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STABLE DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF THE SOURCE IDENTIFICATION PROBLEM FOR HYPERBOLIC EQUATIONS

СТІЙКА РІЗНИЦЕВА СХЕМА ДЛЯ ЧИСЛОВОГО РОЗВ'ЯЗУВАННЯ ЗАДАЧІ ІДЕНТИФІКАЦІЇ ДЖЕРЕЛА ДЛЯ ГІПЕРБОЛІЧНИХ РІВНЯНЬ

We present a stable difference scheme of the second order of accuracy for a one-dimensional hyperbolic equation. The well-posedness of the difference scheme is established. Numerical results are presented.

Наведено стійку різницеву схему другого порядку точності для одновимірного гіперболічного рівняння. Встановлено, що різницева схема є коректно поставленою. Наведено числові результати.

1. Introduction. Identification problems take an important place in applied sciences and engineering applications and have been studied by many authors (see, e.g., [1 – 8] and the references therein). The theory and applications of source identification problems for partial differential equations have been given in various papers (see, e.g., [9 – 25, 28, 29] and the references therein).

In the paper [28], a source identification problem for a class of abstract nonlocal differential equations in separable Hilbert spaces is investigated. The existence of mild solutions and strong solutions for the problem of identifying parameter are obtained. The continuous dependence on the data and the regularity of the mild solutions and strong solutions of nonlocal differential equations is studied. Examples given in anomalous diffusion equations illustrate the existence and regularity results.

In the paper [29], two inverse problems with final overdetermination for diffusion and wave equations containing the Caputo fractional time derivative and a fractional Laplacian of distributed order are considered. Uniqueness of solutions of these problems is proved. Sufficient conditions for the uniqueness are stricter for the problem to recover simultaneously the source term, the order of the time derivative and the fractional Laplacian than for the problem to reconstruct a time-dependent source term.

In the paper [26], the source identification problem

$$\begin{aligned}
&\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) = p(t)q(x) + f(t,x), \quad x \in (0,l), \quad t \in (0,T), \\
&u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x), \quad x \in [0,l], \\
&u(t,0) = u(t,l) = 0, \qquad \int_0^l u(t,x)dx = \zeta(t), \quad t \in [0,T],
\end{aligned} \tag{1}$$

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for a one dimensional hyperbolic equation with local boundary conditions was studied, where $u(t,x)$ and $p(t)$ are unknown functions, $a(x) \geq a > 0,\ f(t,x),\ \zeta(t),\ \varphi(x)$ and $\psi(x)$ are sufficiently smooth functions and $q(x)$ is a sufficiently smooth function assuming $q(0) = q(l) = 0$ and $\int_0^l q(x)dx \neq 0.$ Well-posedness of the source identification problem was established. The first order of accuracy difference scheme for the numerical solution of source identification problem was presented and studied. Numerical results were given.

Of great interest is the study of absolute stable difference schemes of a high order of accuracy for hyperbolic partial differential equations, in which stability was establised without any assumptions in respect of the grid steps τ and h. At the same time we note that the absolute stable high order of accuracy difference schemes for the numerical solution of identification hyperbolic problem (1) are not studied before.

In the present paper, absolute stable second order of accuracy difference scheme for the numerical solution of identification hyperbolic problem (1) is presented:

$$\begin{aligned}
&\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{1}{2h}\left(a(x_{n+1})\frac{u_{n+1}^k - u_n^k}{h} - a(x_n)\frac{u_n^k - u_{n-1}^k}{h}\right) \\
&\quad - \frac{1}{4h}\left(a(x_{n+1})\frac{u_{n+1}^{k+1} - u_n^{k+1} + u_{n+1}^{k-1} - u_n^{k-1}}{h}\right. \\
&\quad \left. - a(x_n)\frac{u_n^{k+1} - u_{n-1}^{k+1} + u_n^{k-1} - u_{n-1}^{k-1}}{h}\right) = p_k q_n + f(t_k, x_n), \\
&N\tau = T, \quad 1 \leq n \leq M-1, \quad Mh = l, \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \\
&u_n^0 = \varphi(x_n), \quad 0 \leq n \leq M, \\
&\frac{u_n^1 - u_n^0}{\tau} - \frac{\tau}{h}\left(a(x_{n+1})\frac{u_{n+1}^1 - u_{n+1}^0 - u_n^1 + u_n^0}{h}\right. \\
&\quad \left. - a(x_n)\frac{u_n^1 - u_n^0 - u_{n-1}^1 + u_{n-1}^0}{h}\right) = \psi(x_n) + \frac{\tau}{2}f(0, x_n) \\
&\quad + \frac{\tau}{2}\left[\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^0 - u_n^0}{h} - a(x_n)\frac{u_n^0 - u_{n-1}^0}{h}\right) + p_0 q_n\right], \quad 1 \leq n \leq M-1, \\
&u_0^{k+1} = u_M^{k+1} = 0, \qquad \sum_{i=1}^{M-1} u_i^{k+1}h = \zeta(t_{k+1}), \quad -1 \leq k \leq N-1.
\end{aligned} \tag{2}$$

Here, it is assumed that $q_M = q_0 = 0$, and $\sum_{i=1}^{M-1} q_i \neq 0$.

The well-posedness for the numerical solution of this difference scheme (2) is established. The theoretical statements for solution of this difference scheme are supported by the result of the numerical experiments.

2. The well-posedness of the difference scheme. To formulate our results on difference problem, we introduce the Banach space $C_\tau(H) = C([0,T]_\tau, H)$ of all abstract grid functions $\phi^\tau = \{\phi(t_k)\}_{k=0}^N$ defined on

$$[0,T]_\tau = \{t_k = k\tau,\ 0 \leq k \leq N,\ N\tau = T\}$$

with values in H equipped with the norm

$$\|\phi^\tau\|_{C_\tau(H)} = \max_{0\le k\le N}\|\phi(t_k)\|_H.$$

Moreover, $L_{2h} = L_2[0,l]_h$ is the Hilbert space of all grid functions $\gamma^h(x) = \{\gamma_n\}_{n=0}^M$ defined on

$$[0,l]_h = \big\{x_n = nh,\ \ 0\le n\le M,\ \ Mh = l\big\},$$

equipped with the norm

$$\left\|\gamma^h\right\|_{L_{2h}} = \left\{\sum_{i=0}^{M}|\gamma_i|^2 h\right\}^{\frac12},$$

and $W_{2h}^2 = W_2^2[0,l]_h$ is the discrete analogy of Sobolev space with norm

$$\left\|\gamma^h\right\|_{W_{2h}^2} = \left\{\sum_{i=0}^{M}|\gamma_i|^2 h + \sum_{i=1}^{M-1}\left|\frac{\gamma_{i+1}-2\gamma_i+\gamma_{i-1}}{h^2}\right|^2 h\right\}^{\frac12}.$$

We introduce the self-adjoint positive definite difference operator A_h defined by the formula

$$A_h\varphi^h(x) = \left\{-\frac{1}{h}\left(a(x_{n+1})\frac{\varphi_{n+1}-\varphi_n}{h} - a(x_n)\frac{\varphi_n-\varphi_{n-1}}{h}\right)\right\}_{n=1}^{M-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_{n=0}^M$ defined on $[0,l]_h$ satisfying the conditions $\varphi_M = \varphi_0 = 0$.

For the numerical solution $\left\{\{u_n^k\}_{k=0}^N\right\}_{n=0}^M$ of problem (1), we consider the second order of accuracy difference scheme.

We have the following theorem on the stability of difference scheme (2).

Theorem 1. *For the solution of difference scheme* (2), *the following stability estimates hold*:

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\left\{\frac{u_{k+1}^h+2u_k^h+u_{k-1}^h}{4}\right\}_{k=1}^{N-1}\right\|_{C_\tau\left(W_{2h}^2\right)}$$
$$\le M_{14}(q)\Bigg[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^1} + \|f_0^h\|_{L_{2h}}$$
$$+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\Bigg], \tag{3}$$

$$\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \le M_{15}(q)\Bigg[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|f_0^h\|_{L_{2h}}$$
$$+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}$$
$$+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\Bigg]. \tag{4}$$

Here and throughout this subsection $f_k^h(x) = \{f(t_k,x_n)\}_{n=0}^M$, $1\le k\le N-1$.

Proof. We will use the substitution

$$u_n^k = w_n^k + \eta_k q_n, \tag{5}$$

where

$$\eta_k = \sum_{i=1}^{k} \left\{ \frac{(k-i)p_i + (k-(i-1))p_{i-1}}{2} \right\} \tau^2, \quad 1 \le k \le N, \quad \eta_0 = 0. \tag{6}$$

It is easy to see that $\left\{ \{w_n^k\}_{k=0}^N \right\}_{n=0}^M$ is the solution of the difference problem

$$\begin{gathered}
\frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} - \frac{1}{2h}\left(a(x_{n+1}) \frac{w_{n+1}^k - w_n^k}{h} - a(x_n) \frac{w_n^k - w_{n-1}^k}{h} \right) \\
- \frac{1}{4h}\left(a(x_{n+1}) \frac{w_{n+1}^{k+1} - w_n^{k+1}}{h} - a(x_n) \frac{w_n^{k+1} - w_{n-1}^{k+1}}{h} \right) \\
- \frac{1}{4h}\left(a(x_{n+1}) \frac{w_{n+1}^{k-1} - w_n^{k-1}}{h} - a(x_n) \frac{w_n^{k-1} - w_{n-1}^{k-1}}{h} \right) \\
= f(t_k, x_n) + \frac{1}{h}\left[a(x_{n+1}) \frac{q_{n+1} - q_n}{h} - a(x_n) \frac{q_n - q_{n-1}}{h} \right] \frac{1}{4}(\eta_{k+1} + 2\eta_k + \eta_{k-1}), \\
1 \le k \le N-1, \quad 1 \le n \le M-1, \\
w_n^0 = \varphi(x_n), \quad 0 \le n \le M, \\
\frac{w_n^1 - w_n^0}{\tau} - \frac{\tau}{h}\left(a(x_{n+1}) \frac{w_{n+1}^1 - w_{n+1}^0 - w_n^1 + w_n^0}{h} \right. \\
\left. - a(x_n) \frac{w_n^1 - w_n^0 - w_{n-1}^1 + w_{n-1}^0}{h} \right) \\
- \frac{\tau}{h}\left(a(x_{n+1}) \frac{q_{n+1} - q_n}{h} - a(x_n) \frac{q_n - q_{n-1}}{h} \right) \eta_1 \\
= \psi(x_n) + \frac{\tau}{2} f(0, x_n) + \frac{\tau}{2}\left[\frac{1}{h}\left(a(x_{n+1}) \frac{w_{n+1}^0 - w_n^0}{h} - a(x_n) \frac{w_n^0 - w_{n-1}^0}{h} \right) \right], \\
1 \le n \le M-1, \\
w_0^{k+1} = w_M^{k+1} = 0, \quad -1 \le k \le N-1.
\end{gathered} \tag{7}$$

Now, we will take an estimate for $|p_k|$.Using the overdetermined condition $\sum_{i=1}^{M-1} u_i^{k+1} h = \zeta(t_{k+1})$ and substitution (5), one can obtain

$$\eta_k = \frac{\zeta_k - \sum_{i=1}^{M-1} w_i^k h}{\sum_{i=1}^{M-1} q_i h}. \tag{8}$$

Then, using the formulas $p_k = \dfrac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}$ and (8), we get

$$p_k = \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1} - \sum_{i=1}^{M-1}\left(w_i^{k+1} - 2w_i^k + w_i^{k-1}\right)h}{\tau^2 \sum_{i=1}^{M-1} q_i h}.$$

Then, applying the discrete analogue of the Cauchy – Schwarz inequality and the triangle inequality, we have

$$|p_k| \le \frac{1}{\left|\sum_{i=1}^{M-1} q_i h\right|}\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \sum_{i=1}^{M-1}\left|\frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2}\right| h\right]$$

$$\le \frac{1}{\left|\sum_{i=1}^{M-1} q_i h\right|}\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \sqrt{l}\left\|\left\{\frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2}\right\}_{i=0}^{M}\right\|_{L_{2h}}\right]$$

$$\le M_{16}(q)\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \left\|\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}\right] \tag{9}$$

for all $1 \le k \le N-1$. From that it follows

$$\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \le M_{16}(q)\left[\left\|\left\{\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} + \left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}\right]. \tag{10}$$

Now, using substitution (5), we get

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} = \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + p_k q(x_n).$$

Applying the triangle inequality, we obtain

$$\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \le \left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\left\|\{q(x_n)\}_{n=0}^{M}\right\|_{L_{2h}}. \tag{11}$$

Therefore, the proof of estimates (3) and (4) is based on equation (2), the triangle inequality, estimates (10), (11) and on the following stability estimate for the solution of difference problem (7):

$$\left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}$$

$$\leq M_{17}(q)\Bigg[\|\varphi^h\|_{W_{2h}^2}+\|\psi^h\|_{W_{2h}^1}+\|f_0^h\|_{L_{2h}}$$

$$+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\Bigg]. \tag{12}$$

Theorem 1 is proved.

Now, we will prove estimate (12) for the solution of difference problem (7). Firstly, it is easy to see that difference scheme (7) can be written as the abstract Cauchy difference problem

$$\begin{aligned}
&\frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}+\frac{1}{4}A(w_{k+1}+2w_k+w_{k-1})=\theta_k,\\
&\theta_k=f_k-\frac{1}{4}(\eta_{k+1}+2\eta_k+\eta_{k-1})Aq,\\
&\theta_k=\theta(t_k),\quad t_k=k\tau,\quad 1\leq k\leq N-1,\quad N\tau=1,\\
&(I+\tau^2A)\tau^{-1}(w_1-w_0)=\frac{\tau}{2}(\theta_0-Aw_0)+\psi,\\
&\theta_0=f(0)+p_0q,\quad w_0=\varphi,
\end{aligned} \tag{13}$$

in a Hilbert space $H=L_{2h}$ with positive-definite self-adjoint operator A. Here,

$$\{w_k\}_{k=0}^N=\{w_k^h\}_{k=0}^N,\qquad \{f_k\}_{k=1}^{N-1}=\{f_k^h\}_{k=1}^{N-1}$$

are unknown and known abstract mesh functions defined on $[0,T]_\tau$ with values in $H=L_{2h}$, respectively, and $\varphi=\varphi^h$, $\psi=\psi^h$, $q=q^h$ are given elements.

Secondly, applying the approaches of [27] we will prove a lemma that will be needed in the sequel.

Lemma 1. *Assume that* $\varphi\in D(A)$, $\psi,q\in D(A^{\frac{1}{2}})$. *Then, for the solution of difference problem* (13), *the following stability estimate holds for any* $1\leq k\leq N-1$:

$$\left\|\frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}\right\|_H\leq M(q)\Bigg[\|A\varphi\|_H+\|A^{\frac{1}{2}}\psi\|_H+\|f_0\|_H+|\eta_1|$$

$$+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C(H)}+\sum_{s=1}^k\left|\frac{\eta_{s+1}-2\eta_s+\eta_{s-1}}{\tau^2}\right|\tau\Bigg] \tag{14}$$

for any $1\leq k\leq N-1$.

Proof. It is clear that there exists a unique solution of this initial value problem, and for the solution of (13), the following formula is satisfied (see [27]):

$$w_0=\varphi,\qquad w_1=(I+\tau^2A)^{-1}\left[\left(I+\frac{\tau^2}{2}A\right)\varphi+\tau\psi+\frac{\tau^2}{2}\theta_0\right],$$

$$w_k=\left[R^k+\frac{1}{2i}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)[R^k-\widetilde{R}^k]\right.$$

$$\times\left(\left(\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)\frac{\tau}{2}A-iA^{\frac{1}{2}}(I+\tau^2A)\right)\left(I+\tau^2A\right)^{-1}\right]\varphi$$

$$+\frac{i}{2}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left[R^k-\widetilde{R}^k\right](I+\tau^2A)^{-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)\psi$$

$$+\frac{i}{2}A^{-\frac{1}{2}}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left[R^k-\widetilde{R}^k\right](I+\tau^2A)^{-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)\frac{\tau}{2}\theta_0$$

$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-\frac{1}{2}}\left[R^{k-s}-\widetilde{R}^{k-s}\right]\theta_s,\quad 2\le k\le N.$$

Applying Abel's formula, we can write

$$ww_k=\left\{R^k-\frac{1}{2}iA^{-\frac{1}{2}}B\left[R^k-\widetilde{R}^k\right]\right.$$

$$\left.\times\left(\widetilde{B}\frac{\tau}{2}A-iA^{\frac{1}{2}}(I+\tau^2A)\right)(I+\tau^2A)^{-1}\right\}\varphi$$

$$+\left\{\frac{i}{2}A^{-\frac{1}{2}}B\widetilde{B}\left[R^k-\widetilde{R}^k\right](I+\tau^2A)^{-1}\right\}\psi$$

$$+\left\{\frac{i}{2}A^{-\frac{1}{2}}B\widetilde{B}\left[R^k-\widetilde{R}^k\right](I+\tau^2A)^{-1}\right\}\frac{\tau}{2}\theta_0$$

$$+\frac{A^{-1}}{2}\left\{\sum_{s=2}^{k-1}\left[BR^{k-s}-\widetilde{B}\widetilde{R}^{k-s}\right](\theta_{s-1}-\theta_s)\right\}$$

$$+\frac{A^{-1}}{2}\left\{\left[B+\widetilde{B}\right]\theta_{k-1}-\left[BR^{k-1}-\widetilde{B}\widetilde{R}^{k-1}\right]\theta_1\right\},\quad 2\le k\le N, \tag{15}$$

where

$$R=B\widetilde{B}^{-1}=\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1},$$

$$\widetilde{R}=\widetilde{B}B^{-1}=\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}.$$

Using the spectral property of the self-adjoint positive-definite operator, we get

$$\|R\|_{H\to H}\le 1,\qquad \|\widetilde{R}\|_{H\to H}\le 1,\qquad \left\|\left(I\pm\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right\|_{H\to H}\le 1, \tag{16}$$

$$\left\|\left(I\pm i\tau A^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\le 1,\qquad \left\|\tau A^{\frac{1}{2}}\left(I\pm i\tau A^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\le 1. \tag{17}$$

Now, we will establish estimates for $\left\|\dfrac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}\right\|_H$, $1\le k\le N-1$. Applying equation (13), formula (15) and identities

$$I+R=2\widetilde{B}^{-1},\qquad I+\widetilde{R}=2B^{-1},$$

we have

$$\begin{aligned}
&A\frac{w_{k+1}+w_k+w_{k-1}}{4}\\
&\quad=\Big\{B^{-1}R^k\widetilde{B}^{-1}-\frac{1}{4}i\tau A^{\frac12}\Big[BR^{k-1}\widetilde{B}^{-1}-B^{-1}\widetilde{R}^{k-1}\widetilde{B}\Big]\\
&\qquad\times(I+\tau^2A)^{-1}-\frac12\Big[\widetilde{B}^{-1}R^k-B^{-1}\widetilde{R}^{k-1}\Big]\Big\}A\varphi\\
&\qquad+\frac12\Big\{\Big[BR^{k-1}\widetilde{B}^{-1}-B^{-1}\widetilde{R}^{k-1}\widetilde{B}\Big](I+\tau^2A)^{-1}\Big\}iA^{\frac12}\psi\\
&\qquad+\frac14\Big\{i\tau A^{\frac12}\Big[BR^{k-1}\widetilde{B}^{-1}-B^{-1}\widetilde{R}^{k-1}\widetilde{B}\Big]\big(I+\tau^2A\big)^{-1}\Big\}\theta_0\\
&\qquad+\frac12\sum_{s=2}^{k-1}\Big[\widetilde{B}^{-1}R^{k-s}+B^{-1}\widetilde{R}^{k-s}\Big](\theta_{s-1}-\theta_s)+\frac14\big[\tau^2AB^{-1}\widetilde{B}^{-1}\big]\theta_k\\
&\qquad+\frac12\big[B^{-1}+\widetilde{B}^{-1}\big]\theta_{k-1}-\frac12\Big[\widetilde{B}^{-1}R^{k-1}+B^{-1}\widetilde{R}^{k-1}\Big]\theta_1\\
&\quad=\sum_{i=1}^{4}J_i(k),
\end{aligned}$$

where

$$\begin{aligned}
J_1(k)&=\Big\{B^{-1}R^k\widetilde{B}^{-1}-\frac14 i\tau A^{\frac12}\Big[BR^{k-1}\widetilde{B}^{-1}-B^{-1}\widetilde{R}^{k-1}\widetilde{B}\Big](I+\tau^2A)^{-1}\\
&\qquad-\frac12\Big[\widetilde{B}^{-1}R^k-B^{-1}\widetilde{R}^{k-1}\Big]\Big\}A\varphi,\\
J_2(k)&=\frac12\Big\{\Big[BR^{k-1}\widetilde{B}^{-1}-B^{-1}\widetilde{R}^{k-1}\widetilde{B}\Big]\big(I+\tau^2A\big)^{-1}\Big\}iA^{\frac12}\psi,\\
J_3(k)&=\frac14\Big\{i\tau A^{\frac12}\Big[BR^{k-1}\widetilde{B}^{-1}-B^{-1}\widetilde{R}^{k-1}\widetilde{B}\Big](I+\tau^2A)^{-1}\Big\}\theta_0\\
&\qquad+\frac14\big[\tau^2AB^{-1}\widetilde{B}^{-1}\big]\theta_k+\frac12\big[B^{-1}+\widetilde{B}^{-1}\big]\theta_{k-1}\\
&\qquad-\frac12\big[\widetilde{B}^{-1}R^{k-1}+B^{-1}\widetilde{R}^{k-1}\big]\theta_1,\\
J_4(k)&=\frac12\sum_{s=2}^{k-1}\big[\widetilde{B}^{-1}R^{k-s}+B^{-1}\widetilde{R}^{k-s}\big](\theta_{s-1}-\theta_s).
\end{aligned}$$

Now, applying the triangle inequality, we obtain

$$\left\| A\frac{w_{k+1}+w_k+w_{k-1}}{4}\right\|_H \le \sum_{i=1}^{4}\|J_i(k)\|_H \tag{18}$$

for any $1\le k\le N-1$. Therefore, we will estimate $\|J_i(k)\|_H,\ i=1,2,3,4,$ separately. Firstly, using estimates (16), (17), we get

$$\begin{aligned}\|J_1(k)\|_H &\le \|B^{-1}R^k\widetilde{B}^{-1}\|_{H\to H}+\frac{1}{4}\Big[\Big\|i\tau A^{\frac12}BR^{k-1}\widetilde{B}^{-1}\big(I+\tau^2A\big)^{-1}\Big\|_{H\to H}\\&\quad+\Big\|i\tau A^{\frac12}B^{-1}\widetilde{R}^{k-1}\widetilde{B}(I+\tau^2A)^{-1}\Big\|_{H\to H}\Big]\\&\quad+\frac12\Big[\|\widetilde{B}^{-1}R^k\|_{H\to H}+\|B^{-1}\widetilde{R}^{k-1}\|_{H\to H}\Big]\|A\varphi\|_H\\&\le M\|A\varphi\|_H,\\\|J_2(k)\|_H &\le \frac12\Big[\Big\|BR^{k-1}\widetilde{B}^{-1}(I+\tau^2A)^{-1}\Big\|_{H\to H}\\&\quad+\Big\|B^{-1}\widetilde{R}^{k-1}\widetilde{B}\big(I+\tau^2A\big)^{-1}\Big\|_{H\to H}\|A^{\frac12}\psi\|_H\Big]\\&\le M\|A^{\frac12}\psi\|_H\end{aligned}$$

for any $1\le k\le N-1$. Secondly, using the triangle inequality and estimates (16), (17), we have

$$\begin{aligned}\|J_3(k)\|_H &\le \frac14\Big[\Big\|BR^{k-1}\widetilde{B}^{-1}\tau A^{\frac12}(I+\tau^2A)^{-1}\Big\|_{H\to H}\\&\quad+\Big\|B^{-1}\widetilde{R}^{k-1}\widetilde{B}\tau A^{\frac12}(I+\tau^2A)^{-1}\Big\|_{H\to H}\Big]\|\theta_0\|_H\\&\quad+\frac14\Big[\|\tau^2AB^{-1}\widetilde{B}^{-1}\|_{H\to H}\|\theta_k\|_H\Big]\\&\quad+\frac12\Big[\|B^{-1}\|_{H\to H}+\|\widetilde{B}^{-1}\|_{H\to H}\Big]\|\theta_{k-1}\|_H\\&\quad+\frac12\Big[\|\widetilde{B}^{-1}R^{k-1}\|_{H\to H}+\|B^{-1}\widetilde{R}^{k-1}\|_{H\to H}\Big]\|\theta_1\|_H\\&\le \frac{11}{4}\left[\|\theta_0\|_H+\sum_{i=1}^{k}\|\theta_i-\theta_{i-1}\|_H\right],\end{aligned}$$

$$\begin{aligned}\|J_4(k)\|_H &\le \sum_{s=2}^{k-1}\frac12\Big[\|\widetilde{B}^{-1}R^{k-s}\|_{H\to H}+\|B^{-1}\widetilde{R}^{k-s}\|_{H\to H}\Big]\|\theta_{s-1}-\theta_s\|_H\\&\le \sum_{s=2}^{k-1}\|\theta_{s-1}-\theta_s\|_H\end{aligned}$$

for any $1 \leq k \leq N-1$. Using the triangle inequality and estimate (18), we obtain

$$A\left\|\frac{w_{k+1}+2w_k+w_{k-1}}{4}\right\|_H \leq M\|A\varphi\|_H + M\left\|A^{\frac{1}{2}}\psi\right\|_H$$

$$+\sum_{s=2}^{k-1}\|\theta_{s-1}-\theta_s\|_H + \frac{11}{4}\left[\|\theta_0\|_H + \sum_{i=1}^{k}\|\theta_i-\theta_{i-1}\|_H\right] \qquad (19)$$

for any $1 \leq k \leq N-1$. Using the triangle inequality and estimate (19), we get

$$\left\|\frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}\right\|_H \leq A\left\|\frac{w_{k+1}+2w_k+w_{k-1}}{4}\right\|_H + \|\theta_k\|_H$$

$$\leq M\|A\varphi\|_H + M\left\|A^{\frac{1}{2}}\psi\right\|_H + \sum_{s=2}^{k-1}\|\theta_{s-1}-\theta_s\|_H$$

$$+\frac{15}{4}\left[\|\theta_0\|_H + \sum_{i=1}^{k}\|\theta_i-\theta_{i-1}\|_H\right]$$

for any $1 \leq k \leq N-1$. Combining these estimates, we have estimate (14) for the solution of difference problem (13) for any $1 \leq k \leq N-1$.

Theorem 2. *For the solution of difference problem* (7), *the stability estimate* (13) *holds.*

Proof. Putting $H = L_{2h}, \varphi = \varphi^h,\ \psi = \psi^h,\ q = q^h,\ A\varphi = A_h\varphi^h,\ w_k = w_k^h,\ f_k = f_k^h$ and applying estimates (9) and (14), we obtain

$$\left\|\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\|_{L_{2h}} \leq M_{20}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1}\right]$$

$$+\left\|f_0^h\right\|_{L_{2h}} + T\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + M_{21}(q)M_{16}(q)$$

$$\times\sum_{s=1}^{k}\left\{\left[\left|\frac{\zeta_{s+1}-2\zeta_s+\zeta_{s-1}}{\tau^2}\right| + \left\|\frac{w_{s+1}^h-2w_s^h+w_{s-1}^h}{\tau^2}\right\|_{L_{2h}}\right]\right\}\tau$$

for any $1 \leq k \leq N-1$. By the difference analogue of Gronwall's inequality, we conclude that

$$\left\|\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}$$

$$\leq \frac{1}{1-M_{21}(q)M_{16}(q)\tau}\left\{M_{20}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1}\right]\right.$$

$$+\left\|f_0^h\right\|_{L_{2h}} + T\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + M_{21}(q)M_{16}(q)T$$

$$\times \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \Bigg\} e^{(k-1)\tau \frac{M_{21}(q)M_{16}(q)}{1-M_{21}(q)M_{16}(q)\tau}}$$

for any $1 \leq k \leq N-1$.

Theorem 2 is proved.

3. Numerical experiments. In this section, we study the numerical solution of the identification problem

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(t)\sin x + e^{-t}\sin x, \quad x \in (0,\pi), \quad t \in (0,1), \\
u(0,x) = \sin x, u_t(0,x) = -\sin x, \quad x \in [0,\pi], \\
u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\
\int_0^\pi u(t,x)dx = 2e^{-t}, \quad t \in [0,1],
\end{gathered} \tag{20}$$

for a hyperbolic differential equation. The exact solution pair of this problem is $(u(t,x), p(t)) = (e^{-t}\sin x, e^{-t})$. For the numerical solution of problem (20), we present the following second order of accuracy difference scheme for the approximate solution for the problem (20):

$$\begin{gathered}
\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{1}{2}\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \\
- \frac{1}{4}\left[\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2}\right] \\
= p_k \sin x_n + e^{-t_k}\sin x_n, \\
t_k = k\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1, \quad x_n = nh, \quad 1 \leq n \leq M-1, \quad Mh = \pi, \\
u_n^0 = \sin x_n, \\
\frac{u_n^1 - u_n^0}{\tau} - \frac{\tau}{h^2}\left(u_{n+1}^1 - u_{n+1}^0 - 2u_n^1 + 2u_n^0 + u_{n-1}^1 - u_{n-1}^0\right) \\
= -\sin x_n + \frac{\tau}{2}\left(\frac{u_{n+1}^0 - 2u_n^0 + u_{n-1}^0}{h^2} + p_0 \sin x_n + \sin x_n\right), \quad 0 \leq n \leq M, \\
u_0^{k+1} = u_M^{k+1} = 0, \qquad \sum_{i=1}^{M-1} u_i^{k+1} h = 2e^{-t_{k+1}}, \quad -1 \leq k \leq N-1.
\end{gathered} \tag{21}$$

The algorithm for obtaining the solution of identification problem (21) contains three stages. Actually, let us define

$$u_n^k = w_n^k + \eta_k \sin x_n, \quad 0 \leq k \leq N, \quad 0 \leq n \leq M. \tag{22}$$

Applying the second order of accuracy difference scheme (21) and formula (22), we obtain

$$\eta_k = \frac{2e^{-t_k} - \sum_{i=1}^{M-1} w_i^k h}{\sum_{i=1}^{M-1} \sin x_i h}, \quad 0 \le k \le N, \tag{23}$$

and the difference scheme

$$\begin{gathered}
\frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} - \frac{1}{2}\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} \\
- \frac{1}{4}\left[\frac{w_{n+1}^{k+1} - 2w_n^{k+1} + w_{n-1}^{k+1}}{h^2} + \frac{w_{n+1}^{k-1} - 2w_n^{k-1} + w_{n-1}^{k-1}}{h^2}\right] \\
+ \frac{1}{2}\frac{\cos h - 1}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n \left(\sum_{i=1}^{M-1} w_i^{k+1} h + 2\sum_{i=1}^{M-1} w_i^k h + \sum_{i=1}^{M-1} w_i^{k-1} h\right) \\
= e^{-t_k} \sin x_n + \left[e^{-t_{k+1}} + 2e^{-t_k} + e^{-t_{k-1}}\right] \frac{\cos h - 1}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n, \\
1 \le k \le N-1, \quad 1 \le n \le M-1, \\
w_n^0 = \sin x_n, \quad 0 \le n \le M, \\
\frac{w_n^1 - w_n^0}{\tau} - \frac{\tau}{h^2}\left(w_{n+1}^1 - 2w_n^1 + w_{n-1}^1\right) \\
+ \frac{2\tau(\cosh - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n \sum_{i=1}^{M-1} w_i^1 h + \frac{\tau}{2h^2}\left(w_{n+1}^0 - 2w_n^0 + w_{n-1}^0\right) \\
= \left(\frac{\tau}{2} - 1\right) \sin x_n + \frac{4\tau e^{-t_1}(\cosh - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n, \quad 0 \le n \le M, \\
w_0^{k+1} = w_M^{k+1} = 0, \quad -1 \le k \le N-1.
\end{gathered} \tag{24}$$

In the first stage, we find the solution $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of the corresponding second order of accuracy difference scheme (24). For obtaining it, we will write difference scheme (24), in matrix form as

$$\begin{gathered}
Aw^{k+1} + Bw^k + Cw^{k-1} = \varphi^k, \quad 1 \le k \le N-1, \\
w^0, \ w^1 \quad \text{are given},
\end{gathered} \tag{25}$$

where $A,\ B,\ C$ are $(M+1)\times(M+1)$ square matrices, $w^s,\ s=k,\ k\pm 1,\ \varphi^k$ are $(M+1)\times 1$ column matrices and

$$A=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \dots & c_1 & c_1 & c_1 \\ 0 & b+c_2 & a+c_2 & \dots & c_2 & c_2 & c_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & c_{M-2} & c_{M-2} & \dots & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \dots & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ r & q+e_1 & r+e_1 & \dots & e_1 & e_1 & e_1 \\ 0 & r+e_2 & q+e_2 & \dots & e_2 & e_2 & e_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & e_{M-2} & e_{M-2} & \dots & q+e_{M-2} & r+e_{M-2} & 0 \\ 0 & e_{M-1} & e_{M-1} & \dots & r+e_{M-1} & q+e_{M-1} & r \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \dots & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \dots & c_2 & c_2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & c_{M-2} & c_{M-2} & \dots & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \dots & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k=\begin{bmatrix} 0 \\ \varphi_1^k \\ \dots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1}, \qquad w^s=\begin{bmatrix} 0 \\ w_1^s \\ \dots \\ w_{M-1}^s \\ 0 \end{bmatrix}_{(M+1)\times 1} \quad \text{for} \quad s=k, \quad k\pm 1,$$

$$w^0 = \begin{bmatrix} \sin x_0 \\ \sin x_1 \\ \dots \\ \sin x_{M-1} \\ \sin x_M \end{bmatrix},$$

where

$$a = \frac{1}{\tau^2} + \frac{1}{2h^2}, \qquad b = -\frac{1}{4h^2}, \qquad r = -\frac{1}{2h^2}, \qquad q = \frac{1}{h^2} - \frac{2}{\tau^2},$$

$$c_n = \frac{1}{2} \sin x_n \frac{\cos h - 1}{dh}, \qquad e_n = \sin x_n \frac{\cos h - 1}{dh},$$

$$d = \sum_{i=1}^{M-1} \sin x_i h, \quad 1 \le n \le M-1,$$

$$\varphi_n^k = e^{-t_k} \sin x_n + \left(e^{-t_{k+1}} + 2e^{-t_k} + e^{-t_{k-1}}\right) \frac{\cos h - 1}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n,$$

$$1 \le k \le N-1, \quad 1 \le n \le M-1.$$

Finally, we obtain w^1. Applying the formula

$$\frac{w_n^1 - w_n^0}{\tau} - \frac{\tau}{h^2}\left(w_{n+1}^1 - 2w_n^1 + w_{n-1}^1\right)$$

$$+ \frac{2\tau(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n \sum_{i=1}^{M-1} w_i^1 h + \frac{\tau}{2h^2}\left(w_{n+1}^0 - 2w_n^0 + w_{n-1}^0\right)$$

$$= \left(\frac{\tau}{2} - 1\right) \sin x_n + \frac{4\tau e^{-t_1}(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n$$

and conditions $w_0^1 = w_M^1 = 0$, we get

$$Ew^1 + Fw^0 = \gamma.$$

Here,

$$E = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ y & x+z_1 & y+z_1 & \ldots & z_1 & z_1 & 0 \\ 0 & y+z_2 & x+z_2 & \ldots & z_2 & z_2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & z_{M-2} & z_{M-2} & \ldots & x+z_{M-2} & y+z_{M-2} & 0 \\ 0 & z_{M-1} & z_{M-1} & \ldots & y+z_{M-1} & x+z_{M-1} & y \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$x = \frac{2\tau}{h^2} + \frac{1}{\tau}, \qquad y = -\frac{\tau}{h^2}, \qquad z_n = \frac{2\tau(\cos h - 1)}{dh}\sin x_n,$$

$$u = -\frac{\tau}{2h^2}, \qquad v = -\frac{1}{\tau} - \frac{\tau}{h^2},$$

$$F = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ u & v & u & \ldots & 0 & 0 & 0 \\ 0 & u & v & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & v & u & 0 \\ 0 & 0 & 0 & \ldots & u & v & u \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\gamma = \begin{bmatrix} 0 \\ \nabla_1 \\ \ldots \\ \nabla_{M-1} \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\nabla_n = \left(\frac{\tau}{2} - 1 + \frac{4\tau e^{-t_1}(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h}\right)\sin x_n.$$

From that it follows

$$w^1 = E^{-1}\big(\gamma - F w^0\big). \tag{26}$$

So, we have the initial value problem for the second order difference equation (25) with respect to k with the matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we have $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ by (25).

Table 1. Error analysis

Error	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
E_u	0.0016	$4.2029e-04$	$1.0648e-04$	$2.6796e-05$
E_p	0.0027	$7.0457e-04$	$1.7835e-04$	$4.4867e-05$

Now, applying formula (6), we obtain

$$p_k = \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}, \quad 1 \le k \le N-1, \quad p_0 = \frac{2}{\tau^2}\eta_1. \tag{27}$$

In the second stage, we get $\{p_k\}_{k=1}^{N-1}$ by formulas (23) and (27). Finally, in the third stage, we obtain $\left\{\{u_n^k\}_{k=0}^{N}\right\}_{n=0}^{M}$ by formulas (22) and (23). The errors are computed by

$$E_u = \max_{0\le k\le N}\left(\sum_{n=1}^{M-1}\left|u(t_k,x_n) - u_n^k\right|^2 h\right)^{\frac{1}{2}},$$

$$E_p = \max_{1\le k\le N-1}|p(t_k) - p_k|,$$

where $u(t,x),\ p(t)$ represent the exact solution, u_n^k represents the numerical solutions at (t_k,x_n), and p_k represents the numerical solutions at t_k. The numerical results are given in Table 1.

As can be seen in Table 1, if N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately $\frac{1}{4}$ for the second order difference scheme (21).

4. Conclusion. In the present paper, the stable two-step difference scheme for the approximate solutions of the time-dependent source identification problem for the hyperbolic equations is presented. Stability of this difference scheme is established. Since problem is linear directly by Laxis theorem we can formulate the important results on convergence of difference schemes from the stability results of this paper. The numerical convergence results are presented.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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References

1. Yu. Ya. Belov, *Inverse problems for partial differential equations*, Inverse and Ill-posed Probl. Ser., VSP (2002).
2. G. Di Blasio, A. Lorenzi, *Identification problems for parabolic delay differential equations with measurement on the boundary*, J. Inverse and Ill-posed Probl., **15**, № 7, 709 – 734 (2007).
3. V. T. Borukhov, P. N. Vabishchevich, *Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation*, Comput. Phys. Commun., **126**, № 1-2, 32 – 36 (2000).
4. Yu. A. Gryazin, M. V. Klibanov, T. R. Lucas, *Imaging the diffusion coefficient in a parabolic inverse problem in optical tomography*, Inverse Problems, **15**, № 2, 373 (1999).

5. V. Isakov, *Inverse problems for partial differential equations*, **127**, Springer, New York (2006).
6. N. I. Ivanchov, *On the determination of unknown source in the heat equation with nonlocal boundary conditions*, Ukr. Math. J., **47**, № 10, 1647 – 1652 (1995).
7. S. I. Kabanikhin, O. I. Krivorotko, *Identification of biological models described by systems of nonlinear differential equations*, J. Inverse and Ill-posed Probl., **23**, № 5, 519 – 527 (2015).
8. A. I. Prilepko, D. G. Orlovsky, I. A. Vasin, *Methods for solving inverse problems in mathematical physics*, CRC Press (2000).
9. A. Ashyralyev, *On the problem of determining the parameter of a parabolic equation*, Ukr. Math. J., **62**, № 9, 1397 – 1408 (2011).
10. A. Ashyralyev, D. Agirseven, *On source identification problem for a delay parabolic equation*, Nonlinear Anal., Model. and Control, **19**, № 3, 335 – 349 (2014).
11. A. Ashyralyev, Ch. Ashyralyyev, *On the problem of determining the parameter of an elliptic equation in a Banach space*, Nonlinear Anal. Model. and Control, **19**, № 3, 350 – 366 (2014).
12. A. Ashyralyev, A. Erdogan, *Well-posedness of the right-hand side identification problem for a parabolic equation*, Ukr. Math. J., **66**, № 2 (2014).
13. A. U. Sazaklioglu, A. Ashyralyev, A. Said Erdogan, *Existence and uniqueness results for an inverse problem for semilinear parabolic equations*, Filomat, **31**, № 4, 1057 – 1064 (2017).
14. Ch. Ashyralyyev, *High order of accuracy difference schemes for the inverse elliptic problem with Dirichlet condition*, Boundary Value Problems, **2014**, № 1 (2014).
15. Ch. Ashyralyyev, *High order approximation of the inverse elliptic problem with Dirichlet – Neumann conditions*, Filomat, **28**, № 5, 947 – 962 (2014).
16. M. Ashyraliyeva, M. Ashyraliyev, *On a second order of accuracy stable difference scheme for the solution of a source identification problem for hyperbolic-parabolic equations*, AIP Conf. Proc., **1759**, № 1, AIP Publ. (2016).
17. M. Choulli, M. Yamamoto, *Generic well-posedness of a linear inverse parabolic problem with diffusion parameters*, J. Inverse and Ill-posed Probl., **7**, № 3, 241 – 254 (1999).
18. M. Dehghan, *An inverse problem of finding a source parameter in a semilinear parabolic equation*, Appl. Math. Model., **25**, № 9, 743 – 754 (2001).
19. Y. S. Eidelman, *The boundary value problem for differential equations with a parameter*, Differents. Uravn., **14**, № 7, 1335 – 1337 (1978).
20. Saitoh Saburou, Vu Kim Tuan, M. Yamamoto, *Reverse convolution inequalities and applications to inverse heat source problems*, J. Inequal. Pure and Appl. Math., **3**, № 5, 80 (2002).
21. A. A. Samarskii, P. N. Vabishchevich, *Numerical methods for solving inverse problems of mathematical physics*, **52**, Walter de Gruyter (2008).
22. F. A. Aliev et al., *A method to determine the coefficient of hydraulic resistance in different areas of pump-compressor pipes*, TWMS J. Pure and Appl. Math., **7**, № 2, 211 – 217 (2016).
23. F. A. Aliev et al., *Algorithm for calculating the parameters of formation of gas-liquid mixture in the shoe of gas lift well*, Appl. and Comput. Math., **15**, № 3, 370 – 376 (2016).
24. M. Grasselli, S. I. Kabanikhin, A. Lorenzi, *An inverse hyperbolic integrodifferential problem arising in geophysics. I*, Sib. Math. J., **33**, № 3, 415 – 426 (1992).
25. M. Grasselli, S. I. Kabanikhin, A. Lorenzi, *An inverse hyperbolic integrodifferential problem arising in geophysics II*, Nonlinear Anal., **15**, № 3, 283 – 298 (1990).
26. A. Ashyralyev, F. Emharab, *Source identification problems for hyperbolic differential and difference equations*, J. Inverse and Ill-posed Probl., **27**, № 3, 301 – 315 (2019).
27. A. Ashyralyev, P. E. Sobolevskii, *New difference schemes for partial differential equations, operator theory advances and applications*, Birkhäuser-Verlag etc. (2004).
28. Nguyen Thi Van Anh, Bui Thi Hai Yen, *Source identification problems for abstract semilinear nonlocal differential equations*, Inverse Probl. and Imaging, **16**, № 5, 1389 – 1428 (2022).
29. J. Janno, *Determination of time-dependent sources and parameters of nonlocal diffusion and wave equations from final data*, Fract. Calc. and Appl. Anal., **23**, № 6, 1678 – 1701 (2020).

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