DOI: 10.3842/umzh.v76i5.7409

UDC 519.1

J. John[1] (Department of Mathematics, Government College of Engineering, Tirunelveli, India),

M. S. Malchijah Raj (Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai, Tamil Nadu, India)

GRAPHS WITH LARGE STEINER NUMBER

ГРАФИ З ВЕЛИКИМ ЧИСЛОМ ШТЕЙНЕРА

In 2002, Gary Chartrand and Ping Zhang [*The Steiner number of a graph*, Discrete Math., **242**, 41–54 (2002)] characterized the connected graphs G of order $p \geq 3$ with Steiner number p, $p-1$, or 2. In our paper, we characterize all connected graphs G of order $p \geq 4$ with Steiner number $s(G) = p-2$. In addition, we obtain some sharp Nordhaus – Gaddum bounds for the Steiner number of connected graphs whose complement is also connected.

У 2002 році Гері Чартранд та Пінг Чжан [*The Steiner number of a graph*, Discrete Math., **242**, 41 – 54 (2002)] охарактеризували зв'язні графи G порядку $p \geq 3$ з числом Штейнера p, $p-1$ або 2. У цій статті охарактеризовано всі зв'язні графи G порядку $p \geq 4$ з числом Штейнера $s(G) = p-2$. Також отримано деякі точні межі Нордгауза – Гаддума для числа Штейнера зв'язних графів, доповнення яких також є зв'язними.

1. Introduction. Let $G = (V, E)$ be a finite undirected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to [2]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an u-v *geodesic*. It is known that this distance is a metric on the vertex set V. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $\operatorname{rad}(G)$ and the maximum eccentricity is its diameter, $\operatorname{diam}(G)$ of G. Two vertices x and y are *antipodal* if $d(x,y) = \operatorname{diam}(G)$. Let $U \subseteq V(G)$. Then $G - U$ denotes the graph obtained from G by deleting all the vertices of U together with all the edges with at least one vertex in U. Set U is called a *cut-set* of G if $G - U$ has more components than G. The subgraph induced by a set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. A vertex of degree $p-1$ is called a *universal vertex*. A *self-complementary graph* is a graph which is isomorphic to its complement. The interval $I[u,v]$ consists of all vertices lying on some u-v geodesic of G. For $S \subseteq V$, $I[S] = \bigcup_{u,v \in S} I[u,v]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number*, $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [1] and further studied in [4, 7].

For a nonempty subset W of V, a Steiner W-tree T in G is a tree containing the vertices in W in such a way that $|V(T)|$ is minimum. There can be more than one Steiner W-tree of G for a nonempty subset W of V. The number of edges in a Steiner W-tree of G is called the *Steiner distance* of W and is denoted by $d(W)$. The Steiner distance of a connected graph G was studied in [6]. Let $W \subseteq V$ and T be a Steiner tree of W in G. If W has exactly two vertices u and v, then its Steiner W-tree is a shortest u-v path or a u-v geodesic. Hence, Steiner trees are natural generalization of geodesics. The Steiner interval $S(W)$ contains all vertices of a Steiner W-tree. A set $W \subseteq V$ is called a *Steiner set* if $S(W) = V$. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply a *s-set* and this cardinality is the *Steiner number*, $s(G)$ of G. The Steiner number of a graph was introduced in [3] and

---

[1] Corresponding author, e-mail: john@gcetly.ac.in.

further studied in [5, 7, 9 – 12, 14, 15, 17, 18]. The edge Steiner number of a graph was introduced and studied in [16]. The forcing geodetic and the forcing Steiner numbers of a graph was studied in [19]. In [20], it was proved that every minimum Steiner set is also a geodetic set of a graph having diameter 2. Later, it was proved that this particular result does not always hold in [8]. The upper restrained Steiner number of a graph was studied in [13]. Steiner tree problem has applications in design of computer circuits, long distance telephone lines, or mail routing, combinatorial optimization, etc. It is also used to construct roads of minimum total length to interconnect several highways.

The following theorems are used in the sequel.

Theorem 1.1 [3].

A. *Each extreme vertex of a graph G belongs to every geodetic (Steiner) set of G.*

B. *If G is a connected noncomplete graph of order p, then $s(G) \leq p - \kappa(G)$.*

C. *Let G be a connected graph of order $p \geq 2$. Then $s(G) = p$ if and only if $G = K_p$.*

D. *Let G be a connected graph of order $p \geq 3$. Then $s(G) = p - 1$ if and only if G contains a cut-vertex of degree $p - 1$.*

E. *Let G be a connected noncomplete graph. Then no cut-vertex of G belongs to any minimum Steiner set of G.*

Throughout the paper, G denotes a connected graph with at least two vertices.

2. Graphs with large Steiner number. The upper bound on Steiner number of a graph G in terms of order p and connectivity $\kappa(G)$ was given by Chartrand and Zhang as stated in Theorem 1.1,B.

In this section, the set of all connected graphs such that $s(G) = p - 2$ are characterized.

Lemma 2.1. *Let G be a noncomplete connected graph with a universal vertex v and W be a Steiner set of G. Then*: (i) $v \notin W$, (ii) $d(W) = |W|$.

Proof. Let v be the universal vertex of G and W be a Steiner set of G.

(i) On the contrary, suppose that $v \in W$. Then $\langle W \rangle$ is connected, which implies $S(W) = W \neq V$. Therefore, $v \notin W$.

(ii) Let x be a universal vertex of G. Then one of the Steiner W-trees T of G is star centered at x such that the end vertices of T are elements of W. Therefore, $V(T) = W \cup \{x\}$. Hence, it follows that $d(W) = |V(T)| = |N_T(x)| = |W|$.

Lemma 2.2. *Let G be a connected graph of order $p \geq 3$ and U be the set of cut-vertices of G. Then $s(G) \leq p - |U|$.*

Proof. This follows from Theorem 1.1, E.

Lemma 2.3. *Let G be a connected graph of order $p \geq 4$ with $s(G) = p - 2$. Then $\kappa(G) \leq 2$.*

Proof. This follows from Theorem 1.1, B.

Lemma 2.4. *Let G be a connected graph of order $p \geq 4$ with $s(G) = p - 2$ and W be a s-set of G. Let $V - W = \{u, v\}$. Then every element of W is adjacent to either u or v or both.*

Proof. By Lemma 2.3, $\kappa(G) \leq 2$. Let $V - W = \{u, v\}$.

Claim. Every element of W is adjacent to either u or v or both.

Case 1: $\kappa(G) = 1$. Therefore, $V - W = \{u, v\}$ is the set of cut-vertices of G. Let $H_1, H_2, \ldots, H_r$, $r \geq 2$, be the components of $G - u$. W.l.o.g., let H_1 be the component of $G - u$ containing y and $A_1 = N_{H_1}(x)$. Now let $W_1 = W - A_1$. Since $|W_1| < |W| = s(G)$, we have $S(W_1) \neq V$. Let $B_1 = V(H_1) - A_1 \neq \phi$. We consider two subcases.

Subcase 1.1. Every vertex of A_1 is adjacent to some vertex in B_1.

Let $Y_1, Y_2, \ldots, Y_k$ be the components of the subgraph $\langle B_1 \rangle$. If every vertex of A_1 is adjacent to some vertex of Y_i, $1 \leq i \leq k$, then each vertex of A_1 lies on some Steiner W_1-tree so that $S(W_1) =$

V, which is a contradiction. If not, $k \geq 2$, then there exists some vertex $a_1 \in A_1$ which is not adjacent to any vertex in Y_i for $1 \leq i \leq k$. Let $W_2 = W_1 \cup \{a_1\}$. Now we relabel the components of $\langle B_1 \rangle$ if needed so that $Y_1, Y_2, \ldots, Y_{t_1}$, $t_1 < k$, are the components of $\langle B_1 \rangle$ which contains no vertex adjacent to a_1. If every vertex in $A_2 = A_1 - \{a_1\}$ is adjacent to some vertex of Y_i, $1 \leq i \leq t_1$, then every vertex of A_2 lies on some Steiner W_2-tree so that $S(W_2) = V$. Since $s(G) \leq |W_2| < p - 3$, we get a contradiction. Therefore, there exists $a_2 \in A_2$ such that a_2 is not adjacent to any vertex in Y_i, $1 \leq i \leq t_1$. Let $W_3 = W_2 \cup \{a_2\} = W_1 \cup \{a_1, a_2\}$. Again we relabel the components Y_i, $1 \leq i \leq t_1$, so that $Y_1, Y_2, \ldots, Y_{t_2}$, $t_2 < t_1$, has no vertex adjacent to a_2. We proceed again for the set $A_3 = A_2 - \{a_2\}$. Since G is connected there exists an integer p with $1 \leq p \leq |A_1| - 1$ such that every vertex in $A_{p+1} = A_1 - \{a_1, a_2, \ldots, a_p\}$ is adjacent to some vertex in Y_i, $1 \leq i \leq t_p$. Let $W_{p+1} = W_1 \cup \{a_1, a_2, \ldots, a_p\}$. Then every vertex of A_{p+1} lies on some Steiner W_{p+1}-tree. This implies that W_{p+1} is a Steiner set of G and so $s(G) \leq |W_{p+1}| < |W|$.

Subcase 1.2. There is a vertex of A_1 which is adjacent to no vertex in A_2.

Let $A_1 = A_{11} \cup A_{12}$ with $A_{11} = \{x_1, x_2, \ldots, x_a\}$ and $A_{12} = \{y_1, y_2, \ldots, y_b\}$ such that every vertex in A_{11} is adjacent to some vertex in A_2 and no vertex in A_{12} is adjacent to any vertex in A_2. Then $|a| \geq 1$ and $|b| \geq 1$. Let $G' = G - A_{12}$. Then G' is of order $p - b$. By the similar argument as in Case 1 to G', we get an s-set W' of G' with $|W'| \leq p - b - 2$. Now $W'' = W' \cup A_{12}$ is a complement connected Steiner set of G. Since $|W''| \leq p - 2$, we get a contradiction. Therefore, every element of W is adjacent to u.

Case 2: $\kappa(G) = 2$. Therefore, $V - W = \{u, v\}$ is the minimum cutset of G. By the similar argument as in Case 1, we can prove that every element of W is adjacent to u. Similarly we can prove that every element of W is adjacent to v or both u and v. Hence, the claim.

Lemma 2.5. *Let G be a noncomplete connected graph of order $p \geq 4$ and $s(G) = p - 2$. Then $d \leq 3$.*

Proof. If $p = 4$ or 5, then $d \leq 3$. So we assume that $p \geq 6$. Let W be an s-set of G. By Lemma 2.3, $\kappa(G) \leq 2$.

Case 1: $\kappa(G) = 1$. Then, by Lemma 2.2, G contains exactly two cut-vertices, say u and v. By Lemma 2.4, each element in W is adjacent to either u or v or both. Therefore, $d \leq 3$.

Case 2: $\kappa(G) = 2$. Let $S = \{u, v\}$ be the minimum cutset of G. Let $G_1, G_2, \ldots, G_r$, $r \geq 2$, be the components of $G - S$. Let W be an s-set of G.

Subcase 2.1. Suppose that G contains universal vertices. Since G is noncomplete, $d = 2 \leq 3$.

Subcase 2.2. Suppose that G contains no universal vertices. Let $G_1, G_2, \ldots, G_r$ be the components of $G - S$. Suppose that $uv \notin E$. Since u and v are adjacent to each element of G_i, $1 \leq i \leq r$, and $p \geq 6$, $|V(G_i)| \geq 2$ for some i, $1 \leq i \leq r$. Let $W_1 = \{u, v\}$. Then W_1 is a Steiner set of G, so that $s(G) = 2 \leq p - 4$, which is a contradiction. Therefore, $uv \in E$. Hence, $d \leq 3$.

Lemma 2.6. *There is no connected graph G of order $p \geq 4$ with $s(G) = p - 2$, $\kappa(G) = 1$ and $d = 2$.*

Proof. Suppose that there exists a graph G with $s(G) = p - 2$ and $\kappa(G) = 1$. If there are more than three cut-vertices, then, by Lemma 2.3, $s(G) \leq p - 3$, which is a contradiction. Therefore, G contains one or two cut-vertices.

Case 1. G contains one cut-vertex, say v. Then, by Theorem 1.1, D, v is not a universal vertex of G. Let u and w be two adjacent vertices of v. Since v is not a universal vertex of G, there exists a vertex x such that x is adjacent to either u or w. Without loss of generality, let us assume that x is adjacent to u. Since v is a cut-vertex of G, w is not adjacent to u or x or both. This implies $d \geq 3$, which is a contradiction.

Case 2. G contains two cut-vertices, say u and v.

Subcase 2.1. Suppose that u and v are adjacent. Since u is a cut-vertex of G, there exists a vertex $x \neq v$ such that x is adjacent to u. Also, since v is a cut-vertex of G, there exists $y \neq u$ such that y is adjacent to v. If x and y are adjacent, then u and v are not cut-vertices of G. This implies $d \geq 3$, which is a contradiction.

Subcase 2.2. Assume that u and v are not adjacent. Since G is connected, there exists a path P of length at least two. Since u and v are cut-vertices, there exist vertices $x \neq v$ and $y \neq u$ such that $xu, yv \in E - E(P)$. Also, x and y are not adjacent, which implies $d \geq 4$ which is a contradiction.

Therefore, there is no connected graph G of order $p \geq 4$ with $s(G) = p - 2$, $\kappa(G) = 1$ and $d = 2$.

Lemma 2.7. *Let G be a connected graph of order $p \geq 4$ with $d = 2$ and $s(G) = p - 2$. Then $\kappa(G) = 2$.*

Proof. This follows from Lemmas 2.2 and 2.3.

Lemma 2.8. *Let G be a connected noncomplete graph of order $p \geq 4$. If G contains at least one universal vertex, then $\kappa(G) = 2$.*

Proof. Since G contains a universal vertex, $d = 2$. Therefore, the result follows from Lemma 2.4.

Lemma 2.9. *Let G be a noncomplete connected graph of order $p \geq 4$ with at least one universal vertex. Then $s(G) = p - 2$ if and only if $\kappa(G) = 2$ and G contains one or two universal vertices.*

Proof. Let $s(G) = p-2$. By Lemma 2.4, G contains one or two universal vertices. By Lemma 2.3, $\kappa(G) = 1$ or 2. Since $s(G) = p - 2$, $\kappa(G) \neq 1$. Therefore, $\kappa(G) = 2$.

Conversely, let $\kappa(G) = 2$ and G contains one or two universal vertices. We prove that $s(G) = p-2$. Let $S = \{u, v\}$ be the minimum cutset of G. By Lemma 2.1, G contains either one or two universal vertices. Let $G_1, G_2, \ldots, G_r$, $r \geq 2$, be the components of $G - S$. Let W be a Steiner set of G.

Case 1. G contains two universal vertices.

Let x, y be the two universal vertices of G. Then, by Observation 2.1, $x, y \notin W$. Hence, it follows that $x = u$ and $y = v$. Let $W = V - \{u, v\}$. Then W is a Steiner set of G so that $s(G) \leq p-2$. We prove that $s(G) = p - 2$. On the contrary, suppose that $s(G) \leq p - 3$, then there exists a Steiner set W' such that $|W'| \leq p - 3$. Let z be a vertex of G such that $z \in W$ and $z \notin W'$ and $z \in S(W')$. By Observation 2.1, $u, v \notin W'$. Let T be a Steiner W'-tree of G such that $z \in V(T)$. W.l.o.g., let us assume that $u \in V(T)$. Let $N_T(z) = \{u, z_1, z_2, z_3, \ldots, z_s\}$, where $s \geq 2$. Since u is a universal vertex of G, $uz_i \in E(G)$ for $1 \leq i \leq s$. Let T' be a tree obtained from T by deleting the vertex z and joining the edge uz_i, $1 \leq i \leq s$. Then T' is a Steiner W'-tree with $|V(T')| = |V(T)| - 1$, which is a contradiction to W' is a Steiner set of G. Therefore, $s(G) = p - 2$.

Case 2. G contains exactly one universal vertex.

Let x be the universal vertex of G. Then by Lemma 2.1, $x \notin W$. Then $x = u$ or v. W.l.o.g., let us assume that $x = u$. Let $W = V - \{x, v\}$. Then W is a Steiner set of G so that $s(G) \leq p-2$. We prove that $s(G) = p - 2$. On the contrary, suppose that $s(G) \leq p - 3$. We prove that $\kappa(G - x) = 2$. If not $\kappa(G - x) = 1$. Let $N_{\langle G-x\rangle}(y) = \{y_1, y_2, \ldots, y_n\}$. We prove that at least one y_i, $1 \leq i \leq n$, not belongs to W. If not $y_i \in W$ for all i, $1 \leq i \leq n$. Then $d(W) \geq p - 2$, which implies $|W| \geq p - 2$, which is a contradiction. Then $\kappa(G - x) = 2$. Hence, it follows that $\kappa(G) = 3$, which is a contradiction. Hence, $s(G) = p - 2$.

Lemma 2.10. *Let G be a connected graph of order $p \geq 5$ with $d = 3$ and $s(G) = p - 2$. Then there exist only two adjacent cut-vertices or there is a minimum cutset $W = \{u, v\}$ such that u and v are adjacent.*

Proof. Since $s(G) = p - 2$, by Theorem 1.1, B, $\kappa(G) \le 2$. Let $\kappa(G) = 1$. Since $s(G) = p - 2$, G contains exactly two cut-vertices, say u and v. Suppose that u and v are not adjacent. Since G is connected, there exists a path P between u and v of length at least 2. Let z be a vertex on the path P. Since u is a cut-vertex of G, there exists a vertex $x \neq v$ such that x is adjacent to u. Also, since v is a cut-vertex of G, there exists a vertex $y \neq x$ such that y is adjacent to v. Also, since u and v are cut-vertices of G, u is not adjacent to y and v is not adjacent to x and $xy \notin E(G)$, which implies $d \ge 4$. This is a contradiction. Therefore, u and v are adjacent.

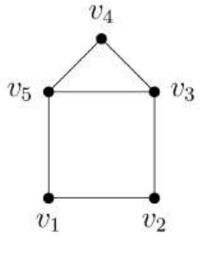
Suppose that $\kappa(G) = 2$. Let $W = \{u, v\}$ be a minimum cutset of G.

Suppose that u and v are not adjacent. Let $G_1, G_2, \ldots, G_r$, $r \ge 2$, be the components of $G - W$. Since $d = 3$, at least one component of $G - W$ has more than one vertex. Let G_1 has two vertices, say x and y and G_2 has one vertex, say z. Since u and v are cut-vertices, u and v are adjacent to z and u and v are adjacent to x or y or both. If u is adjacent to both x and y, then $\{x, y\}$ is a minimum cutset of G, such that x and y are adjacent, which is a contradiction. Hence, u and v are adjacent.

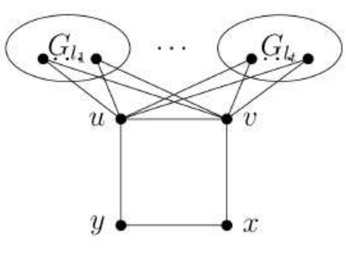
Suppose that u is adjacent to x and not adjacent to y. Then x is a cut-vertex of G, which is not so. If u is adjacent to y and not adjacent to x, then y is a cut-vertex of G, which is a contradiction. Therefore, each $G_i (1 \le i \le k)$ has at least two vertices. Let G_1 contains two vertices and G_2 contains two vertices. Let $V(G_1) = \{x, y\}$ and $V(G_2) = \{w, z\}$. Then $W = \{u, v\}$ is a Steiner set of G so that $s(G) = 2$, which is a contradiction. Thus, u and v are adjacent.

To characterize connected graphs of order $p \ge 4$ with $s(G) = p - 2$, we introduce the following families of graphs (see Fig. 2.1).

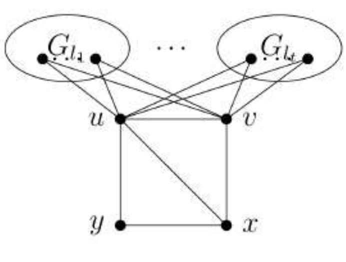
1. Let $\Im_1$ be the collection of all graphs obtained from a cycle $C_4 \colon u, v, x, y, u$ and graphs $G_{l_1}, \ldots, G_{l_t}$ by joining u and v to all vertices of $G_{l_1}, \ldots, G_{l_t}$.
2. Let $\Im_2$ be the collection of all graphs obtained from $\Im_1$ by introducing the edge ux.
3. Let $\Im_3$ be the collection of all graphs obtained from $\Im_1$ by introducing the edges ux and vy.
4. Let $\Im_4$ be the collection of all graphs obtained from $\Im_2$ by removing the edge uv.
5. Let $\Im_5$ be the collection of all graphs obtained from $K_2 : u, v$ and two classes of graphs $G_{m_1}, \ldots, G_{m_r}$ and $G_{n_1}, \ldots, G_{n_w}$ by joining u to all vertices of $G_{m_1}, \ldots, G_{m_q}$ and v to all vertices of $G_{n_1}, \ldots, G_{n_w}$.
6. Let $\Im_6$ be the collection of all graphs obtained from $\Im_5$ and the graphs $G_{l_1}, \ldots, G_{l_t}$ by joining u and v to all vertices of $G_{l_1}, \ldots, G_{l_t}$.
7. Let $\Im_7$ be the collection of all graphs obtained from $\Im_6$ such that there exists edges $x_l y_l$ for some or all l between G_{m_k}, $1 \le k \le r$, and G_{n_i}, $1 \le i \le w$.



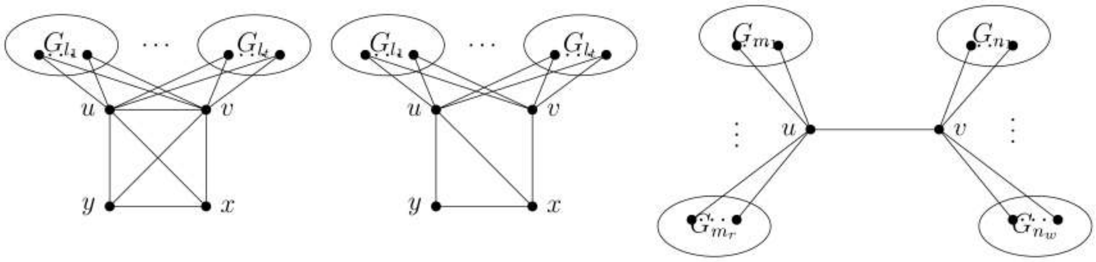
G_1



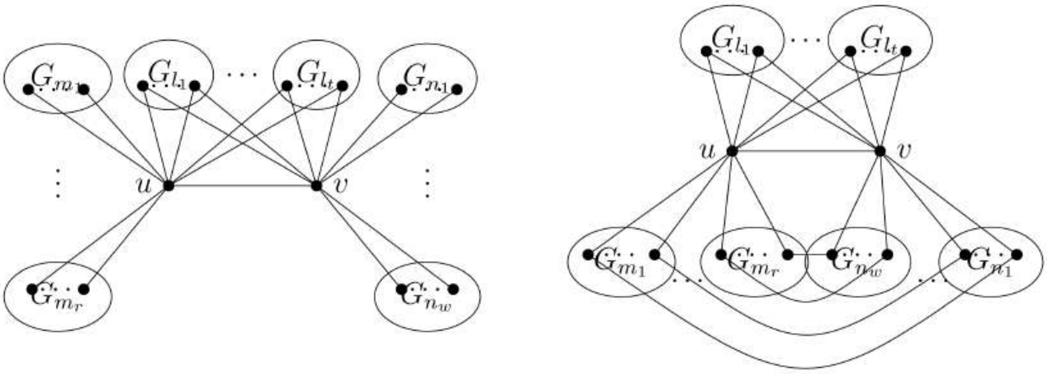
Family $\Im_1$



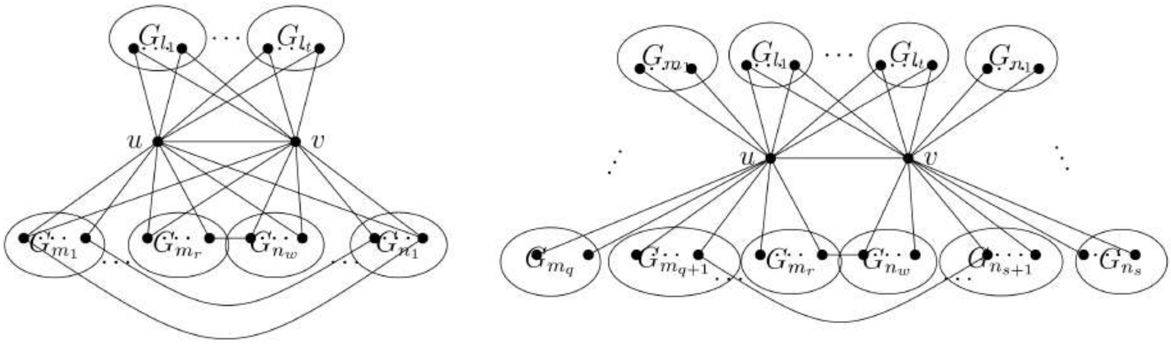
Family $\Im_2$



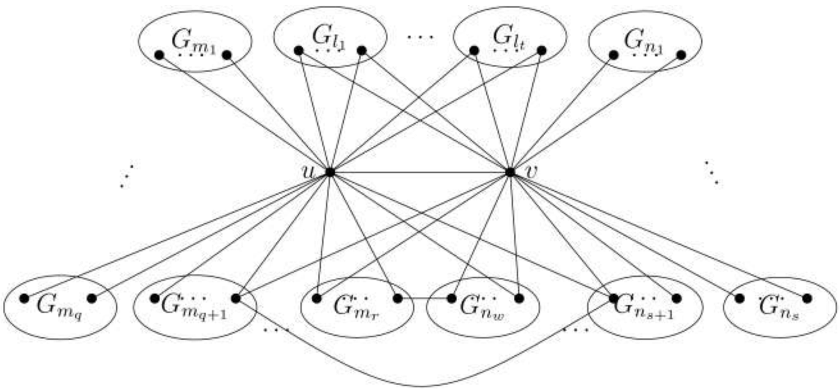
Family $\Im_3$ Family $\Im_4$ Family $\Im_5$



Family $\Im_6$ Family $\Im_7$



Family $\Im_8$ Family $\Im_9$



Family $\Im_{10}$

Fig. 2.1

8. Let $\Im_8$ be the collection of all graphs obtained from $\Im_7$ such that the vertex u is adjacent to each element of G_{n_i}, $1 \leq i \leq w$, and the vertex v is adjacent to each element of G_{m_k}, $1 \leq k \leq r$.

9. Let $\Im_9$ be the collection of all graphs obtained from $\Im_6$ such that for $x_l \in G_{m_k}$, $q+1 \leq k \leq r$, and $y_l \in G_{n_i}$, $s+1 \leq i \leq w$, there exist edges $x_l y_l$ for some or all l, where l is arbitrary.

10. Let $\Im_{10}$ be the collection of all graphs obtained from $\Im_9$ such that the vertex u is adjacent to some or all elements of G_{n_i}, $s+1 \leq i \leq w$, and the vertex v is adjacent to some or all elements of G_{m_k}, $q+1 \leq k \leq r$.

Theorem 2.1. *Let G be a connected graph of order $p \geq 4$ with diameter $d \leq 3$. Then $s(G) = p-2$ if and only if $G \in \Im_1 \cup \Im_2 \cup \Im_3 \cup \Im_4 \cup \Im_5 \cup \Im_6 \cup \Im_7 \cup \Im_8 \cup \Im_9 \cup \Im_{10}$ or G is a connected graph with $\kappa(G) = 2$ and having one or two universal vertices.*

Proof. Let $s(G) = p-2$. By Lemma 2.4, $\kappa(G) = 2$. If $p = 4$, then $G = C_4$ or $G = K_4 - \{e\}$ satisfies the requirements of the theorem. If $p = 5$, then the graphs with $s(G) = p-2$ are $G = C_5$, $G = G_1$ and the graph G that contains at least one universal vertex with $\kappa(G) = 2$. Therefore, we assume that $p \geq 6$.

Case 1: $\kappa(G) = 1$.

Subcase 1.1: $d = 2$. Then, by Lemma 2.6, no such graph exists.

Subcase 1.2: $d = 3$. By Lemma 2.10, G contains two cut-vertices, say u and v, such that u and v are adjacent. Let W be an s-set of G with $|W| = p-2$. Let $G_1, G_2, \ldots, G_r$, $r \geq 2$, be the components of $G - \{u, v\}$. Then, by Lemma 2.4, each element of G_i, $1 \leq i \leq r$, is adjacent to either u or v or both. If each element of G_i, $1 \leq i \leq l$, is adjacent to u and each element of G_j, $l+1 \leq j \leq r$, is adjacent to v, then $G \in \Im_5$. So we are done. Suppose that each element of G_i, $1 \leq i \leq l$, is adjacent to u and each element of G_j, $l+1 \leq j \leq m$, is adjacent to both u and v, and each element of G_k, $m+1 \leq k \leq r$, is adjacent to v. Then $G \in \Im_6$. Suppose that there exist edges $x_l y_l$ for all l between G_{m_k}, $q+1 \leq k \leq r$, and G_{n_i}, $s+1 \leq i \leq w$, where $q = w$ and $G_{m_{q+1}}, \ldots, G_{m_r}, G_{n_1}, \ldots, G_{n_w}, G_{l_1}, \ldots, G_{l_t} \in G_r$ in $G - \{u\}$. Then $G \in \Im_9$. If some or all elements of G_{n_i}, $s+1 \leq i \leq w$, is adjacent to u and some or all elements of G_{m_k}, $q+1 \leq k \leq r$, is adjacent to v, then $G \in \Im_{10}$.

Case 2: $\kappa(G) = 2$. Let $S = \{u, v\}$ be the minimum cutset of G.

Subcase 2.1: $d = 2$. Let W be an s-set of G with $|W| = p-2$. Let $G_1, G_2, \ldots, G_r$, $r \geq 2$, be the components of $G - S$. Then, by Lemma 2.4, each element of G_i, $1 \leq i \leq r$, is adjacent to either u or v or both. If there exist components G_i, $r \geq 2$, such that each element of G_i, $r \geq 2$, is adjacent to both u and v, an edge $xy \in G$ such that $ux, vy \in E$, then $G \in \Im_1$. By Lemma 2.9, if G contains one universal vertex, then $G \in \Im_2 \cup \Im_3 \cup \Im_4$. Suppose that there exists edges $x_l y_l$ for some or all l between G_{m_k}, $1 \leq k \leq r$, and G_{n_i}, $1 \leq i \leq w$, and also the vertex u is adjacent to each element of G_{n_i}, $1 \leq i \leq w$, and the vertex v is adjacent to each element of G_{m_k}, $1 \leq k \leq r$. Then $G \in \Im_8$.

Subcase 2.2: $d = 3$. Let W be an s-set of G with $|W| = p-2$. Let $G_1, G_2, \ldots, G_r$, $r \geq 2$, be the components of $G - S$. Then, by Lemma 2.4, each element of G_i, $1 \leq i \leq r$, is adjacent to either u or v or both. Suppose that each element of G_{m_i}, $1 \leq i \leq r$, is adjacent to u and each element of G_{l_j}, $1 \leq j \leq t$, is adjacent to both u and v, and each element of G_{n_k}, $1 \leq k \leq w$, is adjacent to v such that $i = k$, and there exist at least one edge between G_{m_i}, $1 \leq i \leq r$, and G_{n_k}, $1 \leq k \leq w$. Then $G \in G \in \Im_7$.

Conversely, if $G \in \Im_1 \cup \Im_2 \cup \Im_3 \cup \Im_4 \cup \Im_5 \cup \Im_6 \cup \Im_7 \cup \Im_8 \cup \Im_9 \cup \Im_{10}$ or G is a connected graph with $\kappa(G) = 2$ and having one or two universal vertices, then $s(G) = p-2$.

Proposition 2.1. *For any connected graph G, $|g(G) - s(G)|$ can be arbitrarily large.*

Proof. *Case* 1: $g(G) < s(G)$.

Let $P_i : u_i, v_i, w_i,\ 1 \leq i \leq a$, be a copy of path on three vertices. Let G be the graph obtained from $P_i,\ 1 \leq i \leq a$, by introducing a vertex x and the edges xu_i, xv_i and $xw_i,\ 1 \leq i \leq a$. Let $S = \{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_a\}$ be the set of all extreme vertices of G and so $g(G) \geq 2a$. By Theorem 1.1, A, S is a subset of every geodetic set of G. Since $I[S] = V$, S is a geodetic set of G so that $g(G) = 2a$. Since x is a universal vertex of G, which is a cut-vertex, by Theorem 1.1, D, we have $s(G) = 3a$. Therefore, $s(G) - g(G) = 3a - 2a = a$.

Case 2: $s(G) < g(G)$.

Let $C_i : u_i, v_i, w_i, u_i,\ 1 \leq i \leq a$, be a copy of the cycle C_3. Let R_i be the graph obtained from $C_i,\ 1 \leq i \leq a$, by introducing new vertices $p_i, q_i, r_i, s_i,\ 1 \leq i \leq a$, and the edges $u_ip_i,\ p_iq_i,\ q_ir_i,\ r_iw_i,\ q_is_i,\ s_iv_i,\ 1 \leq i \leq a$. Let G be the graph obtained from $R_i,\ 1 \leq i \leq a$, by joining u_{i-1} with $u_i,\ 2 \leq i \leq a$. Let $W = \{r_i, s_i\},\ 1 \leq i \leq a$. Since $S(W) = V$, W is a Steiner set of G so that $s(G) = 2a$. Since the vertices $p_i, q_i,\ 1 \leq i \leq a$, do not lie on any geodesic joining any pair of vertices of W, W is not a geodetic set of G. Let $S = W \cup \{p_i\},\ 1 \leq i \leq a$. Since $I[S] = V$, S is a geodetic set of G so that $g(G) = 3a$. Therefore, $g(G) - s(G) = 3a - 2a = a$.

3. Some Nordhaus – Gaddum bounds. During the year 1956, Nordhaus and Gaddum studied the chromatic number in a graph G and in its complement $\overline{G}$ together. They proved lower and upper bounds on the sum and on the product of $\chi(G)$ and $\chi(\overline{G})$ in terms of the order p of G. Since then any bound on the sum and/or the product of an invariant in a graph G and the same invariant in the complement $\overline{G}$ of G is called a Nordhaus – Gaddum type inequality. In the following theorem, we study the relations of Nordhaus – Gaddum in obtaining the upper and lower bounds for sum and product for the Steiner number of a graph G and its complement $\overline{G}$.

Theorem 3.1. *Let G and $\overline{G}$ be connected graphs of order $p \geq 4$. Then*:

(i) $4 \leq s(G) + s(\overline{G}) \leq 2p - 4$,

(ii) $2 \leq \sqrt{s(G)s(\overline{G})} \leq p - 2$.

Moreover both upper bounds are sharp if and only if G is either P_4 or C_5.

Proof. (i) Since G and $\overline{G}$ are connected graphs, we have $s(G) \geq 2$ and $s(\overline{G}) \geq 2$ and so $s(G) + s(\overline{G}) \geq 4$. By Theorem 1.1, C, we know that $s(G) = p$ if and only if $G = K_p,\ p \geq 2$. Also, by Theorem 1.1, D, we have $s(G) = p - 1$ if and only if G contains a cut-vertex of degree $p - 1$. Therefore, if $s(G) = p$ or $p - 1$, then $\overline{G}$ is disconnected. So, $s(G) \leq p - 2$, implies $s(\overline{G}) \leq p - 2$. Hence, $s(G) + s(\overline{G}) \leq 2p - 4$. Thus, $4 \leq s(G) + s(\overline{G}) \leq 2p - 4$.

(ii) By the similar argument as in (i), we can prove that $2 \leq \sqrt{s(G)s(\overline{G})} \leq p - 2$.

Also, both upper bounds are sharp if and only if $s(G) = p - 2$ and $s(\overline{G}) = p - 2$,

if and only if G and $\overline{G}$ are self complementary,

if and only if G is either P_4 or C_5.

Remark **3.1.** The inequalities in Theorem 3.1 can be strict. Consider the graph G given in Fig. 3.1. Clearly $G \in \Im_1$. So we have $s(G) = p - 2$. But $W = \{v_3, v_6\}$ is an s-set of $\overline{G}$, which implies $s(\overline{G}) = 2$. Hence, $s(G) + s(\overline{G}) = p < 2p - 2$.

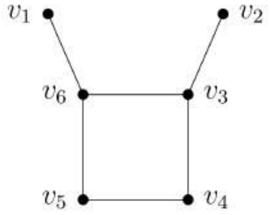


Fig. 3.1

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

Funding. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Author contributions. All authors have contributed equally to the work.

References

1. F. Buckley, F. Harary, *Distance in graphs*, Addison-Wesley, Redwood City, CA (1990).
2. G. Chartrand, L. Lesniak, *Graphs and digraphs*, Fourth ed., Chapman and Hall/CRC, Boca Raton, FL (2005).
3. G. Chartrand, P. Zhang, *The Steiner number of a graph*, Discrete Math., **242**, 41 – 54 (2002).
4. G. Chartrand, F. Harary, P. Zhang, *On the geodetic number of a graph*, Networks, **39**, 1 – 6 (2002).
5. M. Changat, A. K. Lakshmikuttyamma, J. Mathews, *A note on* 3*-Steiner intervals and betweeness*, Discrete Math., **311**, № 22, 2601 – 2609 (2011).
6. G. Chartrand, O. R. Oellermann, S. Tian, H. B. Zou, *Steiner distance in graphs*, Çasopis pro pęstoánímatematiky, **114**, 399 – 410 (1989).
7. C. Hernando, T. Jiang, M. Mora, I. M. Pelayo, C. Seara, *On the Steiner, geodetic and hull number of graphs*, Discrete Math., **293**, 139 – 154 (2005).
8. J. John, *Comment on “Analogies between the geodetic number and the Steiner number of some classes of graphs”*, Filomat, **37**, № 2, 585 – 589 (2023).
9. J. John, *The vertex Steiner number of a graph*, Trans. Combin., **9**, № 2, 115 – 124 (2020); DOI: 10.22108/TOC.2020.116191.1628.
10. J. John, *The total Steiner number of a graph*, Discrete Math., Algorithms and Appl., **12**, № 3 (2020); DOI: 10.1142/S179383092050038X.
11. J. John, *On the vertex monophonic, vertex geodetic and vertex Steiner numbers of graphs*, Asian-Eur. J. Math., **14**, № 10, Article 2150171 (2021); DOI: 10.1142/S1793557121501710.
12. J. John, M. S. Malchijah Raj, *On the complement connected Steiner number of a graph*, Acta Math. Univ. Comenian., **90**, № 4, 377 – 389 (2021).
13. J. John, M. S. Malchijah Raj, *The upper restrained Steiner number of a graph*, Discrete Math., Algorithms and Appl., **12**, № 1 (2020); DOI:10.1142/S1793830920500044.
14. K. M. Kathiresan, S. Arockiaraj, R. Gurusamy, K. Amutha, *On the Steiner radial number of graphs*, Int. Workshop Combin. Algorithms, 65 – 72 (2012).
15. I. M. Pelayo, *Comment on “The Steiner number of a graph” by G. Chartrand and P. Zhang*, Discrete Math., **280**, 259 – 263 (2004).
16. A. P. Santhakumaran, J. John, *The edge Steiner number of a graph*, J. Discrete Math. Sci. and Cryptogr., **10**, 677 – 696 (2007).
17. A. P. Santhakumaran, J. John, *The upper Steiner number of a graph*, Graph Theory Notes of New York, **59**, 9 – 14 (2010).
18. A. P. Santhakumaran, J. John, *The forcing Steiner number of a graph*, Discuss. Math. Graph Theory, **31**, 171 – 181 (2011).
19. A. P. Santhakumaran, J. John, *The forcing geodetic and the forcing Steiner numbers of a graph*, Discuss. Math. Graph Theory, **31**, 611 – 624 (2011).
20. I. G. Yero, J. A. Rodriguez-Velazquez, *Analogies between the geodetic number and the Steiner number of some classes of graphs*, Filomat, **29**, № 8, 1781 – 1788 (2015).

Received 13.12.22