t), \quad z(\zeta_{r-1}) = 0,$$

to find the functions $A_r(A + A_0, \theta_r)$ and $A_r(f, \theta_r)$, respectively. Here $\zeta_{r-1} \in [\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$.

Step **2.** Construct the system of linear algebraic equations (3.3) using the matrices and vectors found in Step 1.

Step **3.** Find the solution $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ to the system constructed in Step 2. Note that the components of λ^* are the values of the solution to problem (1.1), (1.2) at the points ζ_{r-1} of the partition subintervals: $\lambda_r^* = x^*(\zeta_{r-1})$, $r = \overline{1, N}$.

Step **4.** Find the values of the solution $x^*(t)$ at the remaining points of the subintervals by solving the Cauchy problems

$$\frac{dz}{dt} = A(t)z + f(t), \quad z(\zeta_{r-1}) = \lambda_r^*, \quad t \in [\theta_{r-1}, \theta_r).$$

The accuracy of the proposed algorithm depends on that of calculating the coefficients and right-hand parts of system (3.3).

The main auxiliary problem involved in Algorithm A is the Cauchy problem for ordinary differential equations. Depending on the choice of approximate or numerical techniques for auxiliary Cauchy problems, Algorithm A provides an approximate or numerical method of solving the boundary-value problem (1.1), (1.2).

Another important point to notice is that the auxiliary Cauchy problems in Algorithm A are substantially different from Cauchy problems considered in [37]. Firstly, as mentioned above, the initial conditions (2.3) are specified at the points $t = \zeta_{r-1}$ that may be any points of the partition subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$. Secondly, the partition Δ_N depends on the function $\gamma(t)$, where $\gamma(t) = \zeta_j$ for $t \in [\theta_j, \theta_{j+1})$ and $\theta_j \leq \zeta_j \leq \theta_{j+1}$, $j = 0, 1, \ldots, N-1$; $0 = \theta_0 < \theta_1 < \ldots <$

$\theta_{N-1} < \theta_N = T$. The number of subintervals is always equal to or greater than N. Thirdly, each subinterval $[\theta_{r-1}, \theta_r)$ can be divided into m_r parts by points $t = \theta_{i,r}$, $i = \overline{1, m_r}$, $r = \overline{1, N}$: $[\theta_{r-1}, \theta_r) = \bigcup\limits_{i=1}^{m_r} [\theta_{i-1,r-1}, \theta_{i,r})$. Here $\theta_{i,r} = \theta_r + i\dfrac{\theta_r - \theta_{r-1}}{m_r}$, $i = \overline{1, m_r}$, $\theta_{r-1} = \theta_{0,r-1} < \theta_{1,r-1} < \theta_{2,r-1} < \ldots < \theta_{m_r-1,r-1} < \theta_{m_r,r-1} = \theta_r$, $r = \overline{1, N}$. In this case, the number of Cauchy problems and hence the dimension of the system of algebraic equations, is equal to $(m_1 + m_2 + \ldots + m_N)N$. Then, according to Algorithm A, we will solve the Cauchy problems on the subintervals $[\theta_{i-1,r-1}, \theta_{i,r})$ with the initial conditions at the points $t = \theta_{i-1,r-1}$, $i = \overline{1, m_r}$, $r = \overline{1, N}$. The corresponding system of algebraic equations allows one to determine the values of the solution to problem (1.1), (1.2) at the left endpoints of the subintervals: $\lambda^*_{i,r} = x^*(\theta_{i-1,r-1})$, $i = \overline{1, m_r}$, $r = \overline{1, N}$.

4. Algorithms of the parameterization method and the well-posedness of problem (1.1), (1.2). In Section 3, we established conditions for the solvability and the unique solvability of problem (1.1), (1.2) in terms of the invertibility of the matrix $Q_*(\Delta_N)$. This matrix is formed using the Δ_N-general solution and a fundamental matrix $X_r(t)$ of the system of ordinary differential equations

$$\frac{dz_r}{dt} = A(t)z_r(t), \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N}.$$

In this section we establish conditions for the well-posedness of problem (1.1), (1.2) in terms of input data. To this end, the Dzhumabaev parametrization method is directly applied to problem (1.1), (1.2).

We consider the partition Δ_N of the interval $[0, T)$: $[0, T) = \bigcup\limits_{r=1}^{N} [\theta_{r-1}, \theta_r)$. Let $x_r(t)$ be the restriction of the function $x(t)$ to the rth subinterval of Δ_N, i.e., $x_r(t) = x(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$.

Then problem (1.1), (1.2) is transformed into the equivalent multipoint problem

$$\frac{dx_r}{dt} = A(t)x_r(t) + A_0(t)x_r(\zeta_{r-1}) + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}, \tag{4.1}$$

$$Bx_1(0) + C \lim_{t \to T-0} x_N(t) = d, \tag{4.2}$$

$$\lim_{t \to \theta_p - 0} x_p(t) = x_{p+1}(\theta_p), \quad p = \overline{1, N-1}. \tag{4.3}$$

Here, (4.3) are the conditions for the continuity of the solution at the partition points $t = t_p$, $p = \overline{1, N-1}$.

A system of functions $x[t] = \big(x_1(t), x_2(t), \ldots, x_N(t)\big) \in C\big([0, T], \Delta_N, \mathbb{R}^{nN}\big)$ is called a solution to problem (4.1)–(4.3) if its components $x_r(t)$, $r = \overline{1, N}$, satisfy the system of DEPCAG (4.1), the boundary condition (4.2), and the continuity conditions (4.3).

In (4.1), we take into account that $\gamma(t) = \zeta_j$ for $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$.

By introducing the parameters $\lambda_r = x_r(\zeta_{r-1})$, $r = \overline{1, N}$, and making the substitutions $z_r(t) = x_r(t) - \lambda_r$ on each rth subinterval $[\theta_{r-1}, \theta_r)$, we obtain the boundary-value problem for the system of differential equations with parameters

$$\frac{dz_r}{dt} = A(t)z_r(t) + (A(t) + A_0(t))\lambda_r + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}, \tag{4.4}$$

$$z_r(\zeta_{r-1}) = 0, \quad r = \overline{1, N}, \tag{4.5}$$

$$Bz_1(0) + B\lambda_1 + C \lim_{t\to T-0} z_N(t) + C\lambda_N = d, \tag{4.6}$$

$$\lim_{t\to\theta_p-0} z_p(t) + \lambda_p = z_{p+1}(\theta_p) + \lambda_{p+1}, \quad p = \overline{1, N-1}. \tag{4.7}$$

A pair $(\lambda^*, z^*[t])$ with components $\lambda^* = (\lambda_1^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ and $z^*[t] = (z_1^*(t), \ldots, z_N^*(t)) \in C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ is a solution to problem (4.4)–(4.7), if the functions $z_r^*(t),\ r = \overline{1,N}$, are continuously differentiable on $[\theta_{r-1}, \theta_r)$ and satisfy the Cauchy problems (4.4), (4.5) with $\lambda_r = \lambda_r^*$ and conditions (4.6) and (4.7).

The problem (1.1), (1.2) is equivalent to problem (4.4)–(4.7) in the following sense. If a function $x(t)$ is a solution to problem (1.1), (1.2), then the pair $(\lambda, z[t])$ with components $\lambda_r = x(\zeta_{r-1})$ and $z_r(t) = x(t) - x(\zeta_{r-1}),\ t \in [\theta_{r-1}, \theta_r),\ r = \overline{1,N}$, is a solution to problem (4.4)–(4.7). Conversely, if a pair $(\lambda^*, z^*[t])$ is a solution to problem (4.4)–(4.7), then the function $x^*(t)$, defined by the equalities $x^*(t) = \lambda_r^* + z_r^*(t),\ t \in [\theta_{r-1}, \theta_r),\ r = \overline{1,N}$, and $x^*(T) = \lambda_N^* + \lim_{t\to T-0} z_N^*(t)$, is a solution to problem (1.1), (1.2).

For fixed λ_r, the Cauchy problem (4.4), (4.5) is equivalent to the system of Volterra integral equations of the second kind

$$z_r(t) = \int_{\zeta_{r-1}}^{t} A(\tau) z_r(\tau) d\tau + \int_{\zeta_{r-1}}^{t} [A(\tau) + A_0(\tau)] d\tau \lambda_r + \int_{\zeta_{r-1}}^{t} f(\tau) d\tau, \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1,N}. \tag{4.8}$$

In (4.8), replacing $z_r(\tau)$ by the right-hand part of (4.8) at $t = \tau$ and repeating this procedure ν times ($\nu \in \mathbb{N}$), we obtain

$$z_r(t) = \big[E_{\nu,r}(A(\cdot), A(\cdot) + A_0(\cdot), t)\big]\lambda_r + G_{\nu,r}(t, z_r) + E_{\nu,r}(A(\cdot), f(\cdot), t), \tag{4.9}$$

where

$$\begin{aligned} E_{\nu,r}(A(\cdot), A(\cdot) + A_0(\cdot), t) = {} & \int_{\zeta_{r-1}}^{t} [A(\tau_1) + A_0(\tau_1)] d\tau_1 \\ & + \int_{\zeta_{r-1}}^{t} A(\tau_1) \int_{\zeta_{r-1}}^{\tau_1} [A(\tau_2) + A_0(\tau_2)] d\tau_2 d\tau_1 + \ldots \\ & + \int_{\zeta_{r-1}}^{t} A(\tau_1) \int_{\zeta_{r-1}}^{\tau_1} A(\tau_2) \ldots \int_{\zeta_{r-1}}^{\tau_{\nu-2}} A(\tau_{\nu-1}) \\ & \times \int_{\zeta_{r-1}}^{\tau_{\nu-1}} \big[A(\tau_\nu) + A_0(\tau_\nu)\big] d\tau_\nu d\tau_{\nu-1} \ldots d\tau_2 d\tau_1, \end{aligned}$$

$$G_{\nu,r}(t,z_r)=\int\limits_{\zeta_{r-1}}^{t}A(\tau_1)\int\limits_{\zeta_{r-1}}^{\tau_1}A(\tau_2)\ldots\int\limits_{\zeta_{r-1}}^{\tau_{\nu-2}}A(\tau_{\nu-1})\int\limits_{\zeta_{r-1}}^{\tau_{\nu-1}}A(\tau_\nu,x)\widetilde{v}_r(\tau_\nu,x)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_2 d\tau_1,$$

$$E_{\nu,r}(A(\cdot),f(\cdot),t)=\int\limits_{\zeta_{r-1}}^{t}f(\tau_1)d\tau_1+\int\limits_{\zeta_{r-1}}^{t}A(\tau_1)\int\limits_{\zeta_{r-1}}^{\tau_1}f(\tau_2)d\tau_2 d\tau_1+\ldots$$

$$+\int\limits_{\zeta_{r-1}}^{t}A(\tau_1)\int\limits_{\zeta_{r-1}}^{\tau_1}A(\tau_2)\ldots\int\limits_{\zeta_{r-1}}^{\tau_{\nu-2}}A(\tau_{\nu-1})\int\limits_{\zeta_{r-1}}^{\tau_{\nu-1}}f(\tau_\nu)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_2 d\tau_1,$$

$$t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,N}.$$

From (4.9), we find $z_1(0)$, $\lim_{t\to T-0}z_N(t)$, $\lim_{t\to\theta_p-0}z_p(t)$, $z_{p+1}(\theta_p)$, $p=\overline{1,N-1}$. Substituting them into (4.6), (4.7), we get the system of algebraic equations in unknown parameters λ_r, $r=\overline{1,N}$:

$$Q_\nu(\Delta_N)\lambda=-F_\nu(\Delta_N)-G_\nu(\Delta_N,z), \tag{4.10}$$

where

$$Q_\nu(\Delta_N)=\begin{vmatrix} BD_{\nu,1}(0) & 0 & 0 & \ldots & 0 & CD_{\nu,N}(T)\\ D_{\nu,1}(\theta_1) & -D_{\nu,2}(\theta_1) & 0 & \ldots & 0 & 0\\ 0 & D_{\nu,2}(\theta_2) & -D_{\nu,3}(\theta_2) & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & D_{\nu,N-1}(\theta_{N-1}) & -D_{\nu,N}(\theta_{N-1}) \end{vmatrix},$$

$D_{\nu,r}(\theta_{r-1})=I+E_{\nu,r}(A(\cdot),A(\cdot)+A_0(\cdot),\theta_{r-1})$, $D_{\nu,r}(\theta_r)=I+E_{\nu,r}(A(\cdot),A(\cdot)+A_0(\cdot),\theta_r)$, $r=\overline{1,N}$, I is the identity matrix of order n,

$$G_\nu(\Delta_N,z)=\Big(BG_{\nu,1}(0,z_1)+CG_{\nu,N}(T,z_N),G_{\nu,1}(\theta_1,z_1)$$
$$-G_{\nu,2}(\theta_1,z_2),\ldots,G_{\nu,N-1}(\theta_{N-1},z_{N-1})-G_{\nu,N}(\theta_{N-1},z_N)\Big)',$$

$$F_\nu(\Delta_N)=\Big(-d+BE_{\nu,1}(A(\cdot),f(\cdot),0)+CE_{\nu,N}(A(\cdot),f(\cdot),T),E_{\nu,1}(A(\cdot),f(\cdot),\theta_1)$$
$$-E_{\nu,2}(A(\cdot),f(\cdot),\theta_1),\ldots,E_{\nu,N-1}(A(\cdot),f(\cdot),\theta_{N-1})-E_{\nu,N}(A(\cdot),f(\cdot),\theta_{N-1})\Big)'.$$

The matrix $Q_\nu(\Delta_N)$ has a special block-banded structure, maps $\mathbb{R}^{nN}$ into $\mathbb{R}^{nN}$, and admits the following estimate:

$$\|Q_\nu(\Delta_N)\|\le\max(\|B\|+\|C\|,1)\left\{1+\theta[\alpha+\alpha_0]\sum_{i=1}^{\nu-1}\frac{(\alpha\theta)^i}{i!}\right\}.$$

Here,

$$\alpha = \max_{t\in[0,T]} \|A(t)\|, \qquad \alpha_0 = \max_{t\in[0,T]} \|A_0(t)\|,$$

$$\theta = \max\left\{ \max_{r=\overline{1,N}} (\theta_r - \zeta_{r-1}),\ \max_{r=\overline{1,N}} (\zeta_{r-1} - \theta_{r-1}) \right\}.$$

If the components $z_r(t)$ of $z[t] \in C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ are known, then the components λ_r of $\lambda \in \mathbb{R}^{nN}$ can be found from (4.10). Conversely, if $\lambda \in \mathbb{R}^{nN}$ is known, then $z[t] \in C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ can be found from (4.8). However, both $z[t]$ and λ are unknown, so we use an iterative method to find the solution to problem (4.4) – (4.7).

We determine the pair $(\lambda^*, z^*[t]) \in \mathbb{R}^{nN} \times C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ with components $(\lambda_r^*, z_r^*(t))$, $r = \overline{1,N}$, as the limit of the sequence $(\lambda^{(k)}, z^{(k)}[t]) \in \mathbb{R}^{nN} \times C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ with components $(\lambda_r^{(k)}, z_r^{(k)}(t))$, $r = \overline{1,N}$, $k = 0, 1, 2, \ldots$. To this end, we propose ***Algorithm B***.

***Step* 0.** Under assumption that, for a chosen $\nu \in \mathbb{N}$, the matrix $Q_\nu(\Delta_N) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is invertible, we determine the initial approximation to the parameter $\lambda^{(0)} = \big(\lambda_1^{(0)}, \ldots, \lambda_N^{(0)}\big)' \in \mathbb{R}^{nN}$ from the system of linear algebraic equations $Q_\nu(\Delta_N)\lambda = -F_\nu(\Delta_N)$. Then, by solving the Cauchy problems (4.4), (4.5) with $\lambda_r = \lambda_r^{(0)}$ on $[\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$, we get $z^{(0)}[t] = \big(z_1^{(0)}(t), \ldots, z_N^{(0)}(t)\big)' \in C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$.

***Step* 1.** From system (4.10), replacing $z[t]$ by $z^{(0)}[t]$, we determine $\lambda^{(1)} = (\lambda_1^{(1)}, \ldots, \lambda_N^{(1)})' \in \mathbb{R}^{nN}$. Solving the Cauchy problems (4.4), (4.5) with $\lambda_r = \lambda_r^{(1)}$ on $[\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$, we find $z^{(1)}[t] = \big(z_1^{(1)}(t), \ldots, z_N^{(1)}(t)\big)' \in C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$.

***Step* k.** From system (4.10), replacing $z[t]$ by $z^{(k-1)}[t]$, we determine $\lambda^{(k)} = \big(\lambda_1^{(k)}, \ldots, \lambda_N^{(k)}\big)' \in \mathbb{R}^{nN}$. Solving the Cauchy problems (4.4), (4.5) with $\lambda_r = \lambda_r^{(k)}$ on $[\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$, we find $z^{(k)}[t] = \big(z_1^{(k)}(t), \ldots, z_N^{(k)}(t)\big)' \in C\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$, $k = 1, 2, \ldots$.

Thus, according to the parametrization method, the procedure of finding the solution to problem consists of two parts:

(i) the values of the unknown parameters λ_r are found from the system of algebraic equations (4.10);

(ii) the unknown function $z_r(t)$ are found from the Cauchy problems (4.4), (4.5) for ordinary differential equations.

The following statement ensures the feasibility of the proposed algorithm and provides sufficient conditions for the unique solvability of problem (1.1), (1.2).

Theorem 5. *Suppose that, for some* $\nu \in \mathbb{N}$, *the matrix* $Q_\nu(\Delta_N) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ *is invertible and the following inequalities hold*:

(a) $\big\|[Q_\nu(\Delta_N)]^{-1}\big\| \le \chi_\nu(\Delta_N)$, *where* $\chi_\nu(\Delta_N)$ *is a positive constant*;

(b) $q_\nu(\Delta_N) = \chi_\nu(\Delta_N) \max(\|B\| + \|C\|, 2)\left[e^{\alpha\theta} - 1 - \sum_{i=1}^{\nu-1} \dfrac{[\alpha\theta]^i}{i!}\right] < 1.$

Then problem (1.1), (1.2) *has a unique solution* $x^*(t) \in C([0,T], \mathbb{R}^n)$ *and the following estimate holds*:

$$\max_{t\in[0,T]} \|x^*(t)\| \le [k_1(\theta,\nu) + k_2(\theta,\nu)] \max\Big(\max_{t\in[0,T]} \|f(t,x)\|, \|d\|\Big), \tag{4.11}$$

where

$$k_1(\Delta_N,\nu)=\frac{\chi_\nu(\Delta_N)}{1-q_\nu(\Delta_N)}\chi_\nu(\Delta_N)\max(\|B\|+\|C\|,2)\frac{[\alpha\theta]^\nu}{\nu!}k_0(\Delta_N,\nu)$$
$$+\,\theta\chi_\nu(\Delta_N)\max\left\{1+(\|B\|+\|C\|)\sum_{i=0}^{\nu-1}\frac{[\alpha\theta]^i}{i!},2\sum_{i=0}^{\nu-1}\frac{[\alpha\theta]^i}{i!}\right\},$$

$$k_2(\Delta_N,\nu)=\left\{e^{\alpha\theta}\theta(\alpha+\alpha_0)\frac{\chi_\nu(\Delta_N)}{1-q_\nu(\Delta_N)}\max(\|B\|+\|C\|,2)\frac{[\alpha\theta]^\nu}{\nu!}+1\right\}k_0(\Delta_N,\nu),$$

$$k_0(\Delta_N,\nu)=e^{\alpha\theta}\chi_\nu(\Delta_N)(\alpha+\alpha_0)\theta\max\left\{1+(\|B\|+\|C\|)\sum_{i=0}^{\nu-1}\frac{[\alpha\theta]^i}{i!},2\sum_{i=0}^{\nu-1}\frac{[\alpha\theta]^i}{i!}\right\}+\theta e^{\alpha\theta}.$$

Proof. It follows from the invertibility of the matrix $Q_\nu(\Delta_N)$ that there exists a unique vector $\lambda^{(0)}$ with components $\lambda_r^{(0)}\in\mathbb{R}^n$, $r=\overline{1,N}$, that admits the estimate

$$\|\lambda^{(0)}\|_2=\max_{r=\overline{1,N}}\|\lambda_r^{(0)}\|\le\chi_\nu(\Delta_N)\|F_\nu(\Delta_N)\|_2$$
$$\le\chi_\nu(\Delta_N)\max\left\{1+(\|B\|+\|C\|)\sum_{i=0}^{\nu-1}\frac{[\alpha\theta]^i}{i!},\ 2\sum_{i=0}^{\nu-1}\frac{[\alpha\theta]^i}{i!}\right\}\max\Big(\|d\|,\max_{t\in[0,T]}\|f(t)\|\Big)\theta. \tag{4.12}$$

Under assumptions of Theorem 5, the Cauchy problem (4.4), (4.5) with $\lambda_r=\lambda_r^{(0)}$ has a unique solution $z^{(0)}[t]$ with the components $z_r^{(0)}(t)$, $r=\overline{1,N}$. Applying the Gronwall – Bellman inequality to the integral equation (4.8), we obtain

$$\|z_r^{(0)}(t)\|\le e^{\alpha(t-\zeta_{r-1})}(t-\zeta_{r-1})(\alpha+\alpha_0)\|\lambda_r^{(0)}\|$$
$$+\,e^{\alpha(t-\zeta_{r-1})}\sup_{t\in[\theta_{r-1},\theta_r)}\|f(t)\|(t-\zeta_{r-1}),\quad t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,N}. \tag{4.13}$$

Since $\lambda^{(0)}\in\mathbb{R}^{nN}$ and $A(t)$, $A_0(t)$, and $f(t)$ are continuous on $[0,T]$, the system of functions $z^{(0)}[t]$ belongs $C\big([0,T],\Delta_N,\mathbb{R}^{nN}\big)$. Moreover, by (4.12) and (4.13), we have

$$\max_{r=\overline{1,N}}\sup_{t\in[\theta_{r-1},\theta_r)}\|z_r^{(0)}(t)\|\le k_0(\Delta_N,\nu)\max\Big(\|d\|,\max_{t\in[0,T]}\|f(t)\|\Big).$$

Now, following Step 1 of Algorithm B, we determine $\lambda^{(1)}\in\mathbb{R}^{nN}$. Using (4.11), we estimate the difference $\lambda^{(1)}-\lambda^{(0)}$:

$$\|\lambda^{(1)}-\lambda^{(0)}\|_2\le\chi_\nu(\Delta_N)\|G_\nu(\Delta_N,z^{(0)})\|_2$$
$$\le\chi_\nu(\Delta_N)\max(\|B\|+\|C\|,2)\frac{[\alpha\theta]^\nu}{\nu!}k_0(\Delta_N,\nu)\max\Big(\|d\|,\max_{t\in[0,T]}\|f(t)\|\Big). \tag{4.14}$$

Continuing the iterative process, we determine the sequences $\{\lambda_r^{(k)}\}$ and $\{z_r^{(k)}(t)\}$, $r=\overline{1,N}$, $k=1,2,\ldots$. Using again the Gronwall – Bellman inequality, we estimate the differences between the solutions to Cauchy problems through the differences between the corresponding parameters:

$$\big\|z_r^{(k)}(t)-z_r^{(k-1)}(t)\big\|\le e^{\alpha(t-\zeta_{r-1})}(t-\zeta_{r-1})(\alpha+\alpha_0)\big\|\lambda_r^{(k)}-\lambda_r^{(k-1)}\big\|. \tag{4.15}$$

In view of (4.10), we have

$$\begin{aligned}\big\|\lambda^{(k+1)}-\lambda^{(k)}\big\|_2 &\le \chi_\nu(\Delta_N)\max(\|B\|+\|C\|,2)\\ &\times\max\Bigg\{\max_{r=\overline{1,N}}\int\limits_{\zeta_{r-1}}^{\theta_r}\alpha\ldots\int\limits_{\zeta_{r-1}}^{\tau_{\nu-2}}\alpha\int\limits_{\zeta_{r-1}}^{\tau_{\nu-1}}\alpha\|z_r^{(k)}(\tau_\nu)-z_r^{(k-1)}(\tau_\nu)\|d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1,\\ &\max_{r=\overline{1,N}}\int\limits_{\zeta_{r-1}}^{\theta_{r-1}}\alpha\ldots\int\limits_{\zeta_{r-1}}^{\tau_{\nu-2}}\alpha\int\limits_{\zeta_{r-1}}^{\tau_{\nu-1}}\alpha\|z_r^{(k)}(\tau_\nu)-z_r^{(k-1)}(\tau_\nu)\|d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1\Bigg\}.\end{aligned}$$

From this inequality, evaluating the iterated integrals and using (4.15), we obtain

$$\big\|\lambda^{(k+1)}-\lambda^{(k)}\big\|_2\le q_\nu(\Delta_N)\big\|\lambda^{(k)}-\lambda^{(k-1)}\big\|_2,\quad k=1,2,\ldots. \tag{4.16}$$

It follows from condition (b) of Theorem 5 and the inequality

$$\max_{r=\overline{1,N}}\sup_{t\in[\theta_{r-1},\theta_r)}\big\|z_r^{(k)}(t)-z_r^{(k-1)}(t)\big\|\le e^{\alpha\theta}\theta(\alpha+\alpha_0)\max_{r=\overline{1,N}}\big\|\lambda_r^{(k)}-\lambda_r^{(k-1)}\big\|, \tag{4.17}$$

that the sequence of pairs $(\lambda^{(k)},z^{(k)}[t])\in\mathbb{R}^{nN})\times C\big([0,T],\Delta_N,\mathbb{R}^{nN}\big)$ converges to $(\lambda^*,\ z^*[t])$ as $k\to\infty$, and the limit pair is the solution to problem (4.4)–(4.7).

Taking into account (4.12)–(4.14), (4.16), and (4.17), we obtain the estimates

$$\|\lambda^*\|_2\le\|\lambda^*-\lambda^{(0)}\|_2+\|\lambda^{(0)}\|_2\le k_1(\Delta_N,\nu)\max\Big(\|d\|,\max_{t\in[0,T]}\|f(t)\|\Big)$$

and

$$\begin{aligned}\max_{r=\overline{1,N}}\sup_{t\in[\theta_{r-1},\theta_r))}\|z^*(t)\|&\le\max_{r=\overline{1,N}}\sup_{t\in[\theta_{r-1},\theta_r)}\big\|z_r^*(t)-z_r^{(0)}(t)\big\|+\max_{r=\overline{1,N}}\sup_{t\in[\theta_{r-1},\theta_r)}\|z_r^{(0)}(t)\|\\ &\le k_2(\Delta_N,\nu)\max\Big(\|d\|,\max_{t\in[0,T]}\|f(t)\|\Big).\end{aligned}$$

Since the pair $(\lambda^*,z^*[t])$ is a solution to problem (4.4)–(4.7), the function $x^*(t)$, defined by the equalities $x^*(t)=\lambda_r^*+z_r^*(t)$, $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N}$, and $x^*(T)=\lambda_N^*+\lim_{t\to T-0}z_N^*(t)$, is a solution to the original problem (1.1), (1.2). The uniqueness of the solution can be proved by contradiction.

Hence, problem (1.1), (1.2) is uniquely solvable, and the solution $x^*(t)$ admits the estimate (4.11).

Theorem 5 is proved.

The main condition for the unique solvability of problem (1.1), (1.2) is that the matrix $Q_\nu(\Delta_N)$ is invertible for some $\nu\in\mathbb{N}$. As mentioned above, this matrix has a special block-banded structure. This fact implies the following lemmas.

Lemma 3. *Suppose that for some* $\nu\in\mathbb{N}$ *the* $(n\times n)$ *matrices* $D_{\nu,i}(\theta_i)$, $i=\overline{1,N-1}$, *are invertible.*

Then the $(nN\times nN)$ *matrix* $Q_\nu(\Delta_N)$ *is invertible if and only if so is the* $(n\times n)$ *matrix*

$$M_\nu(\Delta_N)=B\prod_{s=1}^{N-1}D_{\nu,s}(\theta_{s-1})[D_{\nu,s}(\theta_s)]^{-1}D_{\nu,N}(\theta_{N-1})+CD_{\nu,N}(T).$$

Lemma 4. *If the matrices* $D_{\nu,i}(\theta_i)$, $i=\overline{1,N-1}$, *and* $M_\nu(\Delta_N)$ *are invertible, then*

$$[Q_\nu(\Delta_N)]^{-1}=\{g_{r,j}\},\quad r,j=\overline{1,N},$$

where

$$g_{N,1}=M_\nu^{-1}(\Delta_N),$$
$$g_{N,k}=-M_\nu^{-1}(\Delta_N)B\prod_{s=1}^{k-1}D_{\nu,s}(\theta_{s-1})[D_{\nu,s}(\theta_s)]^{-1},\quad 1\le k<N,$$
$$g_{r-1,r}=[D_{\nu,r-1}(\theta_{r-1})]^{-1}\big[I+D_{\nu,r}(\theta_{r-1})\big]g_{r,r},\quad r=2,3,\ldots,N,$$
$$g_{r-1,j}=[D_{\nu,r-1}(\theta_{r-1})]^{-1}[D_{\nu,r}(\theta_{r-1})]^{-1}g_{r,j},\quad j\neq r,\quad j=\overline{1,N}.$$

Lemma 5. *Suppose that, for some* $\nu\in\mathbb{N}$, *the* $(n\times n)$ *matrices* $D_{\nu,i}(\theta_{i-1})$, $i=\overline{2,N}$, *are invertible.*

Then the $(nN\times nN)$ *matrix* $Q_\nu(\Delta_N)$ *is invertible if and only if so is the* $(n\times n)$ *matrix*

$$L_\nu(\Delta_N)=BD_{\nu,1}(0)+CD_{\nu,N}(T)\prod_{s=N-1}^{1}[D_{\nu,s+1}(\theta_s)]^{-1}D_{\nu,s}(\theta_s).$$

Lemma 6. *If the matrices* $D_{\nu,i}(\theta_{i-1})$, $i=\overline{2,N}$, *and* $L_\nu(\Delta_N)$ *are invertible, then*

$$[Q_\nu(\Delta_N)]^{-1}=\{g_{r,j}\},\quad r,j=\overline{1,N},$$

where

$$g_{1,1}=L_\nu^{-1}(\Delta_N),$$
$$g_{1,k}=L_\nu^{-1}(\Delta_N)C\prod_{s=N}^{k}D_{\nu,s}(\theta_s)[D_{\nu,s}(\theta_{s-1})]^{-1},\quad 1<k\le N,$$
$$g_{r,r}=[D_{\nu,r}(\theta_{r-1})]^{-1}\big[D_{\nu,r-1}(\theta_{r-1})g_{r-1,r}-I\big],\quad r=2,3,\ldots,N,$$
$$g_{r,j}=[D_{\nu,r}(\theta_{r-1})]^{-1}D_{\nu,r-1}(\theta_{r-1})g_{r-1,j},\quad j\neq r,\quad j=\overline{1,N}.$$

Definition 3. *The boundary-value problem* (1.1), (1.2) *is called well-posed if it has a unique solution* $x(t)\in C([0,T],\mathbb{R}^n)$ *for any* $f(t)\in C([0,T],\mathbb{R}^n)$ *and* $d\in\mathbb{R}^n$, *and the following inequality holds*:

$$\max_{t\in[0,T]}\|x(t)\|\le K\max\Big(\max_{t\in[0,T]}\|f(t)\|,\|d\|\Big),$$

where K *is a constant independent of* $f(t)$ *and* d.

Note that in (4.11) the quantity $k(\Delta_N,\nu)=k_1(\Delta_N,\nu)+k_2(\Delta_N,\nu)$ does not depend on $f(t)$ and d. Hence, by Theorem 5, problem (1.1), (1.2) is well-posed.

Theorem 6. *If problem* (1.1), (1.2) *is well-posed, then, for any* $\nu\in\mathbb{N}$, *there exists a partition* Δ_N *such that the matrix* $Q_\nu(\Delta_N)\colon\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ *is invertible and the inequalities* (*a*) *and* (*b*) *of Theorem* 5 *hold.*

Conclusion. In this paper, we propose a constructive method for solving the boundary-value problem for the system of DEPCAG (1.1), (1.2). Conditions for the unique solvability and well-posedness of problem (1.1), (1.2) are established in terms of solvability of the system of algebraic equations (1.3) and in terms of input data. We develop the algorithms for finding approximate solutions to problem (1.1), (1.2) and prove their convergence to the exact solution. Further, based on the approach proposed in the paper, numerical methods for solving problem (1.1), (1.2) will be developed.

Our further research is focused on the extending the proposed method to families of boundary-value problems for systems of DEPCAG with parameter, boundary-value problems for impulsive DEPCAG [14 – 21], nonlocal problems for hyperbolic DEPCAG [6]. The method will be used for investigation mathematical models of neural networks and dynamical systems that are described by DEPCAG [7 – 9, 16 – 17].

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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