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ON SEMIPERFECT a-RINGS

ПРО НАПІВДОСКОНАЛІ a-КІЛЬЦЯ

A ring is called a right a-ring if every right ideal is automorphism invariant. We describe some properties of a-rings over semiperfect rings. It is shown that an I-finite right a-ring is a direct sum of a semisimple Artinian ring and a basic ring. It is also demonstrated that if R is an indecomposable (as a ring) I-finite right a-ring not simple with nontrivial idempotents such that every minimal right ideal is a right annihilator and $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$ is essential in R_R, then R is a quasi-Frobenius ring and it is also a right q-ring.

Кільце називається правим a-кільцем, якщо кожний правий ідеал є інваріантним щодо автоморфізму. Ми описуємо деякі властивості a-кілець над напівдосконалими кільцями. Показано, що I-скінченне праве a-кільце є прямою сумою напівпростого артинового кільця та базисного кільця. Показано також, що якщо R є нерозкладним (як кільце) I-скінченним правим a-кільцем, яке не є простим з нетривіальними ідемпотентами, тобто таким, що кожен мінімальний правий ідеал є правим анігілятором, а $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$ є істотним в R_R, то R є не лише квазіфробеніусовим кільцем, а й правим q-кільцем.

1. Introduction. Automorphism-invariant modules were first introduced in the short paper written by Dickson and Fuller [4]. They studied algebras for which every indecomposable right module is invariant in its injective hull. Then in [14], Lee and Zhou coined the name automorphism-invariant modules by investigating modules which are invariant under automorphisms of their injective hulls. A recent, nice work and good results in this regard appeared in the paper [21] which is due to Tuganbaev. We have seen in such paper a characterization of all automorphism-invariant nonsingular right modules which are injective in terms of the factor ring (modulo the Goldie radical). For more characterizations of automorphism-invariant modules, we refer to the recent book [19, Chapter 4]. On the other hand, the class of I-finite rings is a generalization of local rings, semilocal rings and rings having finite dimensional. That is we are dealing with a ring with finite orthogonal idempotents. There are many characterizations of I-finite rings which can be seen in [15]. In fact, the main motivation of our work here is to go further for investigating I-finite rings with the light of recent results as in [17]. That is, we are dealing with a-rings which are rings such that all right ideals are automorphism invariant. However, we are with modules case which is more vital in modern algebra.

This paper is organized as follows. In Section 2, we study the situation for I-finite a-rings and prove that any right a-ring is in fact a direct sum of a semisimple Artinian ring and a basic ring. We use the notion of index of a primitive idempotent. That is, a natural number which equals the number of indecomposable direct summands appearing in the decomposition of the right R-module R_R. Also, we make use in this section of the equivalence between being semisimple Artinian for a

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ring R and being a right a-ring via the notion of index of idempotents. We give an example which illustrates that a basic semiperfect right automorphism-invariant ring need not to be a right a-ring. Section 3 contains the results about the invariance in the present of semiperfect a-rings with essential socles. We assume that R to be a I-finite ring in which every minimal right ideal is a right annihilator and both the socle of the R_R and of ${}_RR$ coincide. We assume further that such socle is essential in R_R. Dealing with quasi-Frobenius rings is another elegant approach (see [15] for more details). Our main results assert that nonsimple ring which is indecomposable with nontrivial idempotents must be a quasi-Frobenius ring.

Some generalizations of automorphism-invariant modules and automorphism-coinvariant modules are studied and developed. Some important structures of the rings through this class of modules are investigated [1, 9, 10, 12, 13, 16, 18, 20].

Throughout this article all rings are associative rings with unity and all modules are right unital modules over a ring. For a submodule N of M, we write $N \leq M$ $(N < M, N \leq^e M)$ iff N is a submodule of M (respectively, a proper submodule, an essential submodule). We denote by $J(M)$ and $\mathrm{Soc}(M)$ the radical of the module M and the socle of M, respectively. We also denote by $E(M)$ the injective hull of M. If X is a subset of R, the right (resp., left) annihilator of X in R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp., $l(X)$). For any term which is not defined here, the reader is referred to [3] and [5].

2. On I-finite a-rings. In this section, we study some properties of a-rings with the condition I-finite. The class of I-finite rings infers a decomposition of the identity as a sum of primitive idempotents. From this, we obtain a decompositions of a right a-ring I-finite as a direct sum of a semisimple Artinian ring and a basic ring.

Recall that a ring R is a *right a-ring* (*q-ring*) if every right ideal of R is automorphism-invariant (resp., quasiinjective) (see [11] and [8]). On the other hand, every right q-ring is a right a-ring, but the converse is not true in general. For example, the ring R consisting of all eventually constant sequences of elements from the field $\mathbb{F}_2$ is an a-ring. But R is not a q-ring. A ring R is said to be *right automorphism-invariant* if R_R is an automorphism-invariant module.

We have some characterizations for right a-rings:

Lemma 1 [11, Proposition 3.1]**.** *The following conditions are equivalent for a ring R:*

(1) *R is a right a-ring.*

(2) *Every essential right ideal of R is automorphism-invariant.*

(3) *R is right automorphism-invariant and every essential right ideal of R is a left T-module, where T is a subring of R generated by its unit elements.*

A ring R is called *I-finite* if it contains no infinite orthogonal family of idempotents and it is called *semiperfect* if $R/J(R)$ is semisimple and idempotents can be lifted modulo $J(R)$ [15, p. 256, 257]. It follows, from [15, Theorem B.9], that R is semiperfect if and only if R is I-finite and primitive idempotents in R are local.

The following lemma is useful for the proofs of next results.

Lemma 2. *If R is a right automorphism-invariant I-finite ring, then R is a semiperfect ring.*

Proof. Assume that R is a right automorphism-invariant I-finite ring. Note that every direct summand of an automorphism-invariant module is automorphism-invariant. Let e be a primitive idempotent of R. Then eR is an indecomposable autmorphism-invariant right R-module. It follows that $\operatorname{End}(eR)$ is a local ring by [2, Proposition 2.2], and so e is a local idempotent of R. We deduce that R is semiperfect.

A primitive idempotent e in a semiperfect ring R has *index* h in case in each (hence, any) decomposition of R_R as a direct sum of indecomposable modules, there are exactly h summands isomorphic to eR (see [6, p. 114]).

Lemma 3. *Let R be a right a-ring I-finite. If a primitive idempotent $e \in R$ has index at least 2, then eR is simple.*

Proof. Assume that e is a primitive idempotent of R with index at least 2. Then, by definition of e, there exists an idempotent e' of R such that $eR \cap e'R = 0$ and $eR \cong e'R$. It follows, from [11, Lemma 3.3], that eR is a semisimple module. On the other hand, we have that eR is an idecomposable module and obtain that eR is simple.

***Corollary* 1.** *Assume that R is I-finite such that each primitive idempotent of R has index at least 2. Then the following conditions are equivalent*:

(1) *R is semisimple Artinian.*

(2) *R is a right q-ring.*

(3) *R is a right a-ring.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Assume that R is a right a-ring. Since R is a semiperfect ring, $R = e_1R \oplus e_2R \oplus \ldots \oplus e_mR$, where e_i is primitive idempotent of R for every $i = 1, 2, \ldots, m$. By Lemma 3, e_iR is simple and so R is semisimple Artinian.

Let R be a semiperfect ring. Then there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $1 = e_1 + e_2 + \ldots + e_m$. We may assume that $\{e_iR/e_iJ(R) \mid 1 \leq i \leq n\}$ is a complete set of representatives of the isomorphism classes of the simple right R-modules. In this case, $\{e_1, e_2, \ldots, e_n\}$ is called the set of *basic idempotents* for R, and if $e = e_1 + e_2 + \ldots + e_n$, the ring eRe is called the *basic ring* of R. Note that $eR \cong fR$ if and only if $eR/eJ(R) \cong fR/fJ(R)$ for idempotents e and f of R by Jacobson's lemma (see [15, Lemma B.12]). The ring R is itself called a *basic semiperfect* ring if $m = n$, that is, if $1 = e_1 + e_2 + \ldots + e_n$, where the $\{e_i\}$ is a basic set of local idempotents.

The main result of this section is the following theorem.

Theorem 1. *Let R be a right a-ring I-finite. Then R is a direct sum of a semisimple Artinian ring and a basic semiperfect ring.*

Proof. By Lemma 2, R is a semiperfect ring, and so there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $1 = e_1 + e_2 + \ldots + e_m$. Suppose that $e_iR \ncong e_jR$ for all $i \neq j$ with $i, j \in \{1, 2, \ldots, m\}$. Then we are done. Assume that e_i is a local idempotent of R and has index at least 2 for some $i \in \{1, 2, \ldots, m\}$. It follows that e_iR is simple by Lemma 3. Let eR be the direct sum of all copies of e_iR in the decomposition of $R = e_1R \oplus e_2R \oplus \ldots \oplus e_mR$. Note that eR is a direct summand of R. We can assume that e is an idempotent of R. Then we have a decomposition $R = eR \oplus (1 - e)R$. Next, we show that eR and $(1 - e)R$ are ideals of R. In order to show this, it is necessary to prove that $eR(1 - e) = 0$ and $(1 - e)Re = 0$.

Suppose that $(1 - e)Re \neq 0$. Take $(1 - e)te \neq 0$ for some $t \in R$. Then there are primitive idempotents e_j and e_k such that $e_jR \cong e_iR$, $e_kR \ncong e_iR$ with $j, k \in \{1, 2, \ldots, m\}$, $e_j \in eR$, $e_k \in (1 - e)R$ and $e_kte_j \neq 0$. We consider the following map $\alpha \colon e_jR \to e_kR$ defined by $\alpha(e_jr) = e_kte_jr$ for all $r \in R$. One can check that α is a nonzero homomorphism. Note that e_jR is simple. Thus, α is a monomorphism. On the other hand, we have a direct sum $e_jR \oplus e_kR$. Since R is a right a-ring, $e_jR \oplus e_kR$ is an automorphism-invariant module, and so e_jR is e_kR-injective by

[14, Theorem 5]. From this, it immediately follows that α splits. We have that e_kR is simple and obtain that $e_jR \cong e_kR$, a contradiction. We deduce that $(1-e)Re = 0$, and so eR is an ideal of R.

Similarly to the above proof, suppose that $eR(1-e) \neq 0$. Call $eu(1-e) \neq 0$ for some $u \in R$. Then there are primitive idempotents e_p and e_q of R such that $e_pR \cong e_iR$, $e_qR \ncong e_iR$ with $p, q \in \{1, 2, \ldots, m\}$, $e_p \in eR$, $e_q \in (1-e)R$ and $e_pue_q \neq 0$. We consider the following map $\beta\colon e_qR \to e_pR$ defined by $\beta(e_qr) = e_pue_qr$ for all $r \in R$. Then β is a nonzero epimorphism by the simplicity of e_pR. Since e_pR is projective, β splits. One can check that $e_qR \cong e_pR$. This is a contradiction, and so $eR(1-e) = 0$. We deduce that $(1-e)R$ is an ideal of R.

Thus, eR is a semisimple Artinian ring and $(1-e)R$ is a basic semiperfect ring.

***Corollary* 2.** *If R is a right a-ring and $r(a_1) \subseteq r(a_2a_1) \subseteq \ldots$ terminates for every infinite sequence $a_1, a_2, \ldots$ of elements in R, then R is a direct sum of a semisimple Artinian ring and a basic semiperfect ring.*

There is a basic semiperfect right automorphism-invariant ring but it is not a right a-ring.

***Example* 1.** Let $\mathbb{F}$ be a field. Take R a ring of matrices of the form

$$\begin{pmatrix} a & b & 0 & 0 & 0 & 0 & c & d \\ 0 & a & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & g & h & 0 & 0 \\ 0 & 0 & 0 & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g \end{pmatrix}$$

with $a, b, c, d, e, f, g, h \in \mathbb{F}$. From Example 1.6 in [6], it immediately infers that R is a basic semiperfect right automorphism-invariant ring. We consider the right ideal I of R of the form

$$\begin{pmatrix} 0 & b & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & g & h & 0 & 0 \\ 0 & 0 & 0 & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g \end{pmatrix}.$$

One can check that I is an essential right ideal of R and it is not invariant under the units of R. It follows that R is not a right a-ring by Lemma 1.

Now, let R be a right automorphism-invariant I-finite ring. Then we write $1 = e_1 + e_2 + \ldots + e_m$, where the e_i are orthogonal primitive idempotents. We can assume that e_iR is simple if $1 \leq i \leq n$

and e_jR is not simple if $j > n$. Take $e = e_1 + e_2 + \ldots + e_n$. By the proof of Theorem 1 we have that e is a central idempotent of R. Let I be a simple right ideal of $(1 - e)R$. Suppose that $I = fR$ for some idempotent f of R. Since $fR(1-e)$ is nonzero, fRe_j is nonzero for some $j > n$. Take x a nonzero element of fRe_j and consider $\varphi\colon e_jR \to fR$ defined by $\varphi(e_jt) = xe_jt$ for all $t \in R$. One can check that φ is an epimorphism. Moreover, e_jR is indecomposable, and so φ is an isomorphism. This is a contradiction because e_jR is not simple. It is shown that $I^2 = 0$. Thus, we have the following result.

Proposition 1. *Every right automorphism-invariant I-finite ring R has a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and every simple right ideal of T is nilpotent.*

***Corollary* 3.** *Let R be right automorphism-invariant I-finite with $1 = e_1 + e_2 + \ldots + e_m$, where the e_i are orthogonal primitive idempotents. Then, either some e_iR is simple or $\mathrm{Soc}(R_R)^2 = 0$.*

3. On semiperfect a-rings with essential socles. In this section, we assume that R *is a right a-ring I-finite such that every minimal right ideal is a right annihilator and* $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$, *denoted by S, is essential in R_R*. Then R is semiperfect, and so there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ of R such that $1 = e_1 + e_2 + \ldots + e_m$. Call $\{e_1, e_2, \ldots, e_n\}$ a set of basic idempotents for R with $n \le m$.

Next, we give some properties of minimal right and left ideals of R. Moreover, the self-injectivity of R is considered.

Lemma 4. *Let R be a ring.*

(1) *If xR is a minimal right ideal of R, then $l_Rr_R(x) = Rx$ and Rx is a minimal left ideal of R.*

(2) *If Ry is a minimal left ideal of R, then yR is a minimal right ideal of R and $l_Rr_R(Ry) = Ry$.*

(3) $\mathrm{Soc}(e_iR)$ *and* $\mathrm{Soc}(Re_i)$ *are simple for all $i = 1, 2, \ldots, m$.*

(4) *R_R is finitely cogenerated.*

(5) *R is right self-injective.*

Proof. (1) Assume that xR is a minimal right ideal of R. It is easy to see that $Rx \le l_Rr_R(x)$. For the converse, let $t \in l_Rr_R(x)$ be a nonzero element. Then we have $r_R(x) \le r_R(t)$, and so $r_R(x) = r_R(t)$ by the maximality of $r_R(x)$. It follows that $Rx = Rt$ by [17, Lemma 2.2]. Then $t \in Rx$ and so $l_Rr_R(x) \le Rx$ or $l_Rr_R(x) = Rx$. On the other hand, by our assumption $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$, there is a minimal left ideal Ry of R containing in Rx. It follows that $r_R(x) \le r_R(y)$, and so $r_R(x) = r_R(y)$ by the maximality of $r_R(x)$. Thus, we have $Rx = Ry$ which implies that Rx is a minimal left ideal.

(2) Suppose that Ry is a minimal left ideal of R. For any nonzero element yr in yR with $r \in R$, then $l_R(y) = l_R(yr)$ by the maximality of $l_R(y)$. Note that yrR contains a minimal right ideal mR of R. Thus, $l_R(y) = l_R(m) = l_R(yr)$. It follows that $y \in r_Rl_R(y) = r_Rl_R(m) = mR \le yrR$ by our assumption. We deduce that $y \in yrR$. Thus, yR is a minimal right ideal of R. The rest is followed by (1).

(3) For any $i \in \{1, 2, \ldots, m\}$, take kR a minimal right ideal of e_iR. Then Rk is a minimal left ideal of R. Therefore, $l_R(kR) \ge R(1 - e_i)$ and $l_R(kR) = l_R(k) \ge J(R)$. It follows that $l_R(kR) = J(R) + R(1 - e_i)$ because $J(R) + R(1 - e_i)$ is the unique maximal left ideal containing $R(1 - e_i)$. By our assumption we have

$$kR = r_Rl_R(kR) = r_R[J(R) + R(1 - e_i)] = r_R(J(R)) \cap e_iR = \mathrm{Soc}(R_R) \cap e_iR = \mathrm{Soc}(e_iR).$$

It shows that $\mathrm{Soc}(e_iR)$ is a minimal right ideal of R.

Similarly, we also have $\mathrm{Soc}(Re_i)$ is simple for all $i = 1, 2, \ldots, m$.

(4) is followed from (3) and our assumption.

(5) We have a decomposition $R = e_1R \oplus e_2R \oplus \ldots \oplus e_mR$. By (3), we have that e_iR is uniform for any $i \in \{1, 2, \ldots, m\}$, and so R is right self-injective by [14, Corollary 15].

Lemma 5. *If e and f are two orthogonal idempotents of R, then $eRf \subseteq \mathrm{Soc}(R_R)$.*

Proof. Suppose that e and f are two orthogonal idempotents of R. Then $eR \cap fR = 0$. If $eRf = 0$, we are done. Otherwise, let exf be a nonzero arbitrary element of eRf. We consider a nonzero homomorphism $\alpha\colon fR \to eR$ defined by $\alpha(fr) = exfr$ for all $r \in R$. By [11, Lemma 4.1], we have that $\mathrm{Im}(\alpha) = exfR$ is semisimple. It follows that $exf \in \mathrm{Soc}(R_R)$. We deduce that $eRf \subseteq \mathrm{Soc}(R_R)$.

Let R be a semiperfect ring with basic idempotents $\{e_1, e_2, \ldots, e_n\}$. A permutation σ of $\{1, 2, \ldots, n\}$ is called a *Nakayama permutation* for R if $\mathrm{Soc}(Re_{\sigma(i)}) \cong Re_i/J(R)e_i$ and $\mathrm{Soc}(e_iR) \cong e_{\sigma(i)}R/e_{\sigma(i)}J(R)$ for each $i = \{1, 2, \ldots, n\}$. A ring R is called *quasi-Frobenius* if R is one-sided Artinian one-sided self-injective. It is well-known that every quasi-Frobenius ring has a Nakayama permutation.

Lemma 6. *Let R be an indecomposable ring with nontrivial idempotents. Then R has a Nakayama permutation σ of $\{1, 2, \ldots, n\}$. In particular, $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, n$ if R is not a simple ring.*

Proof. By the hypothesis, R is indecomposable and so R is either semisimple Artinian or basic semiperfect by Theorem 1. If R is a semisimple Artinian ring, then R has a Nakayama permutation. Now, we assume that R is not a simple ring. It follows that R is a basic semiperfect ring.

For any $i \in \{1, 2, \ldots, n\}$, from the simplicity of $\mathrm{Soc}(e_iR)$, it infers that there exists $\sigma(i) \in \{1, 2, \ldots, n\}$ such that $\mathrm{Soc}(e_iR) \cong e_{\sigma(i)}R/e_{\sigma(i)}J(R)$. This map σ is a permutation of $\{1, 2, \ldots, n\}$ because $\sigma(i) = \sigma(j)$ implies that $\mathrm{Soc}(e_iR) \cong \mathrm{Soc}(e_jR)$. By the injectivity of e_iR and e_jR, we infer that $e_iR \cong e_jR$, and so $i = j$ (because the e_i are basic). Let $\alpha\colon e_{\sigma(i)}R/e_{\sigma(i)}J(R) \to \mathrm{Soc}(e_iR)$ be an isomorphism and $s_i = \alpha(e_{\sigma(i)} + e_{\sigma(i)}J(R))$. It follows that $s_iR = \mathrm{Soc}(e_iR)$ is a minimal right ideal of R. One can check that $J(R) + R(1 - e_i) \leq l_R(s_i)$. But $R/[J(R) + R(1 - e_i)] \cong Re_i/J(R)e_i$ is simple, and so $l_R(s_i) = J(R) + R(1 - e_i)$. It follows that $Rs_i \cong Re_i/J(R)e_i$. Now observe that $s_i = s_ie_{\sigma(i)} \in \mathrm{Soc}({}_RR)e_{\sigma(i)} = \mathrm{Soc}(Re_{\sigma(i)})$. We have, from Lemma 4, that $\mathrm{Soc}(Re_{\sigma(i)})$ is simple and obtain that $\mathrm{Soc}(Re_{\sigma(i)}) \cong Re_i/J(R)e_i$. Thus, R has a Nakayama permutation σ of $\{1, 2, \ldots, n\}$.

Next, we suppose that $\sigma(i) = i$ for some $i \in \{1, 2, \ldots, n\}$ or $\mathrm{Soc}(e_iR) \cong e_iR/e_iJ(R)$. Assume that $e_iR(1 - e_i) \neq 0$. Since R is a basic semiperfect ring, there would exist $j \in \{1, 2, \ldots, n\}$ with $j \neq i$ such that $e_iRe_j \neq 0$. Then there exists a nonzero homomorphism $\beta : e_jR \to e_iR$. By [11, Lemma 4.1] and e_iR is uniform, we infer that $\mathrm{Im}(\beta)$ is simple. It follows that $\mathrm{Im}(\beta) = \mathrm{Soc}(e_iR)$ and $\mathrm{Ker}(\beta)$ is maximal in e_jR. Then $\mathrm{Ker}(\beta) = e_jJ(R)$ which implies that $e_jR/e_jJ(R) \cong \mathrm{Soc}(e_iR) \cong e_iR/e_iJ(R)$. From this, it immediately infers that $e_iR \cong e_jR$, a contradiction. It is shown that $e_iR(1 - e_i) = 0$. Similarly, we have $(1 - e_i)Re_i = 0$. In fact, if $(1 - e_i)Re_i \neq 0$, then $e_kRe_i \neq 0$ for some $k \in \{1, 2, \ldots, n\}$ with $k \neq i$. By the above similar proof, we infer that $\mathrm{Soc}(e_iR) \cong e_iR/e_iJ(R) \cong \mathrm{Soc}(e_kR)$. By the injectivivity of e_iR and e_kR, we have $e_iR \cong e_kR$ which is impossible. It is shown that e_i is central, a contradiction. We deduce that $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, n$.

Lemma 7. *Let R be an indecomposable ring not simple with nontrivial idempotents. Then e_iRe_i is a division ring for any $i \in \{1, 2, \ldots, n\}$.*

Proof. By the hypothesis, R is a basic semiperfect ring and $1 = e_1 + e_2 + \ldots + e_n$. For any $i \in \{1, 2, \ldots, n\}$, there exists $j \neq i$ with $j \in \{1, 2, \ldots, n\}$ such that $e_iRe_j \neq 0$ by Lemma 6. Suppose that $e_iR(1 - e_i) = 0$. Then $e_iR\Big(\sum_{k\neq i}^{n} e_k\Big) = 0$, which implies that $e_iRe_j = 0$, a contradiction. Thus, $e_iR(1 - e_i) \neq 0$. Next, we show that $e_iJ(R)e_i = 0$. We have $e_iR(1 - e_i) \subset \operatorname{Soc}(eR)$ by Lemma 5, and so $e_iR(1 - e_i) = Soc(e_iR)(1 - e_i)$. Now, we show that $e_iJ(R)e_i$ is a submodule of e_iR. Since R is right automorphism-invariant, $J(R) = \{a \in \mathrm{R} : r_R(a) \leq^e R_R\}$ by [7, Proposition 1] and so $J(R) \operatorname{Soc}(e_iR) = 0$. Now $(e_iJ(R)e_i) \operatorname{Soc}(e_iR) = e_iJ(R) \operatorname{Soc}(e_iR) = 0$ which implies $(e_iJ(R)e_i)(e_iR(1 - e_i)) = 0$. On the other hand, we have

$$e_iJ(R)e_iR = e_iJ(R)e_i(Re_i + R(1 - e_i)) = e_iJ(R)e_iRe_i \subset e_iJ(R)e_i.$$

Hence $e_iJ(R)e_i$ is an R-submodule of e_iR. We have that $\operatorname{Soc}(e_iR)$ is simple and obtain $e_iJ(R)e_i \cap \operatorname{Soc}(e_iR) = 0$ or $\operatorname{Soc}(e_iR) \leq e_iJ(R)e_i$. Suppose $\operatorname{Soc}(e_iR) \leq e_iJ(R)e_i$. Then $e_iR(1 - e_i) = \operatorname{Soc}(e_iR)(1 - e_i) \leq e_iJ(R)e_i(1 - e_i) = 0$, a contradiction. It follows that $e_iJ(R)e_i \cap \operatorname{Soc}(e_iR) = 0$. Thus, $e_iJ(R)e_i = 0$ because $\operatorname{Soc}(e_iR)$ is essential in e_iR. Note that $e_iRe_i \cong \operatorname{End}(e_iR)$ is a local ring. We deduce that e_iRe_i is a division ring.

Theorem 2. *If R is an indecomposable ring not simple with nontrivial idempotents, then R is a quasi-Frobenius ring.*

Proof. By Lemma 4 and the hypothesis, R is a basic semiperfect right self-injective ring and $S = \operatorname{Soc}(R_R)$ is an Artinian right R-module. We have a decomposition $R = e_1R \oplus e_2R \oplus \ldots \oplus e_nR$. Then

$$R = \sum_{i=1}^{n} e_iRe_i + \sum_{i\neq j}^{n} e_iRe_j.$$

Note that $e_iRe_j \subseteq S$ for all $i \neq j$ by Lemma 5. We consider the following mapping:

$$\phi\colon R/S \to \bigoplus_{i=1}^{n} e_iRe_i$$

via $\phi\Big(\sum_{i=1}^{n} e_ir_ie_i\Big) + S = \sum_{i=1}^{n} e_ir_ie_i$. We show that ϕ is an isomorphism. If $\sum_{i=1}^{n} e_ir_ie_i \in S$, then $e_ir_ie_i \in e_iSe_i$ for all $i = 1, 2, \ldots, n$. Since $e_iJ(R)$ is the unique maximal submodule of e_iR, $e_iS \leq e_iJ(R)$, and so $e_ir_ie_i \in e_iJ(R)e_i$. Note that $e_iJ(R)e_i = 0$ by Lemma 7. It shows that ϕ is a mapping. One can check that ϕ is a ring homomorphism. Moreover, ϕ is a bijection, and so ϕ is a ring isomorphism. It shows that R/S is a semisimple Artinian ring. We deduce that R is a right Artinian ring, and so R is a quasi-Frobenius ring.

***Corollary* 4.** *Let R be an indecomposable ring not simple with nontrivial idempotents. Then the following conditions are equivalent:*

(1) *R is a right q-ring.*

(2) *R is a right a-ring.*

(3) *$eRf \subseteq \operatorname{Soc}(R_R)$ for each pair e, f of orthogonal idempotents of R.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) by Lemma 5.

(3) $\Rightarrow$ (1). By Theorem 2, R is a basic semiperfect quasi-Frobenius ring. It follows that R_R is injective cogenerator. Thus, R is a right q-ring by [6, Theorem 2.9].

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