

APPROXIMATION OF DOUBLE WALSH – FOURIER SERIES BY MEANS OF THE MATRIX TRANSFORM

НАБЛИЖЕННЯ ПОДВІЙНИХ РЯДІВ УОЛША – ФУР'Є ЗА ДОПОМОГОЮ МАТРИЧНОГО ПЕРЕТВОРЕННЯ

We discuss the rate of approximation of partial sums of the double Walsh – Fourier series in the spaces $L^p(G^2)$, $1 \leq p < \infty$, and $C(G^2)$ by the matrix transform.

Проаналізовано швидкість наближення частинних сум подвійних рядів Уолша – Фур'є у просторі $L^p(G^2)$, $1 \leq p < \infty$, та у просторі $C(G^2)$ за допомогою матричного перетворення.

1. Definitions and notations. We follow the standard notions of dyadic analysis introduced by F. Schipp, W. R. Wade, P. Simon and J. Pál [22] and others.

Let \mathbb{P} be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let denote by \mathbb{Z}_2 the discrete cyclic group of order 2, the group operation is the modulo 2 addition. Let be every subset open. The normalized Haar measure μ on \mathbb{Z}_2 is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. $G := \prod_{k=0}^{\infty} \mathbb{Z}_2$, G is called the Walsh group. The elements of Walsh group G are sequences of numbers 0 and 1, that is, $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$, $k \in \mathbb{N}$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. Dyadic intervals are defined in the usual way

$$I_0(x) := G, I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G, n \in \mathbb{P}$. They form a base for the neighbourhoods of G . Let $0 := (0 : i \in \mathbb{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

Let $L^p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|\cdot\|_p$), where $1 \leq p < \infty$.

For the sake of brevity in notation, we agree to write $L^\infty(G)$ instead of $C(G)$ and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$. Of course, it is clear that the space $L^\infty(G)$ is not the same as the space of continuous functions, i.e., it is a proper subspace of it. But since in the case of continuous functions the supremum norm and the $L^\infty(G)$ norm are the same, for convenience we hope the reader will be able to tolerate this simplification in notation. We have analogous definitions and mentions for $L^p(G^2)$, too.

For $x \in G$ we define $|x|$ by

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}.$$

¹ E-mail: blahota.istvan@nye.hu.

The partial modulus of continuity of a function $f \in L^p(G^2)$, $1 \leq p \leq \infty$, are defined by

$$\omega_p^1(f, \delta) := \sup_{|t| < \delta} \|f(\cdot + t, \cdot) - f(\cdot, \cdot)\|_p$$

and

$$\omega_p^2(f, \delta) := \sup_{|t| < \delta} \|f(\cdot, \cdot + t) - f(\cdot, \cdot)\|_p,$$

where $\delta > 0$.

The mixed modulus of continuity is defined as follows:

$$\begin{aligned} \omega_p^{1,2}(f, \delta^1, \delta^2) := \sup\{ & \|f(\cdot + x^1, \cdot + x^2) - f(\cdot + x^1, \cdot) - f(\cdot, \cdot + x^2) \\ & + f(\cdot, \cdot)\|_p : |x^1| \leq \delta^1, |x^2| \leq \delta^2\}, \end{aligned}$$

where $\delta^1, \delta^2 > 0$.

We define the Lipschitz classes in $L^p(G^2)$ for each $\alpha > 0$ in the following way:

$$\text{Lip}(\alpha, p) := \{f \in L^p(G^2) : \omega_p^1(f, \delta) + \omega_p^2(f, \delta) = O(\delta^\alpha) \text{ as } \delta \rightarrow 0\}.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k}, \quad x \in G, \quad k \in \mathbb{N}.$$

The Walsh–Paley functions are defined by the help of Rademacher functions. That is, $w_0 = 1$ and, for $n \geq 1$,

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k},$$

where the natural number n is expressed in the number system based 2, in the form

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad n_k \in \{0, 1\}, \quad k \in \mathbb{N}$$

(in this expression only a finite number of n_i 's different from zero), and the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$.

It is known [14] that the system $(w_n, n \in \mathbb{N})$ is the character system of $(G, +)$.

Let \mathcal{P}_n be the collection of Walsh polynomials of order less than n , that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k \omega_k(x),$$

where $n \in \mathbb{P}$ and $\{a_k\}$ is a sequence of complex numbers, and let $\mathcal{P}_{n,m}$ be the collection of two dimensional Walsh polynomials

$$Q(x^1, x^2) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} b_{k,j} \omega_k(x^1) \omega_j(x^2),$$

where $n, m \in \mathbb{P}$ and $\{b_{k,j}\}$ is a two dimensional sequence of complex numbers.

The Dirichlet kernels are defined by

$$D_n := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbb{P}$, $D_0 := 0$ and the Fejér kernels as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n} \sum_{k=1}^n D_k.$$

The two-dimensional Walsh–Fourier coefficients and the corresponding partial sums are the following:

$$\hat{f}(n^1, n^2) := \int_{G^2} f(x^1, x^2) w_{n^1}(x^1) w_{n^2}(x^2) d\mu(x^1, x^2),$$

$$S_{n^1, n^2}(f; x^1, x^2) := \sum_{k^1=0}^{n^1-1} \sum_{k^2=0}^{n^2-1} \hat{f}(k^1, k^2) w_{k^1}(x^1) w_{k^2}(x^2).$$

Let $T := (t_{k,n})_{k,n=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Since the n th row of the matrix T determines the matrix mean σ_n^T and its definition contains only finite number of entries, for the simplicity we say

$$\{t_{k,n} : 1 \leq k \leq n, k \in \mathbb{P}\}$$

is a finite sequence of numbers for each $n \in \mathbb{P}$. We also use notation

$$\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$$

for $1 \leq k \leq n$, $k, n \in \mathbb{P}$, where $t_{n,n+1} := 0$.

We define the two-dimensional matrix transform mean in this way:

$$\sigma_{n^1, n^2}^{T^1, T^2}(f) := \sum_{k^1=1}^{n^1} \sum_{k^2=1}^{n^2} t_{k^1, n^1}^1 t_{k^2, n^2}^2 S_{k^1, k^2}(f).$$

The one- and two-dimensional matrix transform kernels (T kernels) are defined by

$$K_n^T := \sum_{k=1}^n t_{k,n} D_k,$$

$$\begin{aligned} K_{n^1, n^2}^{T^1, T^2}(x^1, x^2) &:= \sum_{k^1=1}^{n^1} \sum_{k^2=1}^{n^2} t_{k^1, n^1}^1 t_{k^2, n^2}^2 D_{k^1}(x^1) D_{k^2}(x^2) \\ &= \left(\sum_{k^1=1}^{n^1} t_{k^1, n^1}^1 D_{k^1}(x^1) \right) \left(\sum_{k^2=1}^{n^2} t_{k^2, n^2}^2 D_{k^2}(x^2) \right) = K_{n^1}^{T^1}(x^1) K_{n^2}^{T^2}(x^2). \end{aligned}$$

From now, let us denote $x := (x^1, x^2)$ and $u := (u^1, u^2)$. It is easily seen that

$$\begin{aligned} \sigma_{n^1, n^2}^{T^1, T^2}(f; x) &= \int_{G^2} f(u) K_{n^1, n^2}^{T^1, T^2}(u + x) d\mu(u) \\ &= \int_{G^2} f(u + x) K_{n^1, n^2}^{T^1, T^2}(u) d\mu(u). \end{aligned}$$

This equality (and its analogous versions for special means) shows us the necessity of observing kernel functions.

2. Historical overview. Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n th Nörlund mean of the Walsh–Fourier series is defined by

$$\frac{1}{Q_n} \sum_{k=1}^n q_{n-k} S_k(f; x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k$, $n \geq 1$, and $S_k(f; x)$ denotes the k th partial sum of the Walsh–Fourier series of f . It is always assumed that $q_0 > 0$ and

$$\lim_{n \rightarrow \infty} Q_n = \infty.$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [18, 30]) if and only if

$$\lim_{n \rightarrow \infty} \frac{q_{n-1}}{Q_n} = 0.$$

In particular case the Nörlund means are the Fejér means (for all k set $q_k = 1$).

In paper [18], the rate of the approximation by Nörlund means of Walsh–Fourier series of a function f in $L^p(G)$ and in $C(G)$ (in particular, in $\text{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$) was studied. As special cases Móricz and Siddiqi obtained the earlier results given by Yano [29], Jastrebova [15] and Skvortsov [23] on the rate of the approximation by Cesàro means. The approximation properties of the Walsh–Cesàro means of negative order were studied by Goginava [12]. In 2008, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [11]. Recently Nagy, Baramidze, Memic, Persson, Tephnadze, Wall and the author presented some results with respect to this topic [1, 6, 8, 16]. See [10, 25–27], as well.

The two-dimensional definitions are introduced in the usual way. For the two-dimensional results see [7, 19, 20].

Let $\{p_k : k \geq 1\}$ be a sequence of nonnegative numbers. The n th weighted mean of Walsh–Fourier series is defined by

$$\frac{1}{P_n} \sum_{k=1}^n p_k S_k(f; x),$$

where $P_n := \sum_{k=1}^n p_k$, $n \geq 1$. It is always assumed that $p_1 > 0$ and

$$\lim_{n \rightarrow \infty} P_n = \infty,$$

which is the condition for regularity.

In particular case the weighted means are the Fejér means (for all k set $p_k = 1$).

In the paper [17], the rate of the approximation by weighted means of Walsh–Fourier series of a function in $L^p(G)$ and in $C(G)$ (in particular, in $\text{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$) was studied. As special cases Móricz and Rhoades obtained the earlier results given by Yano [29], Jastrebova [15] on the rate of the approximation by Walsh–Cesàro means. A generalization of this two results of Móricz and Siddiqi [18] and Móricz and Rhoades [17] was given by Nagy and the author in the paper [3].

Matrix transforms means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund means, the weighted means, the Fejér (or the $(C, 1)$) and the (C, α) means are special cases of the matrix transform summation method introduced above.

This paper is originally motivated by the work of Móricz, Siddiqi [18] on Walsh–Nörlund mean method and the result of Móricz, Rhoades [17] on Walsh weighted mean method. It is important to note that some ideas are coming from the paper of Chripkó [9].

For some other results on matrix transform means see, e.g., [3–5, 13].

Other aspects of these methods with respect to Walsh–Fourier series are treated in the papers [20, 25].

Our main aim is to investigate the rate of the approximation by two-dimensional matrix transform in terms of modulus of continuity. Moreover, our main theorem (Theorem 1) gives a two-dimensional generalization of the two results of Móricz, Siddiqi on Nörlund means [18] and Móricz, Rhoades on weighted means [17]. Moreover, we generalized the system, as well (see also [20]). The results are stated for nondecreasing and nonincreasing generating sequences $\{t_{k,n} : 1 \leq k \leq n\}$, $n \in \mathbb{P}$. At the end, we present an application for Lipschitz functions.

3. Auxiliary results.

Lemma 1 [21]. For $n \in \mathbb{N}$,

$$D_{2^n}(y) = \begin{cases} 2^n, & \text{if } y \in I_n(0), \\ 0, & \text{if } y \notin I_n(0). \end{cases}$$

Lemma 2 [28]. For $n \in \mathbb{N}$,

$$\|K_n\|_1 \leq 2.$$

In 2018 Toledo [24] improved this result, but for our proof the knowledge of the exact supremum of $\|K_n\|_1$ is not necessary, just its boundedness.

Lemma 3 [20]. Let $P \in \mathcal{P}_{2^A}$, $f \in L^p(G^2)$, $1 \leq p \leq \infty$, and $A, B \in \mathbb{P}$. Then there exists a positive constant c such that

$$\left\| \int_{G^2} (f(\cdot + x) - f(\cdot)) D_{2^B}(x^2) r_A(x^1) P(x^1) d\mu(x) \right\|_p \leq c \|P\|_1 \omega_p^1(f, 2^{-A}).$$

Lemma 4 [20]. Let $Q \in \mathcal{P}_{2^A, 2^B}$, $f \in L^p(G^2)$, $1 \leq p \leq \infty$, and $A, B \in \mathbb{P}$. Then there exists a positive constant c such that

$$\left\| \int_{G^2} (f(\cdot + x) - f(\cdot)) r_A(x^1) r_B(x^2) Q(x) d\mu(x) \right\|_p \leq c \|Q\|_1 \omega_p^{1,2}(f, 2^{-A}, 2^{-B}).$$

Lemma 5 Gát [2]. *Let n be a positive integer, then we have*

$$\begin{aligned}
 K_n^T &= \sum_{j=0}^{|n|-1} \sum_{k=0}^{2^j-1} t_{2^j+k,n} D_{2^j} \\
 &+ \sum_{j=0}^{|n|-1} r_j \sum_{k=1}^{2^j-2} \Delta t_{2^j+k,n} k K_k + \sum_{j=0}^{|n|-1} r_j t_{2^{j+1}-1,n} (2^j - 1) K_{2^j-1} \\
 &+ \sum_{k=0}^{n-2^{|n|}} t_{2^{|n|+k,n} } D_{2^{|n|}} + r_{|n|} \sum_{k=1}^{n-2^{|n|}} \Delta t_{2^{|n|+k,n} } k K_k \\
 &=: \sum_{i=1}^5 K_{i,n}^T.
 \end{aligned}$$

4. Main theorem.

Theorem 1. *Let $f \in C(G^2)$ or $f \in L^p(G^2)$, $1 \leq p < \infty$. For every $n^j \in \mathbb{N}$, $\{t_{k^j,n^j}^j : 1 \leq k^j \leq n^j\}$, where $j \in \{1, 2\}$, be a finite sequence of nonnegative numbers such that equalities*

$$\sum_{k^1=1}^{n^1} t_{k^1,n^1}^1 = \sum_{k^2=1}^{n^2} t_{k^2,n^2}^2 = 1 \tag{1}$$

are satisfied.

(a) *If finite sequences $\{t_{k^j,n^j}^j : 1 \leq k^j \leq n^j\}$ are nondecreasing for fixed n^j , where $j \in \{1, 2\}$, and conditions*

$$t_{n^1,n^1} = O(1/n^1) \quad \text{and} \quad t_{n^2,n^2} = O(1/n^2) \tag{2}$$

are satisfied, then

$$\begin{aligned}
 &\left\| \sigma_{n^1,n^2}^{T^1,T^2}(f) - f \right\|_p \\
 &\leq c \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1,n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2+1}-1,n^2}^2 \omega_p^2(f, 2^{-j^2}) \right. \\
 &\quad \left. + \omega_p^1(f, 2^{-|n^1|}) + \omega_p^2(f, 2^{-|n^2|}) \right).
 \end{aligned}$$

(b) *If finite sequences $\{t_{k^j,n^j}^j : 1 \leq k^j \leq n^j\}$ are nonincreasing for fixed n^j , where $j \in \{1, 2\}$, then*

$$\begin{aligned}
 \left\| \sigma_{n^1,n^2}^{T^1,T^2}(f) - f \right\|_p &\leq c \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1},n^1}^1 \omega_p^1(f, 2^{-j^1}) \right. \\
 &\quad \left. + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2},n^2}^2 \omega_p^2(f, 2^{-j^2}) + \omega_p^1(f, 2^{-|n^1|}) + \omega_p^2(f, 2^{-|n^2|}) \right).
 \end{aligned}$$

(c) If finite sequence $\{t_{k^1, n^1}^1 : 1 \leq k^1 \leq n^1\}$ is nondecreasing for fixed n^1 and the condition

$$t_{n^1, n^1} = O(1/n^1)$$

is satisfied and $\{t_{k^2, n^2}^2 : 1 \leq k^2 \leq n^2\}$ is nonincreasing for fixed n^2 , then

$$\begin{aligned} & \left\| \sigma_{n^1, n^2}^{T^1, T^2}(f) - f \right\|_p \\ & \leq c \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1, n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2}, n^2}^2 \omega_p^2(f, 2^{-j^2}) \right. \\ & \quad \left. + \omega_p^1(f, 2^{-|n^1|}) + \omega_p^2(f, 2^{-|n^2|}) \right) \end{aligned}$$

holds (for $f \in C(G^2)$ we change p by ∞).

Proof. During our proofs c denotes a positive constant, which may vary at different appearances. We make the proof for $L^p(G^2)$ spaces $1 \leq p < \infty$. For $C(G^2)$ the proof is similar, even simpler. Let us set $f \in L^p(G^2)$.

By using condition (1), the usual Minkowski inequality and Lemma 5, we have

$$\begin{aligned} \left\| \sigma_{n^1, n^2}^{T^1, T^2}(f) - f \right\|_p &= \left(\int_{G^2} |\sigma_{n^1, n^2}^{T^1, T^2}(f; x) - f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \\ &= \left(\int_{G^2} \left| \int_{G^2} K_{n^1, n^2}^{T^1, T^2}(u) (f(x+u) - f(x)) d\mu(u) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\ &\leq \sum_{i^1=1}^5 \sum_{i^2=1}^5 \left\| \int_{G^2} K_{i^1, n^1}^{T^1}(u^1) K_{i^2, n^2}^{T^2}(u^2) (f(\cdot + u) - f(\cdot)) d\mu(u) \right\|_p \\ &=: \sum_{i^1=1}^5 \sum_{i^2=1}^5 I_{i^1, i^2, n^1, n^2}. \end{aligned}$$

Because of the symmetry, it is enough to investigate cases, when $i^1 \leq i^2$.

At first let us see $I_{1,1}$, $I_{4,4}$ and $I_{1,4}$. Using generalized Minkowski's inequality [30, vol. 1, p. 19], Lemma 1 and inequality

$$\begin{aligned} & |f(x^1 + u^1, x^2 + u^2) - f(x^1, x^2)| \\ & \leq |f(x^1 + u^1, x^2 + u^2) - f(x^1 + u^1, x^2)| + |f(x^1 + u^1, x^2) - f(x^1, x^2)|, \end{aligned}$$

we write that

$$\begin{aligned} & \left\| \int_{G^2} D_{2^{j^1}}(u^1) D_{2^{j^2}}(u^2) (f(\cdot + u) - f(\cdot)) d\mu(u) \right\|_p \\ & \leq \int_{G^2} D_{2^{j^1}}(u^1) D_{2^{j^2}}(u^2) \left(\int_{G^2} |f(x + u) - f(x)|^p d\mu(x) \right)^{\frac{1}{p}} d\mu(u) \\ & \leq \omega_p^1(f, 2^{-j^1}) + \omega_p^2(f, 2^{-j^2}). \end{aligned}$$

So, by using condition (1), we get

$$\begin{aligned} I_{1,1,n^1,n^2} & \leq \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1},n^1}^1 \sum_{j^2=0}^{|n^2|-1} \sum_{k^2=0}^{2^{j^2}-1} t_{2^{j^2+k^2},n^2}^2 \left(\omega_p^1(f, 2^{-j^1}) + \omega_p^2(f, 2^{-j^2}) \right) \\ & \leq \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1},n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} \sum_{k^2=0}^{2^{j^2}-1} t_{2^{j^2+k^2},n^2}^2 \omega_p^2(f, 2^{-j^2}). \end{aligned}$$

In case (a) we have

$$I_{1,1,n^1,n^2} \leq \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1,n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2+1}-1,n^2}^2 \omega_p^2(f, 2^{-j^2}),$$

in case (b)

$$I_{1,1,n^1,n^2} \leq \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1},n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2},n^2}^2 \omega_p^2(f, 2^{-j^2}),$$

and in case (c)

$$I_{1,1,n^1,n^2} \leq \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1,n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2},n^2}^2 \omega_p^2(f, 2^{-j^2}).$$

Similarly, by using condition (1), we obtain

$$\begin{aligned} I_{4,4,n^1,n^2} & \leq \sum_{k^1=0}^{n^1-2^{|n^1|}} t_{2^{|n^1|+k^1},n^1}^1 \sum_{k^2=0}^{n^2-2^{|n^2|}} t_{2^{|n^2|+k^2},n^2}^2 \left(\omega_p^1(f, 2^{-|n^1|}) + \omega_p^2(f, 2^{-|n^2|}) \right) \\ & \leq \omega_p^1(f, 2^{-|n^1|}) + \omega_p^2(f, 2^{-|n^2|}). \end{aligned}$$

Also similarly we get

$$I_{1,4,n^1,n^2} \leq \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1},n^1}^1 \sum_{k^2=0}^{n^2-2^{|n^2|}} t_{2^{|n^2|+k^2},n^2}^2 \left(\omega_p^1(f, 2^{-j^1}) + \omega_p^2(f, 2^{-|n^2|}) \right)$$

$$\leq \omega_p^2(f, 2^{-|n^2|}) + \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1-1}, n^1}^1 \omega_p^1(f, 2^{-j^1}).$$

So, in cases (a) and (c) we obtain

$$I_{1,4,n^1,n^2} \leq \omega_p^2(f, 2^{-|n^2|}) + \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1-1}, n^1}^1 \omega_p^1(f, 2^{-j^1})$$

and in case (b)

$$I_{1,4,n^1,n^2} \leq \omega_p^2(f, 2^{-|n^2|}) + \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}, n^1}^1 \omega_p^1(f, 2^{-j^1}).$$

Now let us observe $I_{1,2}, I_{1,3}, I_{1,5}, I_{2,4}, I_{3,4}$ and $I_{4,5}$. By using Lemmas 3 and 2, we have

$$\begin{aligned} & \left\| \int_{G^2} D_{2^{j^1}}(u^1) r_{j^2}(u^2) K_{k^2}(u^2) (f(\cdot + u) - f(\cdot)) d\mu(u) \right\|_p \\ & \leq c \|K_{k^2}\|_1 \omega_p^2(f, 2^{-j^2}) \leq c \omega_p^2(f, 2^{-j^2}) \end{aligned} \quad (3)$$

and

$$I_{1,2,n^1,n^2} \leq c \sum_{j^2=0}^{|n^2|-1} \sum_{k^2=1}^{2^{j^2}-2} |\Delta t_{2^{j^2+k^2}, n^2}^2| k^2 \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1}, n^1}^1 \omega_p^2(f, 2^{-j^2}).$$

In case (a)

$$\begin{aligned} \sum_{k^2=1}^{2^{j^2}-2} |\Delta t_{2^{j^2+k^2}, n^2}^2| k^2 &= \sum_{k^2=1}^{2^{j^2}-2} (t_{2^{j^2+k^2+1}, n^2}^2 - t_{2^{j^2+k^2}, n^2}^2) k^2 \\ &= (2^{j^2} - 2) t_{2^{j^2+1-1}, n^2}^2 - \sum_{k^2=1}^{2^{j^2}-2} t_{2^{j^2+k^2}, n^2}^2 \leq 2^{j^2} t_{2^{j^2+1-1}, n^2}^2, \end{aligned}$$

in cases (b) and (c)

$$\begin{aligned} \sum_{k^2=1}^{2^{j^2}-2} |\Delta t_{2^{j^2+k^2}, n^2}^2| k^2 &= \sum_{k^2=1}^{2^{j^2}-2} t_{2^{j^2+k^2}, n^2}^2 - (2^{j^2} - 2) t_{2^{j^2+1-1}, n^2}^2 \\ &\leq \sum_{k^2=1}^{2^{j^2}-2} t_{2^{j^2+k^2}, n^2}^2 \leq 2^{j^2} t_{2^{j^2}, n^2}^2. \end{aligned}$$

Since from condition (1)

$$\sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1}, n^1}^1 \leq 1, \tag{4}$$

so in cases (a) and (c)

$$I_{1,2,n^1,n^2} \leq c \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2+1}, n^2}^2 \omega_p^2(f, 2^{-j^2}),$$

in case (b)

$$I_{1,2,n^1,n^2} \leq c \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2}, n^2}^2 \omega_p^2(f, 2^{-j^2}).$$

Similarly to (3),

$$I_{1,3,n^1,n^2} \leq c \sum_{j^2=0}^{|n^2|-1} t_{2^{j^2+1}, n^2}^2 (2^{j^2} - 1) \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1}, n^1}^1 \omega_p^2(f, 2^{-j^2}).$$

In case (a) we will use inequality

$$\sum_{j^2=0}^{|n^2|-1} t_{2^{j^2+1}, n^2}^2 (2^{j^2} - 1) \leq \sum_{j^2=0}^{|n^2|-1} t_{2^{j^2+1}, n^2}^2 2^{j^2},$$

in cases (b) and (c)

$$\sum_{j^2=0}^{|n^2|-1} t_{2^{j^2+1}, n^2}^2 (2^{j^2} - 1) \leq \sum_{j^2=0}^{|n^2|-1} t_{2^{j^2}, n^2}^2 2^{j^2}.$$

So, in case of $I_{1,3}$ we get the same inequalities in cases (a), (b) and (c), as in case of $I_{1,2}$.

Also similarly to (3), we obtain

$$I_{1,5,n^1,n^2} \leq c \sum_{k^2=1}^{n^2-2|n^2|} \left| \Delta t_{2^{|n^2|+k^2}, n^2}^2 \right| k^2 \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=0}^{2^{j^1}-1} t_{2^{j^1+k^1}, n^1}^1 \omega_p^2(f, 2^{-|n^2|}).$$

In case (a) from condition (2) we get

$$\begin{aligned} \sum_{k^2=1}^{n^2-2|n^2|} \left| \Delta t_{2^{|n^2|+k^2}, n^2}^2 \right| k^2 &= \sum_{k^2=1}^{n^2-2|n^2|-1} \left(t_{2^{|n^2|+k^2+1}, n^2}^2 - t_{2^{|n^2|+k^2}, n^2}^2 \right) k^2 + (n^2 - 2|n^2|) t_{n^2, n^2}^2 \\ &= (n^2 - 2|n^2| - 1) t_{n^2, n^2}^2 - \sum_{k^2=1}^{n^2-2|n^2|-1} t_{2^{|n^2|+k^2}, n^2}^2 + (n^2 - 2|n^2|) t_{n^2, n^2}^2 \\ &\leq 2n^2 t_{n^2, n^2}^2 \leq c, \end{aligned}$$

in cases (b) and (c) from condition (1) we have

$$\sum_{k^2=1}^{n^2-2|n^2|} |\Delta t_{2|n^2|+k^2, n^2}^2| k^2 = \sum_{k^2=1}^{n^2-2|n^2|} t_{2|n^2|+k^2, n^2}^2 = \sum_{k^2=1}^{n^2-2|n^2|} t_{2|n^2|+k^2, n^2}^2 \leq 1.$$

It yields, that from (4)

$$I_{1,5, n^1, n^2} \leq c \omega_p^2(f, 2^{-|n^2|}).$$

From Lemma 3 and condition (1) again we obtain

$$\begin{aligned} I_{2,4, n^1, n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=1}^{2^{j^1}-2} |\Delta t_{2^{j^1}+k^1, n^1}^1| k^1 \sum_{k^2=0}^{n^2-2|n^2|} t_{2|n^2|+k^2, n^2}^2 \omega_p^1(f, 2^{-j^1}) \\ &\leq c \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=1}^{2^{j^1}-2} |\Delta t_{2^{j^1}+k^1, n^1}^1| k^1 \omega_p^1(f, 2^{-j^1}). \end{aligned}$$

It follows, that in cases (a) and (c)

$$I_{2,4, n^1, n^2} \leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}+1-1, n^1}^1 \omega_p^1(f, 2^{-j^1}),$$

in case (b)

$$I_{2,4, n^1, n^2} \leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}, n^1}^1 \omega_p^1(f, 2^{-j^1}).$$

As before,

$$\begin{aligned} I_{3,4, n^1, n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} (2^{j^1} - 1) t_{2^{j^1}+1-1, n^1}^1 \sum_{k^2=0}^{n^2-2|n^2|} t_{2|n^2|+k^2, n^2}^2 \omega_p^1(f, 2^{-j^1}) \\ &\leq c \sum_{j^1=0}^{|n^1|-1} (2^{j^1} - 1) t_{2^{j^1}+1-1, n^1}^1 \omega_p^1(f, 2^{-j^1}). \end{aligned}$$

Similarly to $I_{1,3}$ it implies in cases (a) and (c) that

$$I_{3,4, n^1, n^2} \leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}+1-1, n^1}^1 \omega_p^1(f, 2^{-j^1}),$$

in case (b)

$$I_{3,4, n^1, n^2} \leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}, n^1}^1 \omega_p^1(f, 2^{-j^1}).$$

Similarly to estimating $I_{1,5}$, we obtain

$$\begin{aligned} I_{4,5,n^1,n^2} &\leq c \sum_{k^2=1}^{n^2-2|n^2|} |\Delta t_{2^{|n^2|+k^2},n^2}^2| k^2 \sum_{k^1=0}^{n^1-2|n^1|} t_{2^{|n^1|+k^1},n^1}^1 \omega_p^2(f, 2^{-|n^2|}) \\ &\leq c \omega_p^2(f, 2^{-|n^2|}). \end{aligned}$$

Expressions $I_{2,2}$, $I_{2,5}$, $I_{2,3}$, $I_{3,3}$, $I_{3,5}$ and $I_{5,5}$ follow. Let us use Lemmas 4 and 2 for estimating

$$\begin{aligned} &\left\| \int_{G^2} r_{j^1}(u^1) r_{j^2}(u^2) K_{k^1}(u^1) K_{k^2}(u^2) (f(\cdot + u) - f(\cdot)) d\mu(u) \right\|_p \\ &\leq c \|K_{k^1} K_{k^2}\|_1 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}) \leq c \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}). \end{aligned}$$

Then

$$I_{2,2,n^1,n^2} \leq c \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=1}^{2^{j^1}-2} |\Delta t_{2^{j^1+k^1},n^1}^1| k^1 \sum_{j^2=0}^{|n^2|-1} \sum_{k^2=1}^{2^{j^2}-2} |\Delta t_{2^{j^2+k^2},n^2}^2| k^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}).$$

Next we will use inequality

$$\omega_p^{1,2}(f, \delta^1, \delta^2) \leq \omega_p^1(f, \delta^1) + \omega_p^2(f, \delta^2)$$

several times. In case (a)

$$\begin{aligned} I_{2,2,n^1,n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1},n^1}^1 \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2+1},n^2}^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}) \\ &\leq c \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1},n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2+1},n^2}^2 \omega_p^2(f, 2^{-j^2}) \right), \end{aligned}$$

in case (b)

$$\begin{aligned} I_{2,2,n^1,n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1},n^1}^1 \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2},n^2}^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}) \\ &\leq c \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1},n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2},n^2}^2 \omega_p^2(f, 2^{-j^2}) \right), \end{aligned}$$

and in case (c)

$$I_{2,2,n^1,n^2} \leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1},n^1}^1 \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2},n^2}^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2})$$

$$\leq c \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1, n^1}^1 \omega_p^1(f, 2^{-j^1}) + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2}, n^2}^2 \omega_p^2(f, 2^{-j^2}) \right).$$

On the other hand,

$$\begin{aligned} I_{2,5,n^1,n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=1}^{2^{j^1}-2} |\Delta t_{2^{j^1+k^1}, n^1}^1| k^1 \sum_{k^2=1}^{n^2-2|n^2|} |\Delta t_{2^{|n^2|+k^2}, n^2}^2| k^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|}) \\ &\leq c \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=1}^{2^{j^1}-2} |\Delta t_{2^{j^1+k^1}, n^1}^1| k^1 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|}). \end{aligned}$$

It implies in cases (a) and (c)

$$\begin{aligned} I_{2,5,n^1,n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1, n^1}^1 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|}) \\ &\leq c \left(\omega_p^2(f, 2^{-|n^2|}) + \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1, n^1}^1 \omega_p^1(f, 2^{-j^1}) \right), \end{aligned}$$

in case (b)

$$\begin{aligned} I_{2,5,n^1,n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}, n^1}^1 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|}) \\ &\leq c \left(\omega_p^2(f, 2^{-|n^2|}) + \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1}, n^1}^1 \omega_p^1(f, 2^{-j^1}) \right). \end{aligned}$$

Similarly, Lemmas 4 and 2 give

$$\left\| \int_{G^2} r_{j^1}(u^1) r_{j^2}(u^2) K_{k^1}(u^1) K_{2^{j^2-1}}(u^2) (f(\cdot + u) - f(\cdot)) d\mu(u) \right\|_p \leq c \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}).$$

Thus,

$$I_{2,3,n^1,n^2} \leq c \sum_{j^1=0}^{|n^1|-1} \sum_{k^1=1}^{2^{j^1}-2} |\Delta t_{2^{j^1+k^1}, n^1}^1| k^1 \sum_{j^2=0}^{|n^2|-1} t_{2^{j^2+1}-1, n^2}^2 (2^{j^2} - 1) \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}).$$

As before, it also implies in every case, that estimations for $I_{2,3}$ are the same, as in case of $I_{2,2}$.

Lemmas 4 and 2 give us again

$$\left\| \int_{G^2} r_{j^1}(u^1) r_{j^2}(u^2) K_{2^{j^1-1}}(u^1) K_{2^{j^2-1}}(u^2) (f(\cdot + u) - f(\cdot)) d\mu(u) \right\|_p \leq c \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}).$$

Thus,

$$I_{3,3,n^1,n^2} \leq c \sum_{j^1=0}^{|n^1|-1} (2^{j^1} - 1)t_{2^{j^1+1}-1,n^1}^1 \sum_{j^2=0}^{|n^2|-1} (2^{j^2} - 1)t_{2^{j^2+1}-1,n^2}^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-j^2}).$$

It means, that we get the same estimation again, as in case of $I_{2,2}$.

From

$$\left\| \int_{G^2} r_{j^1}(u^1)r_{|n^2|}(u^2)K_{2^{j^1-1}}(u^1)K_{k^2}(u^2)(f(\cdot + u) - f(\cdot))d\mu(u) \right\|_p \leq c\omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|})$$

we get

$$\begin{aligned} I_{3,5,n^1,n^2} &\leq c \sum_{j^1=0}^{|n^1|-1} (2^{j^1} - 1)t_{2^{j^1+1}-1,n^1}^1 \sum_{k^2=1}^{n^2-2|n^2|} |\Delta t_{2^{|n^2|+k^2},n^2}^2| k^2 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|}) \\ &\leq c \sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1}-1,n^1}^1 \omega_p^{1,2}(f, 2^{-j^1}, 2^{-|n^2|}). \end{aligned}$$

So, it is the same estimate, as $I_{2,5}$.

Finally,

$$\left\| \int_{G^2} r_{|n^1|}(u^1)r_{|n^2|}(u^2)K_{k^1}(u^1)K_{k^2}(u^2)(f(\cdot + u) - f(\cdot))d\mu(u) \right\|_p \leq c\omega_p^{1,2}(f, 2^{-|n^1|}, 2^{-|n^2|}),$$

and

$$\begin{aligned} I_{5,5,n^1,n^2} &\leq c \sum_{k^1=1}^{n^1-2|n^1|} |\Delta t_{2^{|n^1|+k^1},n^1}^1| k^1 \sum_{k^2=1}^{n^2-2|n^2|} |\Delta t_{2^{|n^2|+k^2},n^2}^2| k^2 \omega_p^{1,2}(f, 2^{-|n^1|}, 2^{-|n^2|}) \\ &\leq c\omega_p^{1,2}(f, 2^{-|n^1|}, 2^{-|n^2|}) \\ &\leq c(\omega_p^1(f, 2^{-|n^1|}) + \omega_p^2(f, 2^{-|n^2|})). \end{aligned}$$

Theorem 1 is proved.

5. Application. In this section, we apply our theorem for Lipschitz functions.

Theorem 2. Let $f \in \text{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$. For matrix transform $\sigma_{n^1,n^2}^{T^1,T^2}$ we suppose that the conditions in Theorem 1 in case (a) are satisfied. Then the next estimate holds

$$\left\| \sigma_{n^1,n^2}^{T^1,T^2}(f) - f \right\|_p = \begin{cases} O(1/(n^1)^\alpha + 1/(n^2)^\alpha), & \text{if } 0 < \alpha < 1, \\ O(\log n^1/n^1 + \log n^2/n^2), & \text{if } \alpha = 1, \\ O(1/n^1 + 1/n^2), & \text{if } \alpha > 1. \end{cases}$$

Proof. Condition (2) yields that

$$\begin{aligned} \left\| \sigma_{n^1, n^2}^{T^1, T^2}(f) - f \right\|_p &= O \left(\sum_{j^1=0}^{|n^1|-1} 2^{j^1} t_{2^{j^1+1-1}, n^1}^1 2^{-j^1 \alpha} + 2^{-|n^1| \alpha} \right. \\ &\quad \left. + \sum_{j^2=0}^{|n^2|-1} 2^{j^2} t_{2^{j^2+1-1}, n^2}^2 2^{-j^2 \alpha} + 2^{-|n^2| \alpha} \right) \\ &= O \left(t_{n^1, n^1}^1 \sum_{j^1=0}^{|n^1|-1} 2^{j^1(1-\alpha)} + 2^{-|n^1| \alpha} \right. \\ &\quad \left. + t_{n^2, n^2}^2 \sum_{j^2=0}^{|n^2|-1} 2^{j^2(1-\alpha)} + 2^{-|n^2| \alpha} \right) \\ &= O \left(1/n^1 \sum_{j^1=0}^{|n^1|-1} 2^{j^1(1-\alpha)} + 1/n^2 \sum_{j^2=0}^{|n^2|-1} 2^{j^2(1-\alpha)} \right) \\ &\quad + O(1/(n^1)^\alpha + 1/(n^2)^\alpha). \end{aligned}$$

From this form we get our statement easily.

Theorem 2 is proved.

Conflict of interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

Funding. This paper was supported by the Scientific Council of the University of Nyíregyháza.

References

1. L. Baramidze, L.-E. Persson, G. Tephnadze, P. Wall, *Sharp $H_p - L_p$ type inequalities of weighted maximal operators of Vilenkin–Nörlund means and its applications*, J. Inequal. and Appl., Article 242 (2016).
2. I. Blahota, G. Gát, *Norm and almost everywhere convergence of matrix transform means of Walsh–Fourier series*, Acta Univ. Sapientiae Math., **15**, № 2, 244–258 (2023).
3. I. Blahota, K. Nagy, *Approximation by Θ -means of Walsh–Fourier series*, Anal. Math., **44**, № 1, 57–71 (2018).
4. I. Blahota, K. Nagy, *Approximation by matrix transform of Vilenkin–Fourier series*, Publ. Math. Debrecen, **99**, № 1-2, 223–242 (2021).
5. I. Blahota, K. Nagy, *Approximation by Marcinkiewicz-type matrix transform of Vilenkin–Fourier series*, Mediterr. J. Math., **19**, Article 165 (2022).
6. I. Blahota, K. Nagy, M. Salim, *Approximation by Θ -means of Walsh–Fourier series in dyadic Hardy spaces and dyadic homogeneous Banach spaces*, Anal. Math., **47**, 285–309 (2021).
7. I. Blahota, K. Nagy, G. Tephnadze, *Approximation by Marcinkiewicz Θ -means of double Walsh–Fourier series*, Math. Inequal. Appl., **22**, № 3, 837–853 (2019).
8. I. Blahota, G. Tephnadze, *A note on maximal operators of Vilenkin–Nörlund means*, Acta Math. Acad. Paedagog. Nyházi., **32**, № 2, 203–213 (2016).
9. Á. Chripkó, *Weighted approximation via Θ -summations of Fourier–Jacobi series*, Studia Sci. Math. Hungar., **47**, № 2, 139–154 (2010).
10. T. Eisner, *The Θ -summation on local fields*, Ann. Univ. Sci. Budapest. Sect. Comput., **33**, 137–160 (2011).

11. S. Fridli, P. Manchanda, A. H. Siddiqi, *Approximation by Walsh–Nörlund means*, Acta Sci. Math., **74**, 593–608 (2008).
12. U. Goginava, *On the approximation properties of Cesàro means of negative order of Walsh–Fourier series*, J. Approx. Theory, **115**, 9–20 (2002).
13. U. Goginava, K. Nagy, *Matrix summability of Walsh–Fourier series*, Mathematics, MDPI, **10**, № 14 (2022).
14. E. Hewitt, K. Ross, *Abstract harmonic analysis*, vols. I, II, Springer-Verlag, Heidelberg (1963).
15. M. A. Jastrebova, *On approximation of functions satisfying the Lipschitz condition by arithmetic means of their Walsh–Fourier series*, Mat. Sb., **71**, 214–226 (1966) (in Russian).
16. N. Memić, L.-E. Persson, G. Tephnadze, *A note on the maximal operators of Vilenkin–Nörlund means with non-increasing coefficients*, Stud. Sci. Math. Hungar., **53**, № 4, 545–556 (2016).
17. F. Móricz, B. E. Rhoades, *Approximation by weighted means of Walsh–Fourier series*, Int. J. Math. Sci., **19**, № 1, 1–8 (1996).
18. F. Móricz, A. Siddiqi, *Approximation by Nörlund means of Walsh–Fourier series*, J. Approx. Theory, **70**, 375–389 (1992).
19. K. Nagy, *Approximation by Nörlund means of quadratical partial sums of double Walsh–Fourier series*, Anal. Math., **36**, № 4, 299–319 (2010).
20. K. Nagy, *Approximation by Nörlund means of double Walsh–Fourier series for Lipschitz functions*, Math. Inequal. Appl., **15**, № 2, 301–322 (2012).
21. F. Schipp, W. R. Wade, *Norm convergence and summability of Fourier series with respect to certain product systems*, Pure and Appl. Math. Approx. Theory, Marcel Dekker, New York etc. (1992), p. 437–452.
22. F. Schipp, W. R. Wade, P. Simon, J. Pál, *Walsh series. An introduction to dyadic harmonic analysis*, Adam Hilger, Bristol, New York (1990).
23. V. A. Skvortsov, *Certain estimates of approximation of functions by Cesàro means of Walsh–Fourier series*, Mat. Zametki, **29**, 539–547 (1981) (in Russian).
24. R. Toledo, *On the boundedness of the L^1 -norm of Walsh–Fejer kernels*, J. Math. Anal. and Appl., **457**, № 1, 153–178 (2018).
25. F. Weisz, *Θ -summability of Fourier series*, Acta Math. Hungar., **103**, № 1-2, 139–175 (2004).
26. F. Weisz, *Θ -summation and Hardy spaces*, J. Approx. Theory, **107**, 121–142 (2000).
27. F. Weisz, *Several dimensional Θ -summability and Hardy spaces*, Math. Nachr., **230**, 159–180 (2001).
28. Sh. Yano, *On Walsh–Fourier series*, Tohoku Math. J., **3**, 223–242 (1951).
29. Sh. Yano, *On approximation by Walsh functions*, Proc. Amer. Math. Soc., **2**, 962–967 (1951).
30. A. Zygmund, *Trigonometric series*, 3rd ed., vols. 1, 2, Cambridge Univ. Press (2015).

Received 30.11.22