DOI: 10.3842/umzh.v76i6.7497

UDC 517.9

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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF HAMILTONIAN SYSTEMS

ІСНУВАННЯ ТА МНОЖИННІСТЬ РОЗВ'ЯЗКІВ ДЛЯ ОДНОГО КЛАСУ ГАМІЛЬТОНОВИХ СИСТЕМ

We investigate a class of Hamiltonian systems

$$-q''(t) + (L(t) - \xi)q(t) = a(t)|q(t)|^{p-2}q(t) + \eta f(t),$$
$$q \in H^1(\mathbb{R}, \mathbb{R}^N),$$

where $(t, q) \in \mathbb{R} \times \mathbb{R}^N$, $p > 2$, $a \in C(\mathbb{R}, (0, +\infty))$, $f \in C(\mathbb{R}, \mathbb{R}^N)$, ξ, η are real parameters, and $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$. The main technical approach is based on the Nehari manifold method combined with variational and topological methods. The obtained results extend and complement the results available in the literature.

Ми досліджуємо клас гамільтонових систем

$$-q''(t) + (L(t) - \xi)q(t) = a(t)|q(t)|^{p-2}q(t) + \eta f(t),$$
$$q \in H^1(\mathbb{R}, \mathbb{R}^N),$$

де $(t, q) \in \mathbb{R} \times \mathbb{R}^N$, $p > 2$, $a \in C(\mathbb{R}, (0, +\infty))$, $f \in C(\mathbb{R}, \mathbb{R}^N)$, ξ та η — дійсні параметри і $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ — додатно визначена симетрична матриця для всіх $t \in \mathbb{R}$. Основний технічний підхід базується на методі многовиду Нехарі спільно з варіаційними і топологічними методами. Отримані результати розширюють і доповнюють результати, доступні в літературі.

1. Introduction. In this paper, we investigate the existence and multiplicity of solutions for the class of Hamiltonian system

$$\begin{aligned} &-q''(t) + (L(t) - \xi)q(t) = a(t)|q(t)|^{p-2}q(t) + \eta f(t), \\ &q \in H^1(\mathbb{R}, \mathbb{R}^N), \end{aligned} \tag{1.1}$$

where $(t, q) \in \mathbb{R} \times \mathbb{R}^N$, $p > 2$, $a \in C(\mathbb{R}, (0, +\infty))$, $f \in C(\mathbb{R}, \mathbb{R}^N)$, ξ, η are real parameters and $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$. We assume that L, f and a satisfies the following assumptions:

$(C1)$ For each $t \in \mathbb{R}$, $L(t)$ is a positive definite symmetric matrix. Moreover, there is a function $v_0 \in C(\mathbb{R}, (0, \infty))$ satisfying that $v_0(t) \to \infty$ as $|t| \to \infty$ as well as

$$(L(t)q, q) \geq v_0(t)|q|^2 \quad \text{for any} \quad t \in \mathbb{R}, \quad q \in \mathbb{R}^N.$$

$(C2)$ $a \in C(\mathbb{R}, (0, \infty)) \cap L^\infty$, $f \in C(\mathbb{R}, \mathbb{R}^N) \cap L^1$.

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In recent years, there has been a growing interest among researchers in studying various classes of Hamiltonian systems using variational methods. This is primarily due to the widespread application of partial differential systems in numerous engineering and scientific disciplines, including physics, chemistry, biology, economics, and control theory. It is worth noting that the Nehari manifold and variational methods have been widely employed in investigating the existence and multiplicity of homoclinic solutions for Hamiltonian systems. Several studies, such as [1 – 7, 10, 11, 13 – 15, 17, 18, 21 – 23], have extensively utilized these techniques, along with the references cited therein. In [12], Brown and Zhang have studied the elliptic equation by exploiting the relationship between the Nehari manifold and fibrering maps (i.e., maps of the form $t \to J(tu)$, where J is the Euler functional associated with the equation) they discuss how the Nehari manifold changes as λ changes and show how existence and nonexistence results for positive solutions of the equation are linked to properties of the manifold. In [19, 20], the author studied a semilinear scalar elliptic equation featuring concave-convex nonlinearities. They demonstrated the existence of multiple solutions in relation to the parameter ξ by extracting Palais – Smale sequences from the Nehari manifold. For further information on Nehari manifolds, interested readers can refer to the monographs [16] or [22]. Upon comparing problem (1.1) with those in the aforementioned papers, it becomes apparent that the introduction of the perturbation term ηf poses challenges in studying the existence of multiple solutions. Moreover, the presence of the nonlinear term $a(t)|q(t)|^{p-2}q(t)$ with $p > 2$ adds further intricacy to the analysis of the system using classical variational methods. Consequently, supplementary analytical techniques are required to comprehensively investigate the problem (1.1).

Motivated by the above works, in this paper the associated action integral corresponding to system (1.1) lacks a lower bound. Consequently it is invalid to prove the existence of critical value by applying directly the method of minimization. In order to address this challenge, we propose the utilization of the Nehari manifold $M_{\xi,\eta}$, which we divide into three distinct sections associated with local maxima, local minima, and points of inflection. These sections are denoted as $M^{-}_{\xi,\eta}$, $M^{+}_{\xi,\eta}$, and $M^{0}_{\xi,\eta}$, respectively. To establish the existence and multiplicity of weak solutions, we employ the Nehari manifold methods and various analysis techniques. Specifically, we seek the minimizers of the associated action integral on both $M^{-}_{\xi,\eta}$ and $M^{+}_{\xi,\eta}$. Through this approach, we are able to demonstrate the desired results.

2. Preliminaries and main results. In this section, we will set up some definitions and basic properties of the working spaces and some preliminary results. Let $\mathcal{H}$ be the Hilbert space defined as

$$\mathcal{H} = \left\{ q \in H^1\big(\mathbb{R}, \mathbb{R}^N\big) : \int\limits_{\mathbb{R}} \big[|q'(t)|^2 + (L(t)q(t), q(t))\big] dt < +\infty \right\}$$

endowed with the inner product

$$(q, v) = \int\limits_{\mathbb{R}} [(q'(t), v'(t)) + (L(t)q(t), v(t))] dt$$

and the corresponding norm $\|q\|^2 = (q, q)$. Note that $\mathcal{H} \subset H^1(\mathbb{R}, \mathbb{R}^N) \subset L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, +\infty]$ with the imbedding being continuous. In particular, for $p = +\infty$, there exists a constant $C > 0$ such that $\|q\|_\infty \le C\|q\|$ $\forall q \in \mathcal{H}$. Here, $L^p(\mathbb{R}, \mathbb{R}^N)$, $2 \le p < +\infty$, and $H^1\big(\mathbb{R}, \mathbb{R}^N\big)$ denote the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$\|q\|_p := \left(\int_{\mathbb{R}} |q(t)|^p dt\right)^{1/p} \qquad \text{and} \qquad \|q\|_{H^1} := \left(\|q\|_2^2 + \|q'\|_2^2\right)^{1/2},$$

respectively. $L^\infty\left(\mathbb{R}, \mathbb{R}^N\right)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm $\|q\|_\infty := \operatorname{ess\,sup}\{|q(t)| : t \in \mathbb{R}\}$.

Now, we have the following result.

Lemma 2.1. *Suppose that $(C1)$ is holds. Then the embedding of $\mathcal{H}$ in $L^p(\mathbb{R}, \mathbb{R}^N)$ is compact for $p \in [2, +\infty)$ and is continuous for $p \in [2, +\infty]$.*

Proof. Let $\{q_k\} \subset \mathcal{H}$ be a sequence such that $q_k \rightharpoonup q$ in $\mathcal{H}$. We will prove that $q_k \to q$ in L^p. Without loss of generality, we suppose that $q_k \rightharpoonup 0$ in $\mathcal{H}$. The Banach – Steinhaus theorem implies that

$$A = \sup_k \|q_k\| < +\infty.$$

Let $\varepsilon > 0$. There is $T_0 < 0$ such that $\dfrac{1}{v_0(t)} \leq \varepsilon$ for all t such that $t \leq T_0$. By the same way, there is $T_1 > 0$ such that $\dfrac{1}{v_0(t)} \leq \varepsilon$ for all $t \geq T_1$. Since $v_0(t) > c > 0$ on $]T_0, T_1[\ = I$, the operator defined by $S\colon \mathcal{H} \to H^1(I)\colon q \to q|_I$ is a linear continuous map. So $q_k \to 0$ in $H^1(I)$. Sobolev's theorem (see [8]) implies that $q_k \to 0$ uniformly on $\bar{I}$, so there is a k_0 such that

$$\int_I |q_k(t)|^p dt \leq \varepsilon \quad \text{for all} \quad k \geq k_0 \quad \text{and} \quad p \geq 2. \tag{2.1}$$

Since $\dfrac{1}{v_0(t)} \leq \varepsilon$ on $]-\infty, T_0]$, we have

$$\int_{-\infty}^{T_0} |q_k(t)|^p dt \leq \varepsilon \int_{-\infty}^{T_0} \alpha(t) |q_k(t)|^p dt \leq \varepsilon A^p. \tag{2.2}$$

Similarly, since $\dfrac{1}{v_0(t)} \leq \varepsilon$ on $[T_1, +\infty[$, we have

$$\int_{T_1}^{+\infty} |q_k(t)|^p dt \leq \varepsilon A^p. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we get that $q_k \to 0$ in L^p for all $p \in [2, +\infty)$.

By Lemma 2.1, there exists $C_p > 0$ such that

$$\|q\|_{L^p} \leq C_p \|q\| \quad \forall p \in [2, +\infty], \quad q \in \mathcal{H}.$$

We are going to establish the corresponding variational framework to obtain solutions of (1.1). To this end, we define the functional $\mathcal{U}_{\xi,\eta}\colon \mathcal{H} \to \mathbb{R}$ as

$$\mathcal{U}_{\xi,\eta}(q) = \frac{1}{2}\|q\|^2 - \frac{\xi}{2}\int\limits_{\mathbb{R}} |q(t)|^2 dt$$
$$- \frac{1}{p}\int\limits_{\mathbb{R}} a(t)|q(t)|^p dt - \eta \int\limits_{\mathbb{R}} f(t)q(t)dt. \tag{2.4}$$

Assume that $(C1)$ and $(C2)$ are satisfied. It is not difficult to see that $\mathcal{U}_{\xi,\eta}$ is continuously differentiable, i.e., $\mathcal{U}_{\xi,\eta} \in C^1(\mathcal{H}, \mathbb{R})$, and we have

$$\langle \mathcal{U}'_{\xi,\eta}(q), v\rangle = \int\limits_{\mathbb{R}} \big[q'(t), v'(t) + (L(t)q(t), v(t))\big] dt - \xi \int\limits_{\mathbb{R}} q(t)v(t)dt$$
$$- \int\limits_{\mathbb{R}} a(t)|q(t)|^{p-2} q(t)v(t)dt - \eta \int\limits_{\mathbb{R}} f(t)v(t)dt$$

for $q, v \in \mathcal{H}$. Due to the condition $p > 2$, the energy functional $\mathcal{U}_{\xi,\eta}$ lacks a lower bound on $\mathcal{H}$. We move to considers $\mathcal{U}_{\xi,\eta}$ on the Nehari manifold $M_{\xi,\eta}$ given as

$$M_{\xi,\eta} := \big\{q \in \mathcal{H}\backslash\{0\} \mid \langle \mathcal{U}'_{\xi,\eta}(q), q\rangle = 0\big\}.$$

Obviously, $q \in M_{\xi,\eta}$ if and only if $q \in \mathcal{H}$ fulfill

$$\|q\|^2 - \xi \int\limits_{\mathbb{R}} |q|^2 dt - \int\limits_{\mathbb{R}} a(t)|q|^p dt - \eta \int\limits_{\mathbb{R}} f(t)q(t)dt = 0. \tag{2.5}$$

Firstly, to show a critical points for $\mathcal{U}_{\xi,\eta}$, the Nehari manifold $M_{\xi,\eta}$ needs to be split into three subsets:

$$M^+_{\xi,\eta} := \big\{q \in M_{\xi,\eta} \mid \langle \mathcal{J}'_{\xi,\eta}(q), q\rangle > 0\big\}, \tag{2.6}$$
$$M^0_{\xi,\eta} := \big\{q \in M_{\xi,\eta} \mid \langle \mathcal{J}'_{\xi,\eta}(q), q\rangle = 0\big\}, \tag{2.7}$$
$$M^-_{\xi,\eta} := \big\{q \in M_{\xi,\eta} \mid \langle \mathcal{J}'_{\xi,\eta}(q), q\rangle < 0\big\} \tag{2.8}$$

with $\mathcal{J}_{\xi,\eta}(q) = \langle \mathcal{U}'_{\xi,\eta}(q), q\rangle$. Exactly, we get

$$\langle \mathcal{J}'_{\xi,\eta}(q), q\rangle = 2\|q\|^2 - 2\xi \int\limits_{\mathbb{R}} |q|^2 dt - p \int\limits_{\mathbb{R}} a(t)|q|^p dt - \eta \int\limits_{\mathbb{R}} f(t)q(t)dt. \tag{2.9}$$

In view of Lemma 2.1 for $p \in [2, \infty]$, we define

$$\lambda_p = \inf\{C > 0 \colon \|q\|_{L^p} \le C\|q\| \quad \text{for all} \quad q \in \mathcal{H}\}. \tag{2.10}$$

Let $\xi_0 = \dfrac{1}{\lambda_2^2}$, $a_\infty = \sup_{t\in\mathbb{R}} a(t)$. For $-\infty < \xi < \xi_0$, we denote η_ξ as

$$\eta_\xi = \big(a_\infty \lambda_p^p\big)^{\frac{1}{2-p}} \frac{p-2}{\|f\|_{L^1}\lambda_\infty} \left(\frac{1-\xi\lambda_2^2}{p-1}\right)^{\frac{p-1}{p-2}} > 0. \tag{2.11}$$

Presently, we are prepared to provide our findings regarding the presence and multiple occurrences of solutions to equation (1.1).

Theorem 2.1. *Suppose that* $(C1)$ *and* $(C2)$ *hold. If* $f \equiv 0$, *then problem* (1.1) *has at least one nontrivial weak solution for* $-\infty < \xi < \eta_0$. *If* $f \neq 0$, *then problem* (1.1) *has at least two nontrivial weak solutions for* $-\infty < \xi < \xi_0,\ 0 < \eta < \dfrac{\eta_\xi}{2}$.

We have the following results.

Lemma 2.2. *Assume that* $(C1)$ *and* $(C2)$ *hold. If* $-\infty < \xi < \xi_0$, *then* $\mathcal{U}_{\xi,\eta}$ *is bounded below and coercive on* $M_{\xi,\eta}$.

Proof. For $q \in M_{\xi,\eta}$, from (2.5), we have

$$\int\limits_{\mathbb{R}} a(t)|q|^p dt = \|q\|^2 - \xi \int\limits_{\mathbb{R}} |q|^2 dt - \eta \int\limits_{\mathbb{R}} f(t)q(t)dt.$$

As a result, utilizing equations (2.4) and (2.10), we get

$$\begin{aligned}\mathcal{U}_{\xi,\eta}(q) &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|q\|^2 - \xi \int\limits_{\mathbb{R}} |q|^2 dt\right) - \eta\left(1 - \frac{1}{p}\right) \int\limits_{\mathbb{R}} f(t)q(t)dt \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \xi\lambda_2^2\right)\|q\|^2 - \eta\left(1 - \frac{1}{p}\right) \int\limits_{\mathbb{R}} |f(t)||q(t)|dt \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \xi\lambda_2^2\right)\|q\|^2 - \eta\left(1 - \frac{1}{p}\right)\|f\|_{L^1}\|q\|_\infty \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \xi\lambda_2^2\right)\|q\|^2 - \eta\left(1 - \frac{1}{p}\right)\|f\|_{L^1}\lambda_\infty\|q\|.\end{aligned}$$

Therefore, considering the fact that $\xi < \xi_0$ and $p > 2$, we can conclude that $\mathcal{U}_{\xi,\eta}$ is bounded below and coercive on $M_{\xi,\eta}$.

Lemma 2.3. *Assume that the assumptions* $(C1)$ *and* $(C2)$ *hold. If* $q_0 \in M_{\xi,\eta}$ *is a local minimizer of* $\mathcal{U}_{\xi,\eta}$ *on* $M_{\xi,\eta}$ *with* $q_0 \notin M^0_{\xi,\eta}$, *then* q_0 *is a nontrivial solution to problem* (1.1).

Proof. Since $q_0 \in M_{\xi,\eta}$ is a local minimizer of $\mathcal{U}_{\xi,\eta}$ on $M_{\xi,\eta}$, using Theorem 4.1.1 in [9], there exists a number μ such that

$$\mathcal{U}'_{\xi,\eta}(q_0) = \mu\mathcal{J}'_{\xi,\eta}(q_0), \tag{2.12}$$

where $\mathcal{J}_{\xi,\eta}(q) = \mathcal{U}'_{\xi,\eta}(q)q$. Therefore, if $q_0 \in M_{\xi,\eta}$ it can be inferred that

$$0 = \mathcal{U}'_{\xi,\eta}(q_0)q_0 = \mu\mathcal{J}'_{\xi,\eta}(q_0)q_0.$$

Considering that $q_0 \notin M^0_{\xi,\eta}$, we derive $\mathcal{J}'_{\xi,\eta}(q_0)q_0 \neq 0$, and as a result $\mu = 0$. Thus, $\mathcal{U}'_{\xi,\eta}(q_0) = 0$ follows from (2.12). This implies that q_0 represents a nontrivial solution to problem (1.1).

Lemma 2.4. *Suppose that the assumptions* $(C1)$ *and* $(C2)$ *hold. If* $f \equiv 0$, *then* $M^0_{\xi,\eta} = \varnothing$, *and if* $f \neq 0$, *then* $M^0_{\xi,\eta} = \varnothing$ *for* $-\infty < \xi < \xi_0, 0 < \eta < \eta_\xi$, *where* η_ξ *as in* (2.11).

Proof. In fact, if not, then there exists $q_0 \in M^0_{\xi,\eta}$ with $q_0 \neq 0$. Then, by (2.5)–(2.9), we get

$$\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q_0|^2 dt - \int\limits_{\mathbb{R}} a(t)|q_0|^p dt - \eta \int\limits_{\mathbb{R}} f(t)q_0(t)dt = 0, \tag{2.13}$$

$$2\|q_0\|^2 - 2\xi \int\limits_{\mathbb{R}} |q_0|^2 dt - p \int\limits_{\mathbb{R}} a(t)|q_0|^p dt - \eta \int\limits_{\mathbb{R}} f(t) q_0(t) dt = 0. \tag{2.14}$$

(i) If $f \equiv 0$, then, by (2.13), (2.14), we obtain $\int_{\mathbb{R}} a(t)|q_0|^p dt = 0$, which is a contradiction noting that $q_0 \neq 0$ and $a \in C(\mathbb{R}, (0, \infty))$.

(ii) If $f \neq 0$, then, by subtracting (2.13) from (2.14), we have

$$\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q_0|^2 dt - (p-1) \int\limits_{\mathbb{R}} a(t)|q_0|^p dt = 0. \tag{2.15}$$

Multiplying (2.13) by p and subtracting (2.14), we get

$$(p-2)\left(\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q_0|^2 dt \right) - \eta(p-1) \int\limits_{\mathbb{R}} f(t) q_0(t) dt = 0. \tag{2.16}$$

Thereby by combining equations (2.15), (2.16) along with equation (2.10), we obtain the following:

$$\left(1 - \xi \lambda_2^2\right)\|q_0\|^2 \leq (p-1) a_\infty \int\limits_{\mathbb{R}} |q_0|^p dt \leq (p-1) a_\infty \lambda_p^p \|q_0\|^p \tag{2.17}$$

and

$$(p-2)(1 - \xi \lambda_2^2)\|q_0\|^2 \leq \eta(p-1)\|f\|_{L^1}\|q_0\|_\infty \leq \eta(p-1)\|f\|_{L^1}\lambda_\infty \|q_0\|. \tag{2.18}$$

If $q_0 \neq 0$ and $\xi < \xi_0$, from equations (2.17), (2.18) it can be deduced that $\eta \geq \eta_\xi$. However, this contradicts the assumption $0 < \eta < \eta_\xi$. Therefore, we conclude that $M^0_{\xi,\eta} = \varnothing$.

3. Proof of main results. In this section, we present several existence results concerning solutions to problem (1.1).

Lemma 3.1. *Assume that the assumptions* $(C1)$ *and* $(C2)$ *hold. If* $f \equiv 0$, *then* $M^-_{\xi,\eta} \neq \varnothing$ *for* $-\infty < \xi < \xi_0$, *and if* $f \neq 0$, *then* $M^+_{\xi,\eta} \neq \varnothing$ *and* $M^-_{\xi,\eta} \neq \varnothing$ *for* $-\infty < \xi < \xi_0, 0 < \eta < \eta_\xi$, *where* η_ξ *as in* (2.11).

Proof. *Case* 1. If $f \equiv 0$, then, for any $q_0 \in \mathcal{H} \backslash \{0\}$,

$$\mathcal{U}_{\xi,\eta}(q_0) = \frac{1}{2}\left(\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q_0|^2 dt \right) - \frac{1}{p} \int\limits_{\mathbb{R}} a(t)|q_0|^p dt.$$

Let

$$\phi_{q_0}(s) = \mathcal{U}_{\xi,\eta}(s q_0) = \frac{s^2}{2}\left(\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q_0|^2 dt \right) - \frac{s^p}{p} \int\limits_{\mathbb{R}} a(t)|q_0|^p dt, s \in [0, \infty).$$

So

$$\phi'_{q_0}(s) = s\left(\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q_0|^2 dt \right) - s^{p-1} \int\limits_{\mathbb{R}} a(t)|q_0|^p dt, \quad s \in [0, \infty). \tag{3.1}$$

Write $\psi_{q_0}(s) = \phi'_{q_0}(s),\ s \in [0,\infty)$. Due to that $q_0 \neq 0,\ a(t) > 0,\ t \in \mathbb{R},\ p > 2,\ \xi < \xi_0$, we get

$$\|q_0\|^2 - \xi \int\limits_{\mathbb{R}} |q|^2 dt \geq \left(1 - \xi\lambda_2^2\right)\|q_0\|^2 > 0.$$

Then we have that $\psi_{q_0}(0) = 0,\ \psi_{q_0}(s) > 0$ for small $s > 0$ and $\psi_{q_0}(s) \to -\infty$ as $s \to \infty$. Thus, there exists unique $\bar{s} > 0$ such that

$$\psi'_{q_0}(\bar{s}) = 0, \qquad \psi_{q_0}(\bar{s}) = \max_{s \geq 0} \psi_{q_0}(s) > 0, \qquad \psi'_{q_0}(s) > 0 \quad \text{for} \quad s \in (0, \bar{s}),$$

$$\psi'_{q_0}(s) < 0 \quad \text{for} \quad s \in (\bar{s}, \infty).$$

Furthermore, it follows from (3.1) that

$$\bar{s} = \left(\frac{\|q_0\|^2 - \xi \displaystyle\int_{\mathbb{R}} |q_0|^2 dt}{(p-1) \displaystyle\int_{\mathbb{R}} a(t)|q_0|^p dt} \right)^{\frac{1}{p-2}}.$$

On the contrary, since $\psi_{q_0}(\bar{s}) > 0,\ \psi'_{q_0}(s) < 0$ for $s \in (\bar{s}, \infty)$ and $\psi_{q_0}(s) \to -\infty$, there exists unique $s^- > \bar{s}$ such that $\psi_q(s^-) = 0$. Obviously $\psi'_q(s^-) < 0$. Thus, from

$$\left\langle \mathcal{J}'_{\xi,\eta}(sq_0), sq_0 \right\rangle = (\psi_{q_0}(s)s)' s = \psi'_{q_0}(s)s^2 + \psi_{q_0}(s)s, \quad s \in [0,\infty),$$

we have

$$\left\langle \mathcal{J}'_{\xi,\eta}(s^- q_0), s^- q_0 \right\rangle = \psi'_{q_0}(s^-)(s^-)^2 + \psi_{q_0}(s^-)s^- = \psi'_{q_0}(s^-)(s^-)^2 < 0.$$

Hence,

$$s^- q_0 \in M^-_{\xi,\eta}, \qquad \phi'_{q_0}(s^-) = 0, \qquad \phi''_{q_0}(s^-) < 0,$$

and

$$\phi_{q_0}(s^-) = \sup_{s \geq 0} \mathcal{U}_{\xi,\eta}(sq_0), \tag{3.2}$$

noting that $\phi''_{q_0} = \psi'_{q_0}$. Thus, $M^-_{\xi,\eta} \neq \varnothing$.

Case 2. If $f \neq 0$, then, without loss of generality, we can say that $f(t) = (f_1, f_2, \ldots, f_N)$ satisfies $f_1(t) > 0$ for all $t \in (t_0 - r, t_0 + r)$, for some $t_0 \in \mathbb{R}$ and some $r > 0$. Take $\bar{q} = (q_1, q_2, \ldots, q_N)$ with $q_k = 0$ for $2 \leq k \leq N$, and $q_1(t)$ is given by

$$q_1(t) = \begin{cases} \dfrac{(t-t_0)^2}{(t-t_0)^2 - r^2} & \text{for} \quad |t - t_0| < r, \\ 0 & \text{for} \quad |t - t_0| \geq r. \end{cases}$$

Then $\bar{q} \in C_0^\infty\left(\mathbb{R}, \mathbb{R}^N\right)$ and, therefore, $\bar{q} \in \mathcal{H}$ with $\bar{q} \neq 0$. Otherwise, $\displaystyle\int_{\mathbb{R}} f(t)\bar{q}(t)dt > 0$. Next, we present a similar argument as in Case 1. Let

$$\phi_{\bar q}(s)=\mathcal{U}_{\xi,\eta}(s\bar q)=\frac{s^2}{2}\|\bar q\|^2-\frac{\xi s^2}{2}\int\limits_{\mathbb{R}}|\bar q|^2dt$$
$$-\frac{s^p}{p}\int\limits_{\mathbb{R}}a(t)|\bar q|^pdt-s\eta\int\limits_{\mathbb{R}}f(t)\bar q(t)dt=0,\quad s\in[0,\infty).$$

Then

$$\phi'_{\bar q}(s)=s\left(\|\bar q\|^2-\xi\int\limits_{\mathbb{R}}|\bar q|^2dt\right)$$
$$-s^{p-1}\int\limits_{\mathbb{R}}a(t)|\bar q|^pdt-\eta\int\limits_{\mathbb{R}}f(t)\bar q(t)dt,\quad s\in[0,\infty). \tag{3.3}$$

Let

$$\psi_{\bar q}(s)=s\left(\|\bar q\|^2-\xi\int\limits_{\mathbb{R}}|\bar q|^2dt\right)-s^{p-1}\int\limits_{\mathbb{R}}a(t)|\bar q|^pdt.$$

Then

$$\psi'_{\bar q}(s)=\|\bar q\|^2-\xi\int\limits_{\mathbb{R}}|\bar q|^2dt-(p-1)s^{p-2}\int\limits_{\mathbb{R}}a(t)|\bar q|^pdt,\quad s\in[0,\infty).$$

Clearly, $\psi_{\bar q}(0)=0,\ \psi_{\bar q}(s)>0$ as $s>0$ small enough, and $\psi_{\bar q}(s)\to-\infty$ as $s\to\infty$. Consequently, there exists unique $\bar s>0$ such that

$$\psi_{\bar q}(\bar s)=\max_{s\ge 0}\psi_{\bar q}(s)>0,\qquad \psi'_{\bar q}(\bar s)=0, \tag{3.4}$$

$$\psi'_{\bar q}(s)>0\quad\text{for}\quad s\in(0,\bar s),\qquad \psi'_{\bar q}(s)<0\quad\text{for}\quad s\in(\bar s,\infty). \tag{3.5}$$

Furthermore, through calculation, we obtain

$$\psi_{\bar q}(\bar s)=\frac{p-2}{p-1}\left(\frac{\left(\|\bar q\|^2-\xi\int_{\mathbb{R}}|\bar q|^2dt\right)^{p-1}}{(p-1)\int_{\mathbb{R}}a(t)|\bar q|^pdt}\right)^{\frac{1}{p-2}}$$
$$\ge\frac{p-2}{p-1}\left(\frac{\left(\left(1-\xi\lambda_2^2\right)\|\bar q\|^2\right)^{p-1}}{(p-1)a_\infty\lambda_p^p\|\bar q\|^p}\right)^{\frac{1}{p-2}}$$
$$=\frac{p-2}{p-1}\left(\frac{\left(1-\xi\lambda_2^2\right)^{p-1}}{(p-1)b_\infty\lambda_p^p}\right)^{\frac{1}{p-2}}\|\bar q\|.$$

Since $0 < \eta \int_{\mathbb{R}} f(t)\bar{q}(t)dt \leq \eta\|f\|_{L^1}\|\bar{q}\|_\infty \leq \eta\|f\|_{L^1}\lambda_\infty\|\bar{q}\|$, it can be easily verified that $0 < \eta \int_{\mathbb{R}} f(t)\bar{q}(t)dt < \psi_{\bar{q}}(\bar{s})$ as long as $0 < \eta < \eta_\xi$, where η_ξ is given by (2.11) and, therefore, $\phi'_{\bar{q}}(\bar{s}) > 0$ by (3.3). Now, considering (3.3) in conjunction with the fact that $\phi'_{\bar{q}}(0) < 0$, $\phi'_{\bar{q}}(\bar{s}) > 0$, it follows from (3.4), (3.5) that there exist unique $0 < s^+ < \bar{s}$ and unique $s_2^- > \bar{s}$ such that

$$\phi'_{\bar{q}}\big(s_1^+\big) = \phi'_{\bar{q}}\big(s_2^-\big) = 0, \quad \phi''_{\bar{q}}\big(s_1^+\big) > 0, \quad \phi''_{\bar{q}}\big(s_2^-\big) < 0; \qquad \phi''_{\bar{q}}(s) > 0, \quad s \in (0, \bar{s}),$$
$$\phi''_{\bar{q}}(s) < 0, \quad s \in (\bar{s}, \infty), \tag{3.6}$$

observing that (3.5) together with the fact that $\phi'_{\bar{q}}(s) = \psi_{\bar{q}}(s)$, $\phi''_{\bar{q}}(s) = \psi'_{\bar{q}}(s)$, $s \in [0, \infty)$. Hence, from

$$\big\langle \mathcal{J}'_{\xi,\eta}(s\bar{q}), s\bar{q}\big\rangle = \big(\phi'_{\bar{q}}(s)s\big)'s = \phi''_{\bar{q}}(s)s^2 + \phi'_{\bar{q}}(s)s,$$

together with (3.6), we have, respectively,

$$\big\langle \mathcal{J}'_{\xi,\eta}\big(s_1^+\bar{q}\big), s_1^+\bar{q}\big\rangle = \phi''_{\bar{q}}\big(s_1^+\big)\big(s_1^+\big)^2 + \phi'_{\bar{q}}\big(s_1^+\big)s_1^+ = \phi''_{\bar{q}}\big(s_1^+\big)\big(s_1^+\big)^2 > 0$$

and

$$\big\langle \mathcal{J}'_{\xi,\eta}\big(s_2^-\bar{q}\big), s_2^-\bar{q}\big\rangle = \phi''_{\bar{q}}\big(s_2^-\big)\big(s_2^-\big)^2 + \phi'_{\bar{q}}\big(s_2^-\big)s_2^- = \phi''_{\bar{q}}\big(s_2^-\big)\big(s_2^-\big)^2 < 0.$$

This implies that

$$s_1^+\bar{q} \in M^+_{\xi,\eta}, \quad s_2^-\bar{q} \in M^-_{\xi,\eta}, \qquad \phi_{\bar{q}}\big(s_1^+\big) = \min_{0\leq s\leq\bar{s}} \phi_{\bar{q}}(s), \qquad \phi_{\bar{q}}\big(s_2^-\big) = \max_{s\geq 0} \phi_q(s) \tag{3.7}$$

noting that (3.6).

Lemma 3.2. *Let us consider the assumptions* $(C1)$ *and* $(C2)$ *to be valid. If* $f \neq 0$, *then* $-\infty < C^+_{\xi,\eta} := \inf_{q\in M^+_{\xi,\eta}} \mathcal{U}_{\xi,\eta} < 0$ *for* $-\infty < \xi < \xi_0$, $0 < \eta < \eta_\xi$, *where* η_ξ *as in* (2.11).

Proof. By Lemma 3.1, $M^+_{\xi,\eta} \neq \varnothing$. For any $q \in M^+_{\xi,\eta}$, we have

$$0 = \mathcal{U}'_{\xi,\eta}(q)q = \|q\|^2 - \xi \int_{\mathbb{R}} |q|^2 dt - \int_{\mathbb{R}} a(t)|q|^p dt - \eta \int_{\mathbb{R}} f(t)q(t)dt \tag{3.8}$$

and

$$0 < \big(\mathcal{U}'_{\xi,\eta}(q)q\big)'q = 2\|q\|^2 - 2\xi \int_{\mathbb{R}} |q|^2 dt - p \int_{\mathbb{R}} a(t)|q|^p dt - \eta \int_{\mathbb{R}} f(t)q(t)dt. \tag{3.9}$$

By (3.8), (3.9), we have

$$(p-1) \int_{\mathbb{R}} a(t)|q|^p dt < \|q\|^2 - \xi \int_{\mathbb{R}} |q|^2 dt. \tag{3.10}$$

However, utilizing (2.4) in conjunction with (3.8) and (3.10), we obtain the following:

$$\mathcal{U}_{\xi,\eta}(q) = -\frac{1}{2}\left(\|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt\right) + \frac{p-1}{p}\int\limits_{\mathbb{R}}a(t)|q|^pdt$$

$$< -\frac{1}{2}\left(\|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt\right) + \frac{1}{p}\left(\|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt\right)$$

$$= \left(\frac{1}{p} - \frac{1}{2}\right)\left(\|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt\right) < 0.$$

Note that $p > 2$ and $q \neq 0$. Hence, $C^+_{\xi,\eta} = \inf_{q\in M^+_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q) < 0$. On the other hand, in terms of Lemma 2.2 and the fact that $\mathcal{U}_{\xi,\eta}$ is bounded below, we get that $C^+_{\xi,\eta} > -\infty$.

Lemma 3.3. *Assume that the assumptions* $(C1)$ *and* $(C2)$ *hold. In addition, if*

(i) $f \equiv 0$, *then* $C^-_{\xi,\eta} := \inf_{q\in M^-_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q) > 0$ *for* $-\infty < \xi < \xi_0$,

(ii) $f \neq 0$, *then* $C^-_{\xi,\eta} := \inf_{q\in M^-_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q) > 0$ *for* $-\infty < \xi < \xi_0$, $0 < \eta < \eta_\xi/2$, *where* η_ξ *as in* (2.11).

Proof. By Lemma 3.1, in Case 1 or Case 2 we always have $M^-_{\xi,\eta} \neq \varnothing$. For any $q \in M^-_{\xi,\eta}$, by $\mathcal{U}'_{\xi,\eta}(q)q = 0$, we have

$$\eta\int\limits_{\mathbb{R}}f(t)q(t)dt = \|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt - \int\limits_{\mathbb{R}}a(t)|q|^pdt. \tag{3.11}$$

Thus, by $\Big((\mathcal{U}'_{\xi,\eta}q)q\Big)'q < 0$, it follows from (3.11) that

$$0 > 2\|q\|^2 - 2\xi\int\limits_{\mathbb{R}}|q|^2dt - p\int\limits_{\mathbb{R}}a(t)|q|^qdt - \eta\int\limits_{\mathbb{R}}f(t)q(t)dt$$

$$= \|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt - (p-1)\int\limits_{\mathbb{R}}a(t)|q|^pdt,$$

and, therefore,

$$\big(1-\xi\lambda_2^2\big)\|q\|^2 \le \|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt < (p-1)\int\limits_{\mathbb{R}}a(t)|q|^pdt \le (p-1)a_\infty\lambda_p^p\|q\|^p.$$

Thus,

$$\|q\| > \left(\frac{\big(1-\xi\lambda_2^2\big)}{(p-1)a_\infty\lambda_p^p}\right)^{\frac{1}{p-2}} := d_0 > 0. \tag{3.12}$$

On the other hand, by (2.4) together with (3.11) as well as (2.10), we get

$$\mathcal{U}_{\xi,\eta}(q) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|q\|^2 - \xi\int\limits_{\mathbb{R}}|q|^2dt\right) - \eta\left(1 - \frac{1}{p}\right)\int\limits_{\mathbb{R}}f(t)q(t)dt$$

$$\geq \left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)\|q\|^2-\eta\left(1-\frac{1}{p}\right)\lambda_\infty\|f\|_{L^1}\|q\|$$

$$=\left[\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)\|q\|-\eta\left(1-\frac{1}{p}\right)\lambda_\infty\|f\|_{L^1}\right]\|q\|. \tag{3.13}$$

(i) If $f\equiv 0$, then by (3.12), (3.13), we obtain

$$\mathcal{U}_{\xi,\eta}(q)\geq\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)d_0^2>0$$

and, therefore,

$$\inf_{q\in M_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q)\geq\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)d_0^2>0.$$

(ii) If $f\neq 0$, then $\|f\|_{L^1}>0$. For $0<\eta<\dfrac{\eta_\xi}{2}$, by calculation, we show that

$$\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)\|q\|-\eta\left(1-\frac{1}{p}\right)\lambda_\infty\|f\|_{L^1}$$
$$\geq\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)d_0-\eta\left(1-\frac{1}{p}\right)\lambda_\infty\|f\|_{L^1}>0$$

and

$$\mathcal{U}_{\xi,\eta}(q)\geq\left[\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\xi\lambda_2^2\right)d_0-\eta\left(1-\frac{1}{p}\right)\lambda_\infty\|f\|_{L^1}\right]d_0:=d_1>0.$$

Therefore, $\inf_{q\in M^-_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q)\geq d_1>0$.

Lemma 3.4. *Assume that the assumptions $(C1)$, $(C2)$ hold. If $f\neq 0$, then, for $-\infty<\xi<\xi_0$, $0<\eta<\eta_\xi$ (where η_ξ as in (2.11)), $\mathcal{U}_{\xi,\eta}$ achieves its minimum at some $\bar q\in M^+_{\xi,\eta}$ on $M^+_{\xi,\eta}$, that is,*

$$\mathcal{U}_{\xi,\eta}(\bar q)=\inf_{q\in M^+_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q):=C^+_{\xi,\eta}.$$

Proof. First, under conditions $(C1),(C2)$ and $f\neq 0$, for $\xi<\xi_0$, $0<\eta<\eta_\xi$, by Lemmas 3.1, 3.2, we know that $M^+_{\xi,\eta}\neq\varnothing$, and $-\infty<C^+_{\xi,\eta}<0$. Let $\{q_n\}\subset M^+_{\xi,\eta}$ be a minimizing sequence for $\mathcal{U}_{\xi,\eta}$ on $M^+_{\xi,\eta}$, that is, $\lim_{n\to\infty}\mathcal{U}_{\xi,\eta}(q_n)=C^+_{\xi,\eta}$. Again, it follows from Lemma 2.2 that $\{q_n\}$ is bounded in $\mathcal{H}$. Thus, by the reflexivity of $\mathcal{H}$, there exists a subsequence, still denoted by $\{q_n\}$, such that $q_n\rightharpoonup\bar q$ in $\mathcal{H}$. Hence, employing Lemma 2.1, we can conclude that

$$q_n\to\bar q\quad\text{in}\quad L^p(\mathbb{R}),\quad p\in[2,\infty),\quad q_n(t)\to\bar q(t),\quad\text{a.e.}\quad t\in\mathbb{R}.$$

Again, since $\{q_n\}$ is bounded in $\mathcal{H}$, there is a constant $M>0$ such that $\|q_n\|\leq M$, $n\geq 1$, and, therefore, it follows from (2.10) that

$$|f(t)\cdot q_n(t)|\leq|f(t)|\|q_n\|_\infty\leq|f(t)|\lambda_\infty\|q_n\|\leq|f(t)|\lambda_\infty M\in L^1(\mathbb{R}).$$

Therefore, utilizing the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(t)q_n(t)dt=\int_{\mathbb{R}} f(t)\bar{q}(t)dt. \tag{3.14}$$

Likewise, we additionally possess

$$\lim_{n\to\infty}\int_{\mathbb{R}} a(t)|q_n(t)|^p dt=\int_{\mathbb{R}} a(t)|\bar{q}(t)|^p dt. \tag{3.15}$$

Next, we demonstrate that $\lim_{n\to\infty}\|q_n\|=\|\bar{q}\|$. Indeed, assuming the contrary, by utilizing the weak lower semi-continuity on the norm, we have

$$\|\bar{q}\|<\lim\inf_{n\to\infty}\|q_n\|. \tag{3.16}$$

Using equations (3.14) to (3.16), we can deduce that

$$\mathcal{U}_{\xi,\eta}(\bar{q})<\lim_{n\to\infty}\mathcal{U}_{\xi,\eta}(q_n). \tag{3.17}$$

Alternatively, considering that $q_n\in M^+_{\xi,\eta}$, we can conclude that

$$0=\mathcal{U}'_{\xi,\eta}(q_n)q_n=\|q_n\|^2-\xi\int_{\mathbb{R}}|q_n|^2dt-\int_{\mathbb{R}}a(t)|q_n|^pdt-\eta\int_{\mathbb{R}}f(t)q_n(t)dt, \tag{3.18}$$

$$0<\big(\mathcal{U}'_{\xi,\eta}(q_n)q_n\big)'q_n=2\|q_n\|^2-2\xi\int_{\mathbb{R}}|q_n|^2dt-p\int_{\mathbb{R}}a(t)|q_n|^pdt-\eta\int_{\mathbb{R}}f(t)q_n(t)dt. \tag{3.19}$$

Based on equations (3.18) through (3.19), we can deduce that

$$\begin{aligned}(p-1)\eta\int_{\mathbb{R}}f(t)q_n(t)dt&>(p-2)\left(\|q_n\|^2-\xi\int_{\mathbb{R}}|q_n|^2dt\right)\\&\geq(p-2)\big(1-\xi\lambda_2^2\big)\|q_n\|^2>0.\end{aligned} \tag{3.20}$$

Taking the limit as $n\to\infty$ in the aforementioned inequality, using (3.14) and equation (3.16), we obtain

$$\begin{aligned}(p-1)\eta\int_{\mathbb{R}}f(t)\bar{q}(t)dt&\geq\big[(p-2)\big(1-\xi\lambda_2^2\big)\big]\lim\inf_{n\to\infty}\|q_n\|^2\\&>\big[(p-2)\big(1-\xi\lambda_2^2\big)\big]\|\bar{q}\|^2.\end{aligned}$$

Hence, $\int_{\mathbb{R}}f(t)\bar{q}(t)dt>0$, and, therefore, by proof Case 1 in Lemma 3.1 and with a derivation similar to (3.6) and (3.7), for $\phi_{\bar{q}}(s)=\mathcal{U}_{\xi,\eta}(s\bar{q}),\ s\in[0,\infty)$, we know that there exist unique $0<s_1^+<\bar{s}<s_2^-$ such that

$$\phi'_{\bar{q}}\big(s_1^+\big)=\phi'_{\bar{q}}\big(s_2^-\big)=0,\quad s_1^+\bar{q}\in M^+_{\xi,\eta},\quad s_2^-\bar{q}\in M^-_{\xi,\eta},$$

$$\phi_{\bar q}\big(s_1^+\big) = \inf_{0\le s\le \bar s} \phi_{\bar q}(s), \qquad \phi_{\bar q}\big(s_2^-\big) = \sup_{s\ge 0} \phi_{\bar q}(s). \tag{3.21}$$

Similarly, let $\phi_{q_n}(s) = \mathcal{U}_{\xi,\eta}(sq_n)$. Then, by the same reason as (3.6) and (3.7) together with (3.20), from $\int_{\mathbb{R}} f(t)q_n(t)dt > 0$, we know that there exist unique $s_n^+ < \bar s_n < s_n^-$ such that

$$\phi'_{q_n}(s_n^+) = \phi'_{q_n}(s_n^-) = 0, \quad \phi''_{q_n}(s) > 0, \quad s\in(0,\bar s_n), \qquad \phi''_{q_n}(s) < 0, \quad s\in(\bar s_n,\infty),$$

as well as

$$s_n^+ q_n \in M^+_{\xi,\eta}, \quad s_n^- q_n \in M^-_{\xi,\eta}.$$

Now, the fact that $q_n \in M^+_{\lambda,\mu}$ and the uniqueness for s_n^+ imply that $s_n^+ = 1$. Thus, $\bar s_n > 1$. Now, the fact that $\phi''_{q_n}(s) > 0$, $s\in(0,\bar s_n)$, yields that $\phi'_{q_n}(s)$ is strongly increasing. Combining with $\phi'_{q_n}(1) = 0$ and $1 < \bar s_n$, we immediately obtain that $\phi'_{q_n}(s) \le 0$, $s\in[0,1]$, namely,

$$s\|q_n\|^2 - \xi s\int_{\mathbb{R}} |q_n|^2 dt - s^{p-1}\int_{\mathbb{R}} a(t)|q_n|^p dt - \eta\int_{\mathbb{R}} f(t)q_n(t)dt \le 0, \quad s\in[0,1].$$

By taking the limit of the above inequality and considering equations (3.14) to (3.16), we obtain

$$\phi'_{\bar q}(s) = s\|\bar q\|^2 - \xi s\int_{\mathbb{R}} |\bar q|^2 dt - s^{p-1}\int_{\mathbb{R}} a(t)|\bar q|^p dt - \eta\int_{\mathbb{R}} f(t)\bar q(t)dt < 0, \quad s\in(0,1]. \tag{3.22}$$

However, according to equation (3.21), $\phi'_{\bar q}\big(s_1^+\big) = 0$, which combined with (3.22) implies that $s_1^+ > 1$. Consequently, $\bar s > s_1^+ > 1$. Again, in terms of (3.21) and (3.17), we have

$$\begin{aligned} C^+_{\xi,\eta} &= \inf_{q\in M^+_{\xi,\eta}} \mathcal{U}_{\xi,\eta}(q) \le \mathcal{U}_{\xi,\eta}\big(s_1^+\bar q\big) \\ &= \inf_{0\le s\le\bar s} \phi_{\bar q}(s) \le \phi_{\bar q}(1) = \mathcal{U}_{\xi,\eta}(\bar q) < \lim_{n\to\infty} \mathcal{U}_{\xi,\eta}(q_n) = C^+_{\xi,\eta}, \end{aligned}$$

which is a contradiction. Thus, $\limsup_{n\to\infty}\|q_n\| = \|\bar q\|$, which yields that $q_n \to \bar q$ in $\mathcal{H}$ because $\mathcal{H}$ is a Hilbert space taking into account that $q_n \rightharpoonup q$ in $\mathcal{H}$. Now, by (3.14), (3.15), it is easy to see that

$$\mathcal{U}_{\xi,\eta}(\bar q) = \lim_{n\to\infty} \mathcal{U}_{\xi,\eta}(q_n) = C^+_{\xi,\eta}.$$

On the other hand, from $\mathcal{U}'_{\xi,\eta}(q_n)q_n = 0$ and $\big(\mathcal{U}'_{\xi,\eta}(q_n)q_n\big)'(q_n) > 0$ together with $q_n \to \bar q$ in $\mathcal{H}$, we can readily observe

$$\mathcal{U}'_{\xi,\eta}(\bar q)\bar q = 0, \qquad \big(\mathcal{U}'_{\xi,\eta}(\bar q)\bar q\big)'\bar q \ge 0.$$

Again, by Lemma 3.2, $\mathcal{U}_{\xi,\eta}(\bar q) = C^+_{\xi,\eta} < 0$, and, therefore, $\bar q \ne 0$. By Lemma 2.4, $M^0_{\xi,\eta} = \varnothing$. Hence, $\big(\mathcal{U}'_{\xi,\eta}(\bar q)\bar q\big)'\bar q > 0$. This means that $\bar q \in M^+_{\xi,\eta}$. Hence, $\mathcal{U}_{\xi,\eta}$ arrives at its minimum in $\bar q \in M^+_{\xi,\eta}$ on $M^+_{\xi,\eta}$.

Lemma 3.5. *Suppose that the assumptions $(C1)$ and $(C2)$ are satisfied. Moreover*,

(1) *if $f\equiv 0$, then $\mathcal{U}_{\xi,\eta}$ reaches its minimum at some $\bar q \in M^-_{\xi,\eta}$ on $M^-_{\xi,\eta}$ for $-\infty < \xi < \xi_0$;*

(2) *if* $f \neq 0$, *then* $\mathcal{U}_{\xi,\eta}$ *reaches its minimum at some* $\bar{q} \in M^-_{\xi,\eta}$ *on* $M^-_{\xi,\eta}$ *for* $-\infty < \xi < \xi_0$, $0 < \eta < \eta_\xi/2$.

Proof. First, by Lemma 3.3, $C^-_{\xi,\eta} > 0$. Now, we are going to make an argument similar to that in Lemma 3.4. Let $\{q_n\} \subset M^-_{\xi,\eta}$ be a minimizing sequence of $\mathcal{U}_{\xi,\eta}$ on $M^-_{\xi,\eta}$, we denote, $\lim_{n\to\infty}\mathcal{U}_{\xi,\eta}(q_n) = \inf_{q\in M^-_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q) := C^-_{\xi,\eta}$. In addition, $\{q_n\}$ is bounded in $\mathcal{H}$, and therefore, there exists $\bar{q} \in \mathcal{H}$ such that $q_n \rightharpoonup \bar{q}$ in $\mathcal{H}$. The same rationale as in Lemma 3.4 suggests that (3.14), (3.15) still hold. Now, we show that $\lim_{n\to\infty}\|q_n\| = \|\bar{q}\|$. In fact, if not, then

$$\|\bar{q}\| < \liminf_{n\to\infty}\|q_n\|. \tag{3.23}$$

We claim that $\bar{q} \neq 0$. In fact, by $q_n \in M_{\xi,\eta}$, it follows from (2.4) and (2.5) that

$$\mathcal{U}_{\xi,\eta}(q_n) = -\frac{1}{2}\left(\|q_n\|^2 - \xi\int\limits_{\mathbb{R}}|q_n|^2dt\right) + \left(1-\frac{1}{p}\right)\int\limits_{\mathbb{R}}a(t)|q_n|^pdt.$$

Hence,

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int\limits_{\mathbb{R}}a(t)|q_n|^pdt &= \mathcal{U}_{\xi,\eta}(q_n) + \frac{1}{2}\left(\|q_n\|^2 - \xi\int\limits_{\mathbb{R}}|q_n|^2dt\right)\\ &\geq C^-_{\xi,\eta} + \frac{1}{2}\big(1-\xi\lambda_2^2\big)\|q_n\|.\end{aligned}$$

Taking the limit as $n\to\infty$ in the above inequality and considering equations (3.15) and (3.23), we obtain

$$\left(1-\frac{1}{p}\right)\int\limits_{\mathbb{R}}a(t)|\bar{q}|^pdt > C^-_{\xi,\eta} + \frac{1}{2}\big(1-\xi\lambda_2^2\big)\|\bar{q}\|.$$

By $C^-_{\xi,\eta} > 0$, we have that $\int_{\mathbb{R}}a(t)|\bar{q}|^pdt > 0$ and, therefore, $\bar{q} \neq 0$. Now, based on the proof of Lemma 3.4, and employing a similar argument to (3.2) and (3.7), we can deduce the existence of a unique positive value $s^- > 0$ such that

$$s^-\bar{q} \in M^-_{\xi,\eta}, \qquad \mathcal{U}_{\xi,\eta}\big(s^-\bar{q}\big) = \sup_{s\geq 0}\mathcal{U}_{\xi,\eta}(s\bar{q}). \tag{3.24}$$

Similarly, since $q_n \in M^-_{\xi,\eta}$, we also have that $\mathcal{U}_{\xi,\eta}(q_n) = \sup_{s\geq 0}\mathcal{U}_{\xi,\eta}(sq_n)$ and, therefore,

$$\mathcal{U}_{\xi,\eta}(s^-q_n) \leq \mathcal{U}_{\xi,\eta}(q_n). \tag{3.25}$$

Once more, (3.14), (3.15) and (3.23) imply that

$$\mathcal{U}_{\xi,\eta}\big(s^-\bar{q}\big) < \lim_{n\to\infty}\mathcal{U}_{\xi,\eta}(s^-q_n). \tag{3.26}$$

Hence, by (3.24)–(3.26), we get

$$C^-_{\xi,\eta} = \inf_{q\in M^-_{\xi,\eta}}\mathcal{U}_{\xi,\eta}(q) \leq \mathcal{U}_{\xi,\eta}\big(s^-\bar{q}\big) < \lim_{n\to\infty}\mathcal{U}_{\xi,\eta}(s^-q_n) \leq \lim_{n\to\infty}\mathcal{U}_{\xi,\eta}(q_n) = C^-_{\xi,\eta},$$

a contradiction. Hence, $\lim_{n\to\infty}\|q_n\| = \|\bar{q}\|$. Using a similar approach as demonstrated in the proof of Lemma 3.4, we can conclude that $\mathcal{U}_{\xi,\eta}$ reaches its minimum at $\bar{q}$ on $M^-_{\xi,\eta}$.

***3.1. Proof of Theorem* 2.1.** Given the assumptions $(C1)$ and $(C2)$, along with Lemmas 2.4, 3.1, 3.3, and 3.5, it is established that $M^0_{\xi,\eta} = \varnothing, M^-_{\xi,\eta} \neq \varnothing$, and $\mathcal{U}_{\xi,\eta}$ reaches its minimum at certain $\bar{q} \in M^-_{\xi,\eta}$. Exactly, $\mathcal{U}_{\xi,\eta}(\bar{q}) = \inf_{q\in M^-_{\xi,\eta}} \mathcal{U}_{\xi,\eta}(q) = C^-_{\xi,\eta} > 0$. Since $M^-_{\xi,\eta}$ is an open subset in $M_{\xi,\eta}$ and $\bar{q} \in M^-_{\xi,\eta}$, there exists a ball $B(\bar{q}, r)$ centered at $\bar{q}$ with radius r verifying $B(\bar{q}, r)\cap M_{\xi,\eta} \subset M^-_{\xi,\eta}$. Thus,

$$\mathcal{U}_{\xi,\eta}(\bar{q}) = \inf_{q\in M^-_{\xi,\eta}} \mathcal{U}_{\xi,\eta}(q) = \inf_{q\in M^-_{\xi,\eta}\cap B(\bar{q},r)} \mathcal{U}_{\xi,\eta}(q) = \inf_{q\in M_{\xi,\eta}\cap B(\bar{q},r)} \mathcal{U}_{\xi,\eta}(q).$$

Hence, by Lemma 2.3 and taking account of the fact that $M^0_{\xi,\eta} = \varnothing$, we obtain that $\bar{q}$ is a nontrivial weak solution of (1.1). In addition, using Lemmas 2.4 and 3.1 – 3.5, there exist $\bar{q}_1 \in M^-_{\xi,\eta}$, $\bar{q}_2 \in M^+_{\xi,\eta}$ such that

$$\mathcal{U}_{\xi,\eta}(\bar{q}_1) = \inf_{q\in M^-_{\xi,\eta}} \mathcal{U}_{\xi,\eta}(q) > 0$$

and

$$\mathcal{U}_{\xi,\eta}(\bar{q}_2) = \inf_{q\in M^+_{\xi,\eta}} \mathcal{U}_{\xi,\eta}(q) < 0.$$

As both $M^+_{\xi,\eta}, M^-_{\xi,\eta}$ are open subsets in $M_{\xi,\eta}$ with $M^+_{\xi,\eta} \cap M^-_{\xi,\eta} = \varnothing$ and $M^0_{\xi,\eta} = \varnothing$, by the same way, we can conclude that $\bar{q}_1, \bar{q}_2$ are two nontrivial solutions of (1.1).

The theorem is proved.

Conflict of interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

Funding. The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

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Received 22.02.23