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MEROMORPHIC FUNCTIONS SHARING THREE VALUES WITH THEIR SHIFT

МЕРОМОРФНІ ФУНКЦІЇ, ЩО ПОДІЛЯЮТЬ ТРИ ЗНАЧЕННЯ З ЇХНІМ ЗСУВОМ

We discuss the problem of uniqueness of a meromorphic function $f(z)$, which shares $a_1(z)$, $a_2(z)$, and $a_3(z)$ CM with its shift $f(z+c)$, where $a_1(z)$, $a_2(z)$, and $a_3(z)$ are three c-periodic distinct small functions of $f(z)$ and $c \in \mathbb{C} \setminus \{0\}$. The obtained result improves the recent result of Heittokangas et al. [Complex Var. and Elliptic Equat., **56** № 1–4, 81–92 (2011)] by dropping the assumption about the order of $f(z)$. In addition, we introduce a way of characterizing elliptic functions in terms of meromorphic functions sharing values with two of their shifts. Moreover, we show by a number of illustrating examples that our results are, in certain senses, best possible.

Обговорено проблему єдиності мероморфної функції $f(z)$, що має спільні $a_1(z)$, $a_2(z)$ і $a_3(z)$ CM зі своїм зсувом $f(z+c)$, де $a_1(z)$, $a_2(z)$ і $a_3(z)$ – три c-періодичні різні малі функції від $f(z)$ та $c \in \mathbb{C} \setminus \{0\}$. Отриманий результат покращує останній результат Heittokangas та ін. [Complex Var. and Elliptic Equat., **56** № 1–4, 81–92 (2011)], оскільки відкинуто припущення про порядок $f(z)$. Крім того, запропоновано спосіб характеризації еліптичних функцій у термінах мероморфних функцій, що поділяють значення з двома своїми зсувами. Насамкінець, на основі розгляду низки ілюстративних прикладів, показано, що одержані результати є найкращими у певному сенсі.

1. Introduction and main result. In this paper, a meromorphic function f always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notation and main results of Nevanlinna theory (see, e.g., [5, 13]). By $S(r,f)$ we denote any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside of an exceptional set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$, and we denote by $S(f)$ the set of functions which are small compared to $f(z)$. Let $\hat{S}(f) = S(f) \cup \{\infty\}$. Moreover, we use notations $\rho(f)$ and $\rho_1(f)$ for the order and the hyper-order of a meromorphic function f, respectively. We know that if $f = e^g$, where g is a nonconstant entire function, then $\rho_1(f) = \rho(g)$ (see Theorem 1.45 [13]). For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z+c)$.

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (IM), if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros counting multiplicities (ignoring multiplicities).

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. Denote by $\overline{N}_{(2}(r,a;f)$ the reduced counting function of multiple a-points of f. Also, we denote $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f)$. We define $\overline{N}(r,a;f) = \overline{N}(r,f)$, if $a = \infty$.

Uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. The following two classical results, due to R. Nevanlinna [9], have prompted research activity on shared value problems up until today.

Theorem A [9]. *If two nonconstant meromorphic functions $f(z)$ and $g(z)$ share five distinct values IM, then $f(z) \equiv g(z)$.*

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Theorem B [9]. *If two meromorphic functions $f(z)$ and $g(z)$ share four distinct values CM, then $f(z) \equiv g(z)$ or $f(z) \equiv T \circ g(z)$, where T is a Möbious transformation.*

Rubel and Yang [11] considered the uniqueness of an entire function and its derivative. In 1977, they proved the following result.

Theorem C [11]. *Let $f(z)$ be a nonconstant entire function and $a, b \in \mathbb{C}$ such that $b \neq a$. If $f(z)$ and $f'(z)$ share a and b CM, then $f(z) \equiv f'(z)$.*

In recent years, meromorphic solutions of complex difference equations and the uniqueness of complex differences have become an area of current interest and the study is based on the Nevanlinna value distribution of difference operators established by Halburd and Korhonen [3] and by Chiang and Feng [1], respectively. Recently, many authors (see [6–8, 15]) have started to consider the sharing values problems of meromorphic functions with their difference operators or shifts. Now, we recall the following result due to Heittokangas et al. [7], which is shift analogue of Theorem C.

Theorem D [7]. *Let $f(z)$ be a nonconstant meromorphic function such that $\rho(f) < \infty$ and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $a_1, a_2, a_3 \in \hat{S}(f)$ with period c CM, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.*

Regarding Theorem D, one may ask the following question.

***Question* 1.** What can be said about the conclusion of Theorem D, if we omit the condition "$\rho(f) < \infty$" ?

We prove the following result to resolve Question 1 completely.

Theorem 1.1. *Let $f(z)$ be a nonconstant meromorphic function and let $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ and $f(z+c)$ share three distinct c-periodic functions $a_1, a_2, a_3 \in \hat{S}(f)$ CM, then one of the following cases holds*:

(1) $f(z) \equiv f(z+c)$;

(2) $f(z) \equiv f(z+2c)$ *if* $\rho_1(f) \geq 1$.

Following two corollaries are immediately from Theorem 1.1.

***Corollary* 1.1.** *Let $f(z)$ be a nonconstant meromorphic function such that $\rho_1(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ and $f(z+c)$ share three distinct c-periodic functions $a_1, a_2, a_3 \in \hat{S}(f)$ CM, then $f(z) \equiv f(z+c)$.*

***Corollary* 1.2.** *Let $f(z)$ be a nonconstant entire function and let $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ and $f(z+c)$ share two distinct c-periodic functions $a_1, a_2 \in S(f)$ CM, then one of the following cases holds*:

(1) $f(z) \equiv f(z+c)$;

(2) $f(z) \equiv f(z+2c)$ *if* $\rho_1(f) \geq 1$.

Following examples show the existence of conclusion (2) of Theorem 1.1.

***Example* 1.1.** Let $f(z) = \dfrac{1+e^{\alpha(z)}}{1-e^{\alpha(z)}}$, where $\alpha(z) = \sin z$ and $c = \pi$. Note that $T(r,f) = T(r,e^{\alpha}) + O(1)$ and so $\rho_1(f) = \rho(\alpha) = 1$. Clearly, $f(z)$ and $f(z+c)$ share $1, -1, \infty$ CM. Here, $f(z) \equiv f(z+2c)$ and so conclusion (2) of Theorem 1.1 holds.

***Example* 1.2.** Let $f(z) = \dfrac{2e^{2z} - e^{-2z}(1-e^{\cosh z})}{1+e^{\cosh z}}$ and $c = \pi i$. Note that $\rho_1(f) = \rho(\cosh z) = 1$ and e^{2z} and e^{-2z} are πi-periodic small functions of $f(z)$. Clearly, $f(z)$ and $f(z+c)$ share e^{2z}, e^{-2z} and ∞ CM. Here, $f(z) \equiv f(z+2c)$ and so conclusion (2) of Theorem 1.1 holds.

***Example* 1.3.** Let $f(z) = \dfrac{1}{2}\{e^{2z} + e^{-2z} + (e^{-2z} - e^{2z})e^{-\sinh z}\}$ and $c = \pi i$. Note that $\rho_1(f) = \rho(\sinh z) = 1$ and e^{2z} and e^{-2z} are πi-periodic small functions of $f(z)$. Clearly, $f(z)$ and $f(z+c)$ share e^{2z}, e^{-2z} and ∞. Here, $f(z) \equiv f(z+2c)$ and so conclusion (2) of Theorem 1.1 holds.

Following examples show that the number of shared values cannot be reduced to two in Theorem 1.1.

***Example* 1.4.** Let $f(z) = e^z \sin z + 1$ and $c = 2\pi$. Clearly, $f(z)$ and $f(z+c)$ share 1 and ∞ CM, but $f(z)$ does not satisfy any case of Theorem 1.1.

***Example* 1.5.** Let $f(z) = \dfrac{e^z+1}{e^z-1}$, $e^c = -1$, $a_1 = 1$ and $a_2 = -1$. Clearly, $f(z)$ and $f(z+c)$ share a_1 and a_2 CM, but $f(z)$ does not satisfy any case of Theorem 1.1.

***Example* 1.6.** Let $f(z) = \dfrac{e^z \sin z}{\sin z - 1} + 1$ and $c = 2\pi$. Clearly, $f(z)$ and $f(z+c)$ share 1 and ∞ CM, but $f(z)$ does not satisfy any case of Theorem 1.1.

We observe from Theorem 1.1 that if we drop the assumption "$\rho(f) < \infty$" in Theorem D, then we get two conclusions. Therefore, if $f(z)$ and $f(z+c)$ share $a_1(z), a_2(z)$ and $a_3(z)$ CM and we want to determine the concrete relation between $f(z)$ and $f(z+c)$, i.e., $f(z) \equiv f(z+c)$, it is necessary to add another condition(s) in Theorem 1.1. It is obvious that if we assume $\rho_1(f) < 1$ in Theorem 1.1, then we get $f(z) \equiv f(z+c)$.

In the next result, we show that if we assume $\sum_{j=1}^{3} \overline{N}_{(2}(r, a_j; f) \neq S(r,f)$ in Theorem 1.1, then we have $f(z) \equiv f(z+c)$.

Theorem 1.2. *Let $f(z)$ be a nonconstant meromorphic function and let $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ and $f(z+c)$ share three distinct c-periodic functions $a_1, a_2, a_3 \in \hat{S}(f)$ CM and $\sum_{j=1}^{3} \overline{N}_{(2}(r, a_j; f) \neq S(r,f)$, then $f(z) \equiv f(z+c)$.*

Following examples ensure the necessity of the condition "$\sum_{j=1}^{3} \overline{N}_{(2}(r, a_j; f) \neq S(r,f)$" in Theorem 1.2.

***Example* 1.7.** Let $f(z) = e^{\sin(\frac{\pi z}{c})}$, $c \in \mathbb{C} \setminus \{0\}$. Since $\overline{N}_{(2}\Big(r, 0; \sin\Big(\dfrac{\pi z}{c}\Big)\Big) + \overline{N}_{(2}\Big(r, 0; \cos\Big(\dfrac{\pi z}{c}\Big)\Big) = 0$, we have $\overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, 0; f) + \overline{N}_{(2}(r, 1; f) = S(r,f)$. Clearly, $f(z)$ and $f(z+c)$ share $0, 1, \infty$ CM, but $f(z) \not\equiv f(z+c)$.

***Example* 1.8.** Let $f(z) = \dfrac{1+e^{\cos z}}{1-e^{\cos z}}$ and $c = \pi$. Clearly, $\overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, 1; f) + \overline{N}_{(2}(r, -1; f) = S(r,f)$. Note that $f(z)$ and $f(z+c)$ share $1, -1, \infty$ CM, but $f(z) \not\equiv f(z+c)$.

2. Auxiliary lemmas. We need the following lemmas to prove Theorems 1.1 and 1.2.

Lemma 2.1 [2, Theorem 3]. *If $f(z)$ and $g(z)$ are two nonconstant rational functions that share two values CM and one value IM, then $f(z) \equiv g(z)$.*

Lemma 2.2 [10]. *Let $g_j(z)$, $j = 1, 2, \ldots, p$, be transcendental entire functions and a_j, $j = 1, 2, \ldots, p$, be nonzero constants. If $\sum_{j=1}^{p} a_j g_j(z) = 1$, then $\sum_{j=1}^{p} \delta(0, g_j) \leq p - 1$.*

Lemma 2.3 [4]. *Let $c \in \mathbb{C} \setminus \{0\}$, $\varepsilon > 0$ and $f(z)$ be a nonconstant meromorphic function such that $\rho_1(f) < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_1(f)-\varepsilon}}\right)$$

outside of an exceptional set of finite logarithmic measure.

Lemma 2.4 [10]. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If $f(z)$ and $g(z)$ share $0, 1, c, \infty$ CM, where $c \neq 0, 1, \infty$, then $f(z)$ and $g(z)$ have one of the following conditions*:

(i) $f(z) \equiv g(z)$;

(ii) $f(z) \equiv -g(z)$ *holds if* $c = -1$ *and* $1, -1$ *are Picard exceptional values of* $f(z)$ *and* $g(z)$;

(iii) $f(z) \equiv 2 - g(z)$ *holds if* $c = 2$ *and* $0, 2$ *are Picard exceptional values of* $f(z)$ *and* $g(z)$;

(iv) $f(z) \equiv 1 - g(z)$ *holds if* $c = \dfrac{1}{2}$ *and* $0, 1$ *are Picard exceptional values of* $f(z)$ *and* $g(z)$;

(v) $f(z) \equiv \dfrac{g(z)}{2g(z) - 1}$ *holds if* $c = \dfrac{1}{2}$ *and* $\dfrac{1}{2}, \infty$ *are Picard exceptional values of* $f(z)$ *and* $g(z)$;

(vi) $f(z) \equiv \dfrac{g(z)}{g(z) - 1}$ *holds if* $c = 2$ *and* $1, \infty$ *are Picard exceptional values of* $f(z)$ *and* $g(z)$;

(vii) $f(z) \equiv \dfrac{1}{g(z)}$ *holds if* $c = -1$ *and* $0, \infty$ *are Picard exceptional values of* $f(z)$ *and* $g(z)$.

Lemma 2.5 [10, Theorem 7.10]. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions sharing 1 CM. If $N_2(r, 0; f) + N_2\Big(r, 0; g) + N_2(r, f) + N_2(r, g) < (\mu + o(1))T(r)$, $r \in I$, where $\mu < 1$, $T(r) = \max\{T(r, f), T(r, g)\}$, I is a set of infinite linear measure of $r \in (0, \infty)$, then $f \equiv g$ or $fg \equiv 1$.*

Following lemma follows from Lemma 4 of [14].

Lemma 2.6. *Let $f(z)$ and $g(z)$ be two distinct nonconstant meromorphic functions such that $f(z)$ and $g(z)$ share 0, 1 and ∞ CM. Then $\overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, 0; f) + \overline{N}_{(2}(r, 1; f) = S(r, f)$.*

3. Proofs of Theorems 1.1 and 1.2. ***Proof of Theorem* 1.1.** We have $a_1, a_2, a_3 \in \hat{S}(f)$.

(A) Suppose first that $a_3(z) = \infty$, while $a_1, a_2 \in S(f)$.

By the given conditions, $f(z)$ and $f(z+c)$ share $a_1(z), a_2(z), \infty$ CM, where $a_1(z)$ and $a_2(z)$ are c-periodic small functions of f. Now, by the first fundamental theorem and the second fundamental theorem for small function (see [12]), we get

$$\begin{aligned} T(r, f(z)) &\leq \overline{N}(r, f(z)) + \overline{N}(r, a_1(z); f(z)) + \overline{N}(r, a_2(z); f(z)) + S(r, f(z)) \\ &\leq \overline{N}(r, f(z+c)) + \overline{N}(r, a_1(z); f(z+c)) + \overline{N}(r, a_2(z); f(z+c)) + S(r, f(z)) \\ &\leq 3T(r, f(z+c)) + S(r, f(z)). \end{aligned} \tag{3.1}$$

Similarly, we have

$$T(r, f(z+c)) \leq 3T(r, f(z)) + S(r, f(z+c)). \tag{3.2}$$

Now from (3.1) and (3.2) one can easily conclude that $S(r, f(z)) = S(r, f(z+c))$.

First we suppose that $f(z)$ is a nonconstant rational function. We know that a small function of a rational function must be a constant. Therefore, $a_1(z)$ and $a_2(z)$ are constants. Also, from (3.2), we deduce that $f(z+c)$ is also a nonconstant rational function. Now, by Lemma 2.1, we have $f(z) \equiv f(z+c)$ and so conclusion (1) of Theorem 1.1 holds.

Next we suppose that $f(z)$ is a transcendental meromorphic function. Then, from (3.1), we deduce that $f(z+c)$ is also a transcendental meromorphic function. Since $f(z)$ and $f(z+c)$ share $a_1(z), a_2(z), \infty$ CM, there exist two entire functions $P(z)$ and $Q(z)$ such that

$$\frac{f(z+c) - a_1(z)}{f(z) - a_1(z)} = e^{P(z)} \tag{3.3}$$

and

$$\frac{f(z+c)-a_2(z)}{f(z)-a_2(z)}=e^{Q(z)}. \tag{3.4}$$

Now we consider following four cases.

Case 1. Suppose that $P(z)$ is a constant and $Q(z)$ is nonconstant. Then (3.3) gives

$$\frac{f(z+c)-a_1(z)}{f(z)-a_1(z)}=c_0, \tag{3.5}$$

where $c_0\in\mathbb{C}\setminus\{0\}$.

First we suppose that $c_0=1$. Then, from (3.5), we get $f(z)\equiv f(z+c)$.

Next we suppose that $c_0\neq 1$. Now solving (3.4) and (3.5), for $f(z)$ we get

$$f(z)=\frac{(c_0-1)a_1(z)+a_2(z)(1-e^{Q(z)})}{c_0-e^{Q(z)}} \tag{3.6}$$

and

$$f(z+c)=\frac{(c_0-1)a_1(z)+a_2(z)(1-e^{Q(z+c)})}{c_0-e^{Q(z+c)}}. \tag{3.7}$$

From (3.6), we have $T(r,f(z))+S(r,f(z))=T(r,e^{Q(z)})$ and so $\rho(f)=\rho(e^Q)$ and $\rho_1(f)=\rho(Q)$. Now we prove that $N(r,a_2(z);f(z))=S(r,f(z))$. If not, suppose that $N(r,a_2(z);f(z))\neq S(r,f(z))$. Then $z_1\in\mathbb{C}$ be a zero of $f(z)-a_2(z)$ such that $a_1(z_1)-a_2(z_1)\neq 0,\infty$. Then we get $f(z_1)=a_2(z_1)$ and $f(z_1+c)=a_2(z_1)$. Therefore, from (3.5), we obtain $a_2(z_1)-a_1(z_1)=c_0(a_2(z_1)-a_1(z_1))$, i.e., $c_0=1$, which is impossible. Hence, $N(r,a_2(z);f(z))=S(r,f(z))$.

Let $\rho_1(f)<1$. Now, from (3.5), we have

$$\frac{f(z+c)-a_2(z+c)}{f(z)-a_2(z)}=\frac{(1-c_0)(a_1(z)-a_2(z))}{f(z)-a_2(z)}+c_0. \tag{3.8}$$

Applying Lemma 2.3 to (3.8), we get $m\left(r,\dfrac{1}{f(z)-a_2(z)}\right)=S(r,f(z))$. Since $N(r,a_2(z);f(z))=S(r,f(z))$, we have $T(r,f(z))=T\left(r,\dfrac{1}{f(z)-a_2(z)}\right)+O(1)=S(r,f(z))$, which is impossible.

Let $\rho_1(f)\geq 1$. Then $\rho(Q)\geq 1$. Also, from (3.6) and (3.7) we obtain, respectively,

$$\frac{f(z)-a_1(z)}{a_2(z)-a_1(z)}=\frac{1-e^{Q(z)}}{c_0-e^{Q(z)}} \tag{3.9}$$

and

$$\frac{f(z+c)-a_1(z)}{a_2(z)-a_1(z)}=\frac{1-e^{Q(z+c)}}{c_0-e^{Q(z+c)}}. \tag{3.10}$$

Since $f(z)$ and $f(z+c)$ share $a_1(z)$ and ∞ CM, from (3.9) and (3.10), we conclude that $e^{Q(z)}$ and $e^{Q(z+c)}$ share 1 and $c_0(\neq 1)$ CM. Note that 0 and ∞ are the Picard exceptional values of $e^{Q(z)}$ and $e^{Q(z+c)}$. Then, by using Lemma 2.4, we obtain either $e^{Q(z)}\equiv e^{Q(z+c)}$ or $e^{Q(z+c)}\equiv e^{-Q(z)}$ and $c_0=-1$.

If $e^{Q(z)} \equiv e^{Q(z+c)}$, then, from (3.6) and (3.7), we have $f(z) \equiv f(z+c)$ and so, from (3.5), we get $c_0 = 1$, which is impossible. Hence, $e^{Q(z+c)} \equiv e^{-Q(z)}$ and $c_0 = -1$. Now, from (3.4), we obtain

$$\frac{f(z+2c) - a_2(z)}{f(z+c) - a_2(z)} = e^{Q(z+c)}. \tag{3.11}$$

Since $e^{Q(z+c)} \equiv e^{-Q(z)}$, from (3.4) and (3.11), we conclude that $f(z) \equiv f(z+2c)$.

Case 2. Suppose that $P(z)$ is nonconstant and $Q(z)$ is a constant. Now proceeding in the same way as done in Case 1, we conclude that $f(z)$ has one of the following conditions:

(i) $f(z) \equiv f(z+c)$,

(ii) $f(z) \equiv f(z+2c)$ and $\rho_1(f) \geq 1$.

Case 3. Suppose that $P(z)$ and $Q(z)$ both are constants. Let $e^P = c_1$ and $e^Q = d_1$. Then, from (3.3) and (3.4), we have, respectively,

$$\frac{f(z+c) - a_1(z)}{f(z) - a_1(z)} = c_1 \quad \text{and} \quad \frac{f(z+c) - a_2(z)}{f(z) - a_2(z)} = d_1. \tag{3.12}$$

If $c_1 \neq d_1$, then, from (3.12), we get

$$f(z) = \frac{c_1 a_1(z) - d_1 a_2(z) + a_2(z) - a_1(z)}{c_1 - d_1},$$

which is impossible. Hence, $c_1 = d_1$ and so, from (3.12), we obtain $f(z) \equiv f(z+c)$.

Case 4. Suppose that $P(z)$ and $Q(z)$ both are nonconstant entire functions. If $e^{P(z)} \equiv e^{Q(z)}$, then, from (3.3) and (3.4), we have

$$\frac{f(z+c) - a_1(z)}{f(z) - a_1(z)} \equiv \frac{f(z+c) - a_2(z)}{f(z) - a_2(z)}$$

and so $f(z) \equiv f(z+c)$. Henceforth we assume that $e^{P(z)} \not\equiv e^{Q(z)}$. Now solving (3.3) and (3.4), for $f(z)$ we get

$$f(z) = \frac{a_1(z)e^{P(z)} - a_2(z)e^{Q(z)} + a_2(z) - a_1(z)}{e^{P(z)} - e^{Q(z)}}. \tag{3.13}$$

Consequently,

$$f(z+c) = \frac{a_1(z)e^{P(z+c)} - a_2(z)e^{Q(z+c)} + a_2(z) - a_1(z)}{e^{P(z+c)} - e^{Q(z+c)}}. \tag{3.14}$$

Therefore, from (3.3), (3.13) and (3.14), we obtain

$$e^{P(z+c)} + e^{Q(z)+Q(z+c)} - e^{P(z+c)+Q(z)} + e^{-P(z)+Q(z)} - e^{-P(z)+Q(z)+Q(z+c)} = 1. \tag{3.15}$$

First we suppose that $P(z) - Q(z)$ is a constant. Let $-P(z) + Q(z) = c_2$. As $e^{P(z)} \not\equiv e^{Q(z)}$, we have $e^{c_2} \neq 1$. Now (3.15) gives

$$e^{P(z+c)} + (1 - e^{-c_2})e^{Q(z)+Q(z+c)} - e^{c_2}e^{Q(z+c)} = 1 - e^{c_2}. \tag{3.16}$$

Let $g_1(z) = e^{P(z+c)}$, $g_2(z) = e^{Q(z)+Q(z+c)}$, $g_3(z) = e^{Q(z+c)}$, $b_1 = \dfrac{1}{1-e^{c_2}}$, $b_2 = \dfrac{1-e^{-c_2}}{1-e^{c_2}}$ and $b_3 = -\dfrac{e^{c_2}}{1-e^{c_2}}$. Then, from (3.16), we get $\sum_{j=1}^{3} b_j g_j = 1$.

If $Q(z)+Q(z+c)$ is nonconstant, then g_1, g_2 and g_3 are transcendental entire functions. Therefore, $\delta(0; g_j) = 1$ for $j = 1, 2, 3$ and so $\sum_{j=1}^{3} \delta(0; g_j) = 3$. On the other hand, applying Lemma 2.2 to $\sum_{j=1}^{3} b_j g_j = 1$, we have $\sum_{j=1}^{3} \delta(0; g_j) \le 2$. Therefore, we obtain a contradiction. Hence, $Q(z) + Q(z + c)$ is a constant, say c_3. Since $-P(z + c) + Q(z + c) = c_2$, from (3.16), we get

$$(1 - e^{2c_2})e^{P(z+c)} = 1 - e^{c_2} - (1 - e^{-c_2})e^{c_3}. \tag{3.17}$$

If $e^{c_2} \ne -1$, then, from (3.17), we immediately get contradiction. Hence, $e^{c_2} = -1$ and so, from (3.17), we get $e^{c_3} = 1$. Therefore, $e^{P(z)} \equiv -e^{Q(z)}$ and $e^{Q(z+c)} \equiv e^{-Q(z)}$. Now, from (3.13), we obtain

$$f(z) = \frac{1}{2}\Big\{a_1(z) + a_2(z) + (a_1(z) - a_2(z))e^{-Q(z)}\Big\}. \tag{3.18}$$

Clearly, from (3.18), we have $T(r, e^{Q(z)}) = T(r, e^{-Q(z)}) + O(1) = T(r, f(z)) + S(r, f(z))$ and so $\rho_1(f) = \rho_1(e^{Q}) = \rho(Q)$.

Let $\rho_1(f) < 1$ and $g_1(z) = e^{Q(z)}$. Then $\rho_1(g_1) < 1$. Since $e^{Q(z)+Q(z+c)} \equiv 1$, we obtain

$$\frac{g_1(z + c)}{g_1(z)} \equiv \frac{1}{g_1^2(z)}. \tag{3.19}$$

Now applying Lemma 2.3 to (3.19), we get $m\left(r, \dfrac{1}{g_1(z)}\right) = S(r, g_1(z))$. Since $N(r, 0; g_1(z)) = 0$, we have $T(r, g_1(z)) = S(r, g_1(z))$, which is impossible.

Let $\rho_1(f) \ge 1$. Now, from (3.4), we obtain

$$\frac{f(z + 2c) - a_2(z)}{f(z + c) - a_2(z)} = e^{Q(z+c)}. \tag{3.20}$$

Since $e^{Q(z+c)} \equiv e^{-Q(z)}$, from (3.4) and (3.20), we conclude that $f(z) \equiv f(z + 2c)$.

Next we suppose that $P(z) - Q(z)$ is nonconstant. Then $P(z + c) - Q(z + c)$ is also. Now applying Lemma 2.2 to (3.15), we deduce that at least one of $Q(z) + Q(z + c)$, $P(z + c) + Q(z)$ and $-P(z) + Q(z) + Q(z + c)$ is a constant.

Now we divide following three subcases.

Subcase 4.1. Suppose that $Q(z) + Q(z + c)$ is a constant. Let $Q(z) + Q(z + c) = c_4 \in \mathbb{C}$. Then, from (3.15), we have

$$e^{P(z+c)} - e^{P(z+c)+Q(z)} + e^{-P(z)+Q(z)} - e^{c_4}e^{-P(z)} = 1 - e^{c_4}. \tag{3.21}$$

First we suppose that $e^{c_4} \ne 1$. Then, from (3.21), we obtain

$$\frac{1}{1-e^{c_4}}e^{P(z+c)} + \frac{-1}{1-e^{c_4}}e^{P(z+c)+Q(z)} + \frac{1}{1-e^{c_4}}e^{-P(z)+Q(z)} + \frac{-e^{c_4}}{1-e^{c_4}}e^{-P(z)} = 1.$$

Now, from Lemma 2.2, we deduce that $P(z + c) + Q(z)$ is a constant. Let $P(z + c) + Q(z) = c_5 \in \mathbb{C}$. Then, from (3.21), we get

$$e^{P(z+c)} + e^{-P(z)+Q(z)} - e^{c_4}e^{-P(z)} = 1 - e^{c_4} + e^{c_5}. \tag{3.22}$$

If $1-e^{c_4}+e^{c_5} \neq 0$, then, applying Lemma 2.2 to (3.22), we get a contradiction. Hence, $1-e^{c_4}+e^{c_5} = 0$. Now, from (3.22), we have

$$e^{-c_4}e^{P(z)+P(z+c)} + e^{-c_4}e^{Q(z)} = 1. \tag{3.23}$$

If $P(z) + P(z+c)$ is not a constant, then, applying Lemma 2.2 to (3.23), we obtain a contradiction. Hence, $P(z) + P(z+c)$ is a constant, say c_6 and so, from (3.23), we get $e^{Q(z)} = e^{c_4}(1 - e^{-c_4+c_6})$. Since $Q(z)$ is nonconstant, we arrive at a contradiction.

Next we suppose that $e^{c_4} = 1$. Then, from (3.21), we have

$$e^{P(z)+P(z+c)} - e^{P(z)+P(z+c)+Q(z)} + e^{Q(z)} = 1. \tag{3.24}$$

Now applying Lemma 2.2 to (3.24), we obtain that at least one of $P(z) + P(z+c)$ and $P(z) + P(z+c) + Q(z)$ is constant.

Let $P(z) + P(z+c) = c_7$, where $c_7 \in \mathbb{C}$. Then, from (3.24), we get $(1 - e^{c_7})e^{Q(z)} = 1 - e^{c_7}$. If $e^{c_7} \neq 1$, then we have $e^{Q(z)} \equiv 1$, which is impossible. Hence, $e^{c_7} = 1$, i.e., $e^{P(z)+P(z+c)} \equiv 1$. Now, from (3.2) and (3.3), we obtain

$$T(r, e^{P(z)}) \leq 4T(r, f(z)) + S(r, f(z))$$

and so $\rho(P) = \rho_1(e^P) \leq \rho_1(f)$.

Let $\rho_1(f) < 1$. Then $\rho_1(e^P) < 1$. Let $g_2(z) = e^{P(z)}$. Then $\rho_1(g_2) < 1$. Since $e^{P(z)+P(z+c)} \equiv 1$, we have

$$\frac{g_2(z+c)}{g_2(z)} \equiv \frac{1}{g_2^2(z)}. \tag{3.25}$$

Now applying Lemma 2.3 to (3.25), we get $m\left(r, \dfrac{1}{g_2(z)}\right) = S(r, g_2(z))$. Since $N(r, 0; g_2(z)) = 0$, we have $T(r, g_2(z)) = S(r, g_2(z))$, which is impossible.

Let $\rho_1(f) \geq 1$. Now, from (3.3), we obtain

$$\frac{f(z+2c) - a_1(z)}{f(z+c) - a_1(z)} = e^{P(z+c)}. \tag{3.26}$$

Since $e^{P(z+c)} \equiv e^{-P(z)}$, from (3.3) and (3.26), we conclude that $f(z) \equiv f(z+2c)$.

Let $P(z) + P(z+c) + Q(z) = c_8$. Now, from (3.24), we have

$$e^{P(z)+P(z+c)} + e^{Q(z)} = 1 + e^{c_8}. \tag{3.27}$$

If $e^{c_8} \neq -1$, then, applying Lemma 2.2 to (3.27), we conclude that $P(z) + P(z+c)$ is a constant, say c_9. Finally, from (3.27) we get $e^{Q(z)} \equiv 1 + e^{c_8} - e^{c_9}$, which is impossible. Hence, $e^{c_8} = -1$. Now, from (3.27), we have $e^{Q(z)} = -e^{P(z)+P(z+c)}$ and so $e^{2Q(z)} = -e^{P(z)+P(z+c)+Q(z)}$. Since $P(z) + P(z+c) + Q(z) = c_8$, we obtain $e^{2Q(z)} \equiv -e^{c_8}$, which is impossible.

Subcase 4.2. Suppose that $P(z+c) + Q(z)$ is a constant. Let $P(z+c) + Q(z) = c_{10}$. Then, from (3.15), we have

$$e^{P(z+c)} + e^{Q(z)+Q(z+c)} + e^{-P(z)+Q(z)} - e^{-P(z)+Q(z)+Q(z+c)} = 1 + e^{c_{10}}. \tag{3.28}$$

First we suppose that $e^{c_{10}} \neq -1$. Then, from (3.28), we get

$$\frac{1}{1+e^{c_{10}}}e^{P(z+c)} + \frac{1}{1+e^{c_{10}}}e^{Q(z)+Q(z+c)} + \frac{1}{1+e^{c_{10}}}e^{-P(z)+Q(z)} + \frac{-1}{1+e^{c_{10}}}e^{-P(z)+Q(z)+Q(z+c)} = 1$$

and so, by Lemma 2.2, we deduce that at least one of $Q(z)+Q(z+c)$ and $-P(z)+Q(z)+Q(z+c)$ is a constant.

Let $Q(z)+Q(z+c) = c_{11}$. Then, from (3.28), we obtain

$$e^{P(z+c)} + e^{-P(z)+Q(z)} - e^{c_{11}}e^{-P(z)} = 1 + e^{c_{10}} - e^{c_{11}}$$

and so we arrive at (3.22). Now, from the proof of Subcase 4.1, we can easily get a contradiction.

Let $-P(z)+Q(z)+Q(z+c) = c_{12}$. Then, from (3.28), we have

$$e^{P(z+c)} + e^{c_{12}}e^{P(z)} + e^{c_{12}}e^{-Q(z+c)} = 1 + e^{c_{10}} + e^{c_{12}}. \tag{3.29}$$

If $1+e^{c_{10}}+e^{c_{12}} \neq 0$, then, from Lemma 2.2 and (3.29), we immediately get a contradiction. Hence, $1+e^{c_{10}}+e^{c_{12}} = 0$ and so, from (3.29), we obtain

$$-e^{-c_{12}}e^{P(z+c)-P(z)} - e^{-P(z)-Q(z+c)} = 1. \tag{3.30}$$

If both $P(z+c)-P(z)$ and $-P(z)-Q(z+c)$ are nonconstants, then, applying Lemma 2.2, we arrive at a contradiction from (3.30). Hence, at least one of $P(z+c)-P(z)$ and $-P(z)-Q(z+c)$ is a constant. Now we see that if one of $P(z+c)-P(z)$ and $-P(z)-Q(z+c)$ is a constant and other one is nonconstant, then, from (3.30), we get a contradiction. Hence, both $P(z+c)-P(z)$ and $-P(z)-Q(z+c)$ are constants. Let $P(z+c)-P(z) = c_{13}$ and $P(z)+Q(z+c) = c_{14}$, where $c_{13}, c_{14} \in \mathbb{C}$. Also, we have $P(z+c)+Q(z) = c_{10}$ and $-P(z)+Q(z)+Q(z+c) = c_{12}$. Eliminating $Q(z+c)$ from $P(z)+Q(z+c) = c_{14}$ and $-P(z)+Q(z)+Q(z+c) = c_{12}$, we obtain

$$-2P(z)+Q(z) = c_{12} - c_{14}. \tag{3.31}$$

Again eliminating $P(z+c)$ from $P(z+c)-P(z) = c_{13}$ and $P(z+c)+Q(z) = c_{10}$, we have

$$P(z)+Q(z) = c_{10} - c_{13}. \tag{3.32}$$

Now, from (3.31) and (3.32), we get $3P(z) = c_{10} - c_{13} - c_{12} + c_{14}$, which is impossible.

Next we suppose that $e^{c_{10}} = -1$. Then, from (3.28), we obtain

$$-e^{P(z)+P(z+c)-Q(z)} - e^{P(z)+Q(z+c)} + e^{Q(z+c)} = 1. \tag{3.33}$$

Clearly, at least one of $P(z)+P(z+c)-Q(z)$ and $P(z)+Q(z+c)$ is a constant, otherwise applying Lemma 2.2 to (3.33), we get a contradiction.

Let $P(z)+P(z+c)-Q(z) = c_{15}$. Then, from (3.33), we have

$$-e^{P(z)+Q(z+c)} + e^{Q(z+c)} = 1 + e^{c_{15}}. \tag{3.34}$$

If $1+e^{c_{15}} = 0$, then, from (3.34), we get $e^{P(z)+Q(z+c)} = e^{Q(z+c)}$, i.e., $e^{P(z)} = 1$, which is impossible. Hence, $1+e^{c_{15}} \neq 0$. Now using Lemma 2.2, we conclude, from (3.34), that $P(z)+Q(z+c)$ is a constant, say c_{16}. Therefore, from (3.34), we obtain $e^{Q(z+c)} = 1 + e^{c_{15}} + e^{c_{16}}$, which is impossible.

Let $P(z)+Q(z+c)=c_{17}$. Then, from (3.33), we get

$$-e^{P(z)+P(z+c)-Q(z)}+e^{Q(z+c)}=1+e^{c_{17}}. \tag{3.35}$$

If $1+e^{c_{17}}\neq 0$, then, using Lemma 2.2, we conclude, from (3.35), that $P(z)+P(z+c)-Q(z)$ is a constant, say c_{18}. Consequently, from (3.35), we have $e^{Q(z+c)}=1+e^{c_{17}}+e^{c_{18}}$, which is impossible. Hence, $1+e^{c_{17}}=0$. Then, from (3.35), we obtain

$$e^{P(z)+P(z+c)}=e^{Q(z)+Q(z+c)}. \tag{3.36}$$

Also, we know that $P(z+c)+Q(z)=c_{10}$ and $P(z)+Q(z+c)=c_{17}$, where $e^{c_{10}}=e^{c_{17}}=-1$. Therefore, $P(z)+P(z+c)+Q(z)+Q(z+c)=c_{10}+c_{17}$. Again from (3.36), we have $P(z)+P(z+c)-Q(z)-Q(z+c)=c_{19}$, where $c_{19}\in\mathbb{C}$. Solving we get

$$P(z)+P(z+c)=\frac{1}{2}(c_{10}+c_{17}+c_{19}). \tag{3.37}$$

Since $P(z+c)+Q(z)=c_{10}$, from (3.37), we obtain $P(z)-Q(z)=\dfrac{1}{2}(c_{10}+c_{17}+c_{19})-c_{10}$ and so we may assume that $e^{P(z)}=de^{Q(z)}$, where $d\in\mathbb{C}\setminus\{0\}$. Consequently, we have $e^{P(z)+P(z+c)}=d^2e^{Q(z)+Q(z+c)}$. Therefore, from (3.36), we get $d^2=1$, i.e., $d=\pm 1$. Since $e^{P(z)}\not\equiv e^{Q(z)}$, it follows that $d=-1$ and so $e^{P(z)}\equiv -e^{Q(z)}$. Again since $P(z+c)+Q(z)=c_{10}$, where $e^{c_{10}}=-1$, we have $e^{P(z+c)}\equiv -e^{-Q(z)}$. Consequently, $e^{P(z+c)}\equiv e^{-P(z)}$. Then, from (3.13), we obtain

$$f(z)=\frac{1}{2}\Big\{a_1(z)+a_2(z)+(a_1(z)-a_2(z))e^{-P(z)}\Big\}.$$

Now proceeding in the same way as done in the proof of Subcase 4.1 one can easily conclude that $\rho_1(f)\geq 1$ and $f(z)\equiv f(z+2c)$.

Subcase 4.3. Suppose that $-P(z)+Q(z)+Q(z+c)$ is a constant. Let $-P(z)+Q(z)+Q(z+c)=c_{20}\in\mathbb{C}$. Then, from (3.15), we have

$$e^{P(z+c)}-e^{P(z+c)+Q(z)}+e^{c_{20}}e^{P(z)}+e^{c_{20}}e^{-Q(z+c)}=1+e^{c_{20}}. \tag{3.38}$$

First we suppose that $1+e^{c_{20}}\neq 0$. Now applying Lemma 2.2 to (3.38), we deduce that $P(z+c)+Q(z)$ is a constant. Let $P(z+c)+Q(z)=c_{21}\in\mathbb{C}$. Then, from (3.38), we get

$$e^{P(z+c)}+e^{c_{20}}e^{P(z)}+e^{c_{20}}e^{-Q(z+c)}=1+e^{c_{20}}+e^{c_{21}}. \tag{3.39}$$

If $1+e^{c_{20}}+e^{c_{21}}\neq 0$, then, by using Lemma 2.2, we obtain a contradiction from (3.39). Hence, $1+e^{c_{20}}+e^{c_{21}}=0$ and so, from (3.39), we have

$$-e^{-c_{20}}e^{P(z+c)+Q(z+c)}-e^{P(z)+Q(z+c)}=1. \tag{3.40}$$

Now applying Lemma 2.2, we conclude, from (3.40), that both $P(z+c)+Q(z+c)$ and $P(z)+Q(z+c)$ are constants. Let $P(z+c)+Q(z+c)=c_{22}$ and $P(z)+Q(z+c)=c_{23}$, where $c_{22},c_{23}\in\mathbb{C}$. Now adding $-P(z)+Q(z)+Q(z+c)=c_{20}$, $P(z+c)+Q(z+c)=c_{22}$ and $P(z)+Q(z+c)=c_{23}$, we get $P(z+c)+Q(z)+3Q(z+c)=c_{20}+c_{22}+c_{23}$. Since $P(z+c)+Q(z)=c_{21}$, we deduce that $3Q(z+c)=c_{20}+c_{22}+c_{23}-c_{21}$. Now $P(z)+Q(z+c)=c_{23}$ implies that $3P(z)=2c_{23}+c_{21}-c_{20}-c_{22}$, which is impossible.

Next we suppose that $1 + e^{c_{20}} = 0$. Then, from (3.38), we have

$$e^{Q(z)} + e^{P(z)-P(z+c)} + e^{-P(z+c)-Q(z+c)} = 1. \tag{3.41}$$

Now applying Lemma 2.2 to (3.41), we deduce that at least one of $P(z) - P(z + c)$ and $-P(z + c) - Q(z + c)$ is a constant.

Let $P(z) - P(z + c) = c_{24} \in \mathbb{C}$. Now, from (3.41), we have

$$e^{Q(z)} + e^{-P(z+c)-Q(z+c)} = 1 - e^{c_{24}}. \tag{3.42}$$

If $1 - e^{c_{24}} \neq 0$, then, by using Lemma 2.2, we conclude, from (3.42), that $-P(z+c)-Q(z+c)$ is a constant. Finally, from (3.42), we get that $e^{Q(z)}$ is a constant, which is impossible. Hence, $1-e^{c_{24}} = 0$. Now, from (3.42), we obtain $e^{P(z+c)+Q(z)+Q(z+c)} = -1$. Let $P(z + c) + Q(z) + Q(z + c) = c_{25}$, where $e^{c_{25}} = -1$. Since $P(z) - P(z + c) = c_{24}$, we have

$$P(z) + Q(z) + Q(z + c) = c_{24} + c_{25}.$$

Also, we know that $-P(z) + Q(z) + Q(z + c) = c_{20}$. Therefore, we get $2P(z) = c_{24} + c_{25} - c_{20}$, which is impossible.

Let $-P(z + c) - Q(z + c) = c_{26} \in \mathbb{C}$. Now, from (3.41), we have

$$e^{Q(z)} + e^{P(z)-P(z+c)} = 1 - e^{c_{26}}. \tag{3.43}$$

If $1 - e^{c_{26}} \neq 0$, then, by using Lemma 2.2, we conclude, from (3.43), that $P(z) - P(z + c)$ is a constant. Finally, from (3.43), we get that $e^{Q(z)}$ is a constant, which is impossible. Hence, $1-e^{c_{26}} = 0$. Now, from (3.43), we obtain $e^{P(z)-P(z+c)-Q(z)} = -1$. Let $P(z) - P(z + c) - Q(z) = c_{27}$, where $e^{c_{27}} = -1$. Since $-P(z + c) - Q(z + c) = c_{26}$, we have

$$P(z) - Q(z) + Q(z + c) = c_{27} - c_{26}.$$

Also, we know that $P(z) - Q(z) - Q(z + c) = -c_{20}$. Therefore, we immediately have $Q(z + c) = \dfrac{1}{2}(c_{20} + c_{27} - c_{26})$, which is impossible.

(B) Suppose that $a_1, a_2, a_3 \in S(f)$. Let

$$F(z) = \frac{f(z) - a_1(z)}{f(z) - a_3(z)} \frac{a_2(z) - a_3(z)}{a_2(z) - a_1(z)}. \tag{3.44}$$

Since a_1, a_2, a_3 are c-periodic functions, from (3.44), we have

$$F(z + c) = \frac{f(z + c) - a_1(z)}{f(z + c) - a_3(z)} \frac{a_2(z) - a_3(z)}{a_2(z) - a_1(z)}. \tag{3.45}$$

Note that $\rho_1(f) = \rho_1(F)$. Since $f(z)$ and $f(z + c)$ share $a_1(z)$, $a_2(z)$ and $a_3(z)$ CM, from (3.44) and (3.45), we conclude that $F(z)$ and $F(z + c)$ share 0, 1 and ∞ CM. Then, by the above result, we have either $F(z) \equiv F(z + c)$ or $F(z) \equiv F(z + 2c)$ if $\rho_1(F) \geq 1$, i.e., either $f(z) \equiv f(z + c)$ or $f(z) \equiv f(z + 2c)$ if $\rho_1(f) \geq 1$.

The theorem is proved.

***Proof of Theorem* 1.2.** Let

$$F(z) = \begin{cases} \dfrac{f(z) - a_2(z)}{a_1(z) - a_2(z)}, & \text{if} \quad a_3(z) = \infty \quad \text{and} \quad a_1, a_2 \in S(f), \\ \dfrac{f(z) - a_1(z)}{f(z) - a_3(z)} \dfrac{a_2(z) - a_3(z)}{a_2(z) - a_1(z)}, & \text{if} \quad a_1, a_2, a_3 \in S(f). \end{cases} \tag{3.46}$$

Since $f(z)$ and $f(z+c)$ share $a_1(z)$, $a_2(z)$ and $a_3(z)$ CM, from (3.46), we can conclude that $F(z)$ and $F(z+c)$ share 0, 1 and ∞ CM. Also, from the proof of Theorem 1.1, we have $S(r, f(z)) = S(r, f(z+c))$. Therefore, we get $S(r, F(z)) = S(r, f(z)) = S(r, f(z+c)) = S(r, F(z+c))$. Since $\sum_{j=1}^{3} \overline{N}_{(2}(r, a_j; f) \neq S(r, f)$, from (3.46), we deduce that $\overline{N}_{(2}(r, F) + \overline{N}_{(2}(r, 0; F) + \overline{N}_{(2}(r, 1; F) \neq S(r, F)$. If $F(z) \not\equiv F(z+c)$, then, from Lemma 2.6, we get a contradiction. Hence, $F(z) \equiv F(z+c)$ and so $f(z) \equiv f(z+c)$.

The theorem is proved.

4. An application. In the same paper, Heittokangas et al. [7] proved that the assumption on the order of $f(z)$ in Theorem D can be removed if the target values are shared by two appropriate shifted functions instead of one. Indeed they got the following result.

Theorem E [7]. *Let $f(z)$ be a nonconstant meromorphic function and let $c_1, c_2 \in \mathbb{C}$ be linearly independent over the real numbers. If $f(z)$, $f(z+c_1)$ and $f(z+c_2)$ share three distinct values $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ CM, then $f(z)$ is an elliptic function with periods c_1 and c_2.*

Clearly, any elliptic function $f(z)$ with periods c_1 and c_2 shares all values with its shifts $f(z+c_1)$ and $f(z+c_2)$. Therefore, Theorem E leads to a new way of characterizing elliptic functions in terms of meromorphic functions which share values with two of their shifts.

In the paper, we now introduce another way of characterizing elliptic functions in terms of meromorphic functions which share values with two of their shifts. Now as an application of Theorem 1.1, we present the following proposition.

Proposition 4.1. *Let $f(z)$ be a nonconstant meromorphic function, $c_1, c_2 \in \mathbb{C}$ be linearly independent over the real numbers and let $a, a_1, a_2, b_1, b_2 \in \mathbb{C} \cup \{\infty\}$ such that $a \neq a_1 \neq b_1$ and $a \neq a_2 \neq b_2$. If $f(z)$ and $f(z+c_1)$ share a, a_1, b_1 CM and $f(z)$ and $f(z+c_2)$ share a, a_2, b_2 CM, then $f(z)$ is an elliptic function with periods c_1 and c_2.*

Proof. We know that $f(z)$ and $f(z+c_1)$ share a, a_1, b_1 CM and $f(z)$ and $f(z+c_2)$ share a, a_2, b_2 CM, where $a, a_1, a_2, b_1, b_2 \in \mathbb{C} \cup \{\infty\}$ such that $a \neq a_1 \neq b_1$ and $a \neq a_2 \neq b_2$.

We consider two cases.

Case 1. Let $\rho_1(f) < 1$. Then, by Theorem 1.1, we get $f(z) \equiv f(z+c_1)$ and $f(z) \equiv f(z+c_2)$. Therefore, $f(z) \equiv f(z+c_1) \equiv f(z+c_2)$ and so $f(z)$ is an elliptic function with periods c_1 and c_2.

Case 2. Let $\rho_1(f) \geq 1$. Then, by Theorem 1.1, we have $f(z) \equiv f(z+2c_1)$ and $f(z) \equiv f(z+2c_2)$. Therefore, $f(z) \equiv f(z+2c_1) \equiv f(z+2c_2)$. Note that $2c_1$ and $2c_2$ are also linearly independent over the real numbers. Consequently, $f(z)$ is an elliptic function with periods $2c_1$ and $2c_2$ and thereby $\rho(f) < \infty$. Therefore, we get a contradiction.

If $a_1 = a_2$ and $b_1 = b_2$, then from Proposition 4.1 we get Theorem E.

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