DOI: 10.3842/umzh.v76i6.7466

UDC 517.9

Yanping Chen, Tian Tian[1] (School of Mathematics and Physics, University of Science and Technology Beijing, China)

ON BLOOM-TYPE CHARACTERIZATIONS OF THE HIGHER-ORDER COMMUTATORS OF MARCINKIEWICZ INTEGRALS

ПРО ХАРАКТЕРИСТИКИ ТИПУ БЛУМА КОМУТАТОРІВ ВИЩОГО ПОРЯДКУ ІНТЕГРАЛІВ МАРЦИНКЕВИЧА

Let Ω be homogeneous of degree zero, have mean value zero, and integrable on the unit sphere. For $m \in \mathbb{N}$, let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and let the higher-order commutator of the Marcinkiewicz integral $\mu^m_{\Omega,b}$ be defined by

$$\mu^m_{\Omega,b}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)]^m f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

We establish a sparse domination of $\mu^m_{\Omega,b}$ for $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$. Moreover, we also give Bloom-type characterizations of the two-weighted boundedness of the higher-order commutators $\mu^m_{\Omega,b}, \mu^{*,m}_{\Omega,\alpha,b}$, and $\mu^m_{\Omega,S,b}$, where the higher-order commutators $\mu^{*,m}_{\Omega,\alpha,b}$ and $\mu^m_{\Omega,S,b}$ are defined, respectively, by

$$\mu^{*,m}_{\Omega,\alpha,b}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\alpha} \left| \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} [b(x)-b(z)]^m f(z)dz \right|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}}, \quad \alpha > 1,$$

and

$$\mu^m_{\Omega,S,b}(f)(x) = \left(\iint_{|x-y|<t} \left| \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} [b(x)-b(z)]^m f(z)dz \right|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}}.$$

Нехай Ω є однорідною нульового степеня, має нульове середнє значення та інтегровна на одиничній сфері. Для $m \in \mathbb{N}$ нехай $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, а комутатор вищого порядку інтеграла Марцинкевича $\mu^m_{\Omega,b}$ визначається таким чином:

$$\mu^m_{\Omega,b}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)]^m f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Отримано розріджене домінування $\mu^m_{\Omega,b}$ при умові, що $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$. Крім того, наведено характеристики типу Блума для двовагової обмеженості комутаторів вищого порядку $\mu^m_{\Omega,b}, \mu^{*,m}_{\Omega,\alpha,b}, \mu^m_{\Omega,S,b}$, де комутатори вищого порядку $\mu^{*,m}_{\Omega,\alpha,b}$ і $\mu^m_{\Omega,S,b}$ визначаються, відповідно, таким чином:

$$\mu^{*,m}_{\Omega,\alpha,b}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\alpha} \left| \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} [b(x)-b(z)]^m f(z)dz \right|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}}, \quad \alpha > 1,$$

і

$$\mu^m_{\Omega,S,b}(f)(x) = \left(\iint_{|x-y|<t} \left| \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} [b(x)-b(z)]^m f(z)dz \right|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}}.$$

[1] Corresponding author, e-mail: ttianhk@126.com.

1. Introduction and main results. For $n \geq 2$, the Marcinkiewicz integral of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t} f(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{1.1}$$

where

$$F_{\Omega,t} f(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy \quad \text{for} \quad f \in \mathcal{S}(\mathbb{R}^n)$$

and Ω satisfies the following conditions:

(a) Ω is homogeneous function of degree zero on $\mathbb{R}^n \backslash \{0\}$, i.e.,

$$\Omega(tx) = \Omega(x) \quad \text{for any} \quad t > 0 \quad \text{and} \quad x \in \mathbb{R}^n \backslash \{0\}; \tag{1.2}$$

(b) Ω has mean value zero on $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, the unit sphere in $\mathbb{R}^n$, i.e.,

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{1.3}$$

Formally, by setting $\phi(x) = \Omega(x)|x|^{-n+1}\chi_B(x)$ and $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$, where $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$, we can know the Marcinkiewicz integral is just the Littlewood – Paley g function, that is,

$$\mu_\Omega(f)(x) = g(f)(x) = \left(\int_0^\infty |\phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

The Marcinkiewicz integral operator $\mu^*_{\Omega,\alpha}$ related to the Littlewood – Paley g^*-function is defined by

$$\mu^*_{\Omega,\alpha}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x-y|} \right)^{n\alpha} |F_{\Omega,t}(y)|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}}, \quad \alpha > 1.$$

The Marcinkiewicz integral operator $\mu_{\Omega,S}$ is related to the Lusin area integral S, and it is defined by

$$\mu_{\Omega,S}(f)(x) = \left(\int_{\Gamma(x)} |F_{\Omega,t}(y)|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

Stein [20] showed that if $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1}), 0 < \gamma \leq 1$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$, and also proved the operator μ_Ω is weak (1,1).

Theorem A. *Let* Ω *satisfy* (1.2), (1.3) *and* $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1}),\ 0 < \gamma \le 1$. *Then*

$$\|\mu_\Omega(f)\|_{L^p(\mathbb{R}^n)} \lesssim C_\Omega \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \le 2,$$

and

$$|\{x \in \mathbb{R}^n : \mu_\Omega(f)(x) > \lambda\}| \lesssim C_\Omega \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)} \quad \textit{for any} \quad \lambda > 0,$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}_\gamma}$.

Subsequently, Benedek, Calderón, Panzone [2] established the boundedness on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ under the stronger condition $\Omega \in C^1(\mathbb{S}^{n-1})$. Ding, Fan and Pan [6] proved that if $\Omega \in H^1(\mathbb{S}^{n-1})$, then μ_Ω (for $1 < p < \infty$), $\mu^*_{\Omega,\alpha}$ and $\mu_{\Omega,S}$ (for $2 \le p < \infty$) are strong (p,p). For the weighted bounded estimates of the operator μ_Ω, Torchinsky and Wang [22] proved that if Ω is continuous and $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1}), 0 < \gamma \le 1$, then, for $1 < p < \infty$, $w \in A_p$, μ_Ω is bounded on $L^p(w)$. In 1999, Ding, Fan and Pan [5] considered weighted boundedness for a class of rough Marcinkiewicz integral with $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q \in (1, \infty]$. Recently, Tao and Hu [21] used a sparse domination for the Marcinkiewicz integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ to obtain some quantitaitve weighted bounded estimates for μ_Ω, and proved the weighted norm inequality of Coifman – Fefferman type of μ_Ω.

Now let us consider the higher-order commutators of Marcinkiewicz integral. For $m \in \mathbb{N}$, $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the higher-order commutators $\mu^m_{\Omega,b}, \mu^{*,m}_{\Omega,\alpha,b}, \mu^m_{\Omega,S,b}$ are defined, respectively, by

$$\mu^m_{\Omega,b}(f)(x) = \left(\int\limits_0^\infty \left| \int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)]^m f(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

$$\mu^{*,m}_{\Omega,\alpha,b}(f)(x)$$

$$= \left(\iint\limits_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x-y|} \right)^{n\alpha} \left| \int\limits_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} [b(x) - b(z)]^m f(z) dz \right|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}}, \quad \alpha > 1,$$

and

$$\mu^m_{\Omega,S,b}(f)(x) = \left(\iint\limits_{\Gamma(x)} \left| \int\limits_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} [b(x) - b(z)]^m f(z) dz \right|^2 \frac{dydt}{t^{n+3}} \right)^{\frac{1}{2}},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. Ding, Lu and Yabuta [7] proved that the weighted L^p boundedness for the higher-order commutators $\mu^m_{\Omega,b}, \mu^{*,m}_{\Omega,\alpha,b}, \mu^m_{\Omega,S,b}$ with rough kernels when $b \in BMO(\mathbb{R}^n)$. Chen and Ding [3] gave a characterization of the L^p-boundedness for the commutators of the Marcinkiewicz integral μ_Ω, the Lusin area integral and Littlewood – Paley g^*_α function via $b \in \mathrm{BMO}(\mathbb{R}^n)$. For more details about other works, see [1, 4, 9, 23].

Before starting our results, let us recall some definitions. Let w be a nonnegative locally integrable function on $\mathbb{R}^n$. For $1 < p < \infty$, we call that $w \in A_p$ if

$$[w]_{A_p} = \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int\limits_Q w(x)dx\right)\left(\frac{1}{|Q|}\int\limits_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty.$$

For a given weight η, we say that $f \in BMO_\eta$ if

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{\eta(Q)}\int\limits_Q |f(x) - f_Q|dx < \infty,$$

where f_Q denotes the average of f on $Q\colon f_Q = \dfrac{1}{|Q|}\displaystyle\int_Q f(x)dx,$ $\eta(Q)$ is defined by $\eta(Q) = \displaystyle\int_Q \eta(x)dx$. The dyadic lattice $\mathcal{D}$ in $\mathbb{R}^n$ is any collection of cubes such that

(i) if $Q \in \mathcal{D}$, then each child of Q is in $\mathcal{D}$ as well;

(ii) for any two cubes $Q_1, Q_2 \in \mathcal{D}$, there is $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$;

(iii) for every compact set $N \subset \mathbb{R}^n$, there is a cube $Q \in \mathcal{D}$ with $N \subset Q$,

where $\mathcal{D}(Q)$ obtained by passing from Q to its children, then to the children of the children, etc. We also recall that a family of dyadic cubes $\mathcal{S}$ is an α-sparse family with $\alpha \in (0,1)$ if for each $Q \in \mathcal{S}$ there exists a measurable set $E_Q \subset Q$ such that

$$\alpha|Q| \le |E_Q|$$

and the E_Q are pairwise disjoint.

Lerner, Ombrosi and Rivera – Ríos [17] established a Bloom-type characterization of the two-weighted boundedness for iterated commutators T_b^m, which are defined by

$$T_b^m f = [b, T_b^{m-1}]f, \quad T_b^1 f = [b, T]f,$$

where T is an ω-Calderón – Zygmund operator that is linear and L^2 bounded, and it can be represented as follows:

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x,y)f(y)dy,$$

where the kernel K satisfies the size condition

$$|K(x,y)| \le \frac{C_K}{|x-y|^n}, \quad x \ne y,$$

and the smoothness condition

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le \omega\left(\frac{|x-z|}{|x-y|}\right)\frac{1}{|x-y|^n}$$

for $|x-y| > 2|x-z|$, where $\omega\colon [0,1] \to [0,\infty)$ is a continuously increasing subadditive function with $\omega(0) = 0$. We say that ω satisfies the Dini condition if $\|\omega\|_{Dini} := \displaystyle\int_0^1 \omega(t)\frac{dt}{t} < \infty$. Lerner, Ombrosi and Rivera-Ríos [17] proved the following theorem.

Theorem B. *Let* $1 < p < \infty, u, \lambda \in A_p$. *Further, let* $v = \left(\frac{u}{\lambda}\right)^{\frac{1}{p}}$ *and* $m \in \mathbb{N}$.

(a) *If* $b \in BMO_{v^{1/m}}$, *then for the* ω*-Calderón – Zygmund operator* T *with* ω *satisfying the Dini condition*

$$\|T_b^m(f)\|_{L^p(\lambda)} \leq c_{n,m,T} \|b\|_{BMO_{v^{1/m}}}^m ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(u)},$$

where $c_{n,m,T} = c_{n,m}(\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini} + C_K)$.

(b) *Let* T_Ω *be a* ω*-Calderón – Zygmund operator with* $K(x,y) = \Omega\left(\frac{x-y}{|x-y|}\right)\frac{1}{|x-y|^n}$, *where* Ω *is a measurable function on* $\mathbb{S}^{n-1}$, *which does not equivalent to zero and not change sign on some open subset from* $\mathbb{S}^{n-1}$. *If there is* $c > 0$ *such that, for every bounded measurable set* $E \subset \mathbb{R}^n$,

$$\|(T_\Omega)_b^m(\chi_E)\|_{L^p(\lambda)} \leq c u(E)^{\frac{1}{p}},$$

then $b \in BMO_{\nu^{1/m}}$.

Recently, Li and Sun [19] established weak and strong type weighted estimates of mixed A_p-A_∞-type for multilinear Calderón – Zygmund operators. Li, Martikainen and Vuorinen [18] obtained a two-weighted Bloom-type inequality for iterated commutators of bi-parameter singular integrals. Duong, Gong, Kuffner, Li and others [8] gave the two-weighted commutators theorem of Calderón – Zygmund operators on spaces of homogeneous type.

In this paper we shall establish the sparse domination of $\mu_{\Omega,b}^m$ with $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and obtain the characterizations of the two-weighted boundedness for $\mu_{\Omega,b}^m, \mu_{\Omega,\alpha,b}^{*,m}, \mu_{\Omega,S,b}^m$. Note that we are vrey much motivated by the works of Lerner [14, 16].

Our main results can be stated as follows.

Theorem 1.1. *Let* $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *satisfying* (1.2) *and* (1.3). *For each bounded function* f *with compact support, there exist a sparse family of cubes* $\mathcal{S}$ *such that, for almost everywhere* $x \in \mathbb{R}^n$,

$$[\mu_{\Omega,b}^m(f)(x)]^2 \leq c_n C_\Omega^2 \sum_{Q \in \mathcal{S}} \left[\sum_{k=0}^m \binom{m}{k} |b(x) - b_Q|^{m-k} \langle |f| b - b_Q|^k| \rangle_Q \right]^2 \chi_Q(x), \tag{1.4}$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}}$.

Theorem 1.2. *Let* $1 < p < \infty, u, \lambda \in A_p$ *and* $v = \left(\frac{u}{\lambda}\right)^{\frac{1}{p}}$, $m \in \mathbb{N}$.

(a) *Suppose that* $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *satisfying* (1.2) *and* (1.3). *If* $b \in BMO_{v^{1/m}}$, *then*

$$\|\mu_{\Omega,b}^m(f)\|_{L^p(\lambda)} \leq c_{n,m} C_\Omega \|b\|_{BMO_{v^{1/m}}}^m ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(u)},$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}}$.

(b) *Let* Ω *be a measurable function on* $\mathbb{S}^{n-1}$, *which does not equivalent to zero and not change sign on some open subset from* $\mathbb{S}^{n-1}$. *If there exists* $c > 0$ *such that, for every bounded measurable set* $E \subset \mathbb{R}^n$,

$$\|\mu_{\Omega,b}^m(\chi_E)\|_{L^p(\lambda)} \leq c u(E)^{\frac{1}{p}},$$

then $b \in BMO_{\nu^{1/m}}$.

Theorem 1.3. *Let* $2 < p < \infty, u, \lambda \in A_p$ *and* $v = \left(\frac{u}{\lambda}\right)^{\frac{1}{p}}$, $\alpha > 1$, $m \in \mathbb{N}$.

(a) Suppose that $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *satisfying* (1.2) *and* (1.3). *If* $b \in BMO_{v^{1/m}}$, *then*

$$\|\mu^m_{\Omega,S,b}(f)\|_{L^p(\lambda)} \le c_{n,m,\alpha} C_\Omega \|b\|^m_{BMO_{v^{1/m}}} ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(u)},$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}}$.

(b) Let Ω *be a measurable function on* $\mathbb{S}^{n-1}$, *which does not equivalent to zero and not change sign on some open subset from* $\mathbb{S}^{n-1}$. *If there exists* $c > 0$ *such that, for every bounded measurable set* $E \subset \mathbb{R}^n$,

$$\|\mu^m_{\Omega,S,b}(\chi_E)\|_{L^p(\lambda)} \le c u(E)^{\frac{1}{p}},$$

then $b \in BMO_{\nu^{1/m}}$.

Theorem 1.4. *Let* $2 < p < \infty, u, \lambda \in A_p$ *and* $v = \left(\frac{u}{\lambda}\right)^{\frac{1}{p}}$, $\alpha > 1$, $m \in \mathbb{N}$.

(a) *Suppose that* $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *satisfying* (1.2) *and* (1.3). *If* $b \in BMO_{v^{1/m}}$, *then*

$$\|\mu^{*,m}_{\Omega,\alpha,b}(f)\|_{L^p(\lambda)} \le c_{n,m,\alpha} C_\Omega \|b\|^m_{BMO_{v^{1/m}}} ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(u)},$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}}$.

(b) *Let* Ω *be a measurable function on* $\mathbb{S}^{n-1}$, *which does not equivalent to zero and not change sign on some open subset from* $\mathbb{S}^{n-1}$. *If there exists* $c > 0$ *such that, for every bounded measurable set* $E \subset \mathbb{R}^n$,

$$\|\mu^{*,m}_{\Omega,\alpha,b}(\chi_E)\|_{L^p(\lambda)} \le c u(E)^{\frac{1}{p}},$$

then $b \in BMO_{\nu^{1/m}}$.

***Remark* 1.1.** Theorems 1.2, 1.3 and 1.4 give the Bloom-type characterizations of the L^p-boundedness of commutators $\mu^m_{\Omega,b}, \mu^{*,m}_{\Omega,\alpha,b}, \mu^m_{\Omega,S,b}$, respectively.

2. Proof of Theorem 1.1. The proof of Theorem 1.1 follows the scheme in [10]. We first recall some basic definitions. We define the grand maximal truncated operator $\mathcal{M}_{\mu_\Omega}$ by

$$\mathcal{M}_{\mu_\Omega} f(x) = \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi \in Q} |\mu_\Omega(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,$$

where the supremum is taken over all the cubes $Q \subset \mathbb{R}^n$ containing x. We also consider a local version of this operator

$$\mathcal{M}_{\mu_\Omega, Q_0} f(x) = \sup_{x \in Q \subset Q_0} \operatorname*{ess\,sup}_{\xi \in Q} |\mu_\Omega(f\chi_{3Q_0 \setminus 3Q})(\xi)|.$$

The proof of Theorem 1.1 is based on the following lemmas.

Lemma 2.1 [10]**.** *Let* $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *satisfying* (1.2) *and* (1.3). *Then*

$$\mathcal{M}_{\mu_\Omega} f(x) \lesssim C_\Omega M f(x) + \mu_\Omega f(x).$$

Furthermore,

$$|\{x \in \mathbb{R}^n : \mathcal{M}_{\mu_\Omega} f(x) > \lambda\}| \le c_n C_\Omega \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)},$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}}$.

For $t \in [1,2]$ and $j \in \mathbb{Z}$, set

$$K_t^j(x) = \frac{1}{2^j} \frac{\Omega(x)}{|x|^{n-1}} \chi_{\{2^{j-1}t<|x|\le 2^j t\}}(x).$$

Let

$$\widetilde{\mu}_\Omega(f)(x) = \left(\int_1^2 \sum_{j\in\mathbb{Z}} |F_j f(x,t)|^2 dt \right)^{1/2}$$

with

$$F_j f(x,t) = \int_{\mathbb{R}^n} K_t^j(x-y) f(y) dy.$$

It is easy to see that

$$\mu_\Omega(f)(x) \approx \widetilde{\mu}_\Omega(f)(x) = \left(\int_1^2 \sum_{j\in\mathbb{Z}} |F_j f(x,t)|^2 dt \right)^{1/2}.$$

Further, the higher-order commutators $\mu^m_{\Omega,b}$ satisfy

$$\mu^m_{\Omega,b}(f)(x) \approx \widetilde{\mu^m_{\Omega,b}}(f)(x) = \left(\int_1^2 \sum_{j\in\mathbb{Z}} |F^m_{j,b} f(x,t)|^2 dt \right)^{1/2}, \tag{2.1}$$

where

$$F^m_{j,b} f(x,t) = \int_{\mathbb{R}^n} K_t^j(x-y) [b(x)-b(y)]^m f(y) dy.$$

Employing the ideas from [13], we define the operator $\mathscr{M}^*_{\mu_\Omega,Q_0}$ as

$$\mathscr{M}^*_{\mu_\Omega,Q_0} f(x) = \sup_{Q\ni x, Q\subset Q_0} \left\| \left(\int_1^2 \sum_{j=J_Q}^\infty |F_j(f\chi_{3Q_0})(\cdot,t)|^2 dt \right)^{1/2} \right\|_{L^\infty(Q)},$$

where and in what follows, for cube $Q \subset \mathbb{R}^n$, J_Q is the integer such that $2^{J_Q} \le \ell(Q) < 2^{J_Q+1}$, and $J^*_Q \in \mathbb{Z}$ such that $2^{J^*_Q - 1} \le 4\sqrt{n}\ell(Q) < 2^{J^*_Q}$.

Lemma 2.2 [10]. *Let* $\Omega \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *satisfying* (1.2) *and* (1.3). *Then*

$$\mathscr{M}^*_{\mu_\Omega,Q_0} f(x) \lesssim \|\Omega\|_{L^\infty} M(f\chi_{3Q_0})(x) + \mathcal{M}_{\mu_\Omega}(f\chi_{3Q_0})(x).$$

Furthermore,

$$|\{x \in \mathbb{R}^n : \mathscr{M}^*_{\mu_\Omega,Q_0} f(x) > \lambda\}| \le c_n C_\Omega \lambda^{-1} \|f\chi_{3Q_0}\|_{L^1(\mathbb{R}^n)},$$

where $C_\Omega = \|\Omega\|_{L^\infty} + \|\Omega\|_{\mathrm{Lip}}$.

To prove Theorem 1.1, it suffices to prove that for every cube $Q_0 \subset \mathbb{R}^n$, there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that, for a.e. $x \in Q_0$,

$$\big[\mu_{\Omega,b}^m(f\chi_{3Q_0})(x)\big]^2 \le c_n C_\Omega^2 \sum_{Q\in\mathcal{F}} \left[\sum_{k=0}^m \binom{m}{k} \big|b(x)-b_{R_Q}\big|^{m-k} \Big\langle \big|f|b-b_{R_Q}|^k\big| \Big\rangle_{3Q}\right]^2 \chi_Q(x). \quad (2.2)$$

First, suppose that we have already proved (2.2). Let us show how to prove Theorem 1.1 using this inequality. Let us take a partition of $\mathbb{R}^n$ by cubes Q_l such that $\operatorname{supp}(f) \subset 3Q_l$ for each l. We can do it as following way. First, take a cube Q_0 such that $\operatorname{supp}(f) \subset Q_0$, and cover $3Q_0 \backslash Q_0$ by $3^n - 1$ congruent cubes Q_l. Each of them satisfies $Q_0 \subset 3Q_l$. Next, cover $9Q_0 \backslash 3Q_0$ in the same way, and so on (see [12]). Then the union of those cubes, including Q_0, will satisfy the desired properties.

Applying (2.2) to each Q_l, we obtain a $\frac{1}{2}$-sparse family $\mathcal{F}_l \subset \mathcal{D}(Q_l)$ such that the following holds for a.e. $x \in Q_l$:

$$\begin{aligned}|\mu_{\Omega,b}^m f(x)|^2 \chi_{Q_l}(x) &= |\mu_{\Omega,b}^m (f\chi_{3Q_l})(x)|^2 \\ &\le c_n C_\Omega^2 \sum_{Q\in\mathcal{F}_l} \left[\sum_{k=0}^m \binom{m}{k} \big|b(x)-b_{R_Q}\big|^{m-k} \Big\langle \big|f|b-b_{R_Q}|^k\big| \Big\rangle_{3Q}\right]^2 \chi_Q(x).\end{aligned}$$

Taking $\mathcal{F} = \cup_l \mathcal{F}_l$, we have that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family and, for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}|\mu_{\Omega,b}^m f(x)|^2 &= \sum_l |\mu_{\Omega,b}^m f(x)|^2 \chi_{Q_l} \\ &\le c_n C_\Omega^2 \sum_l \sum_{Q\in\mathcal{F}_l} \left[\sum_{k=0}^m \binom{m}{k} \big|b(x)-b_{R_Q}\big|^{m-k} \Big\langle \big|f|b-b_{R_Q}|^k\big| \Big\rangle_{3Q}\right]^2 \chi_Q(x) \\ &\le c_n C_\Omega^2 \sum_{Q\in\mathcal{F}} \left[\sum_{k=0}^m \binom{m}{k} \big|b(x)-b_{R_Q}\big|^{m-k} \Big\langle \big|f|b-b_{R_Q}|^k\big| \Big\rangle_{3Q}\right]^2 \chi_Q(x).\end{aligned}$$

Now since $3Q \subset R_Q$ and $|R_Q| \le 3^n|3Q|$, we have $|f|_{3Q} \le c_n |f|_{R_Q}$. Setting

$$\mathcal{S} = \cup_l \{R_Q \in \mathcal{D}_l : Q \in \mathcal{F}\},$$

then we obtain a $\frac{1}{2\cdot 9^n}$-sparse, and

$$|\mu_{\Omega,b}^m f(x)|^2 \le c_n C_\Omega^2 \sum_{R_Q\in\mathcal{S}} \left[\sum_{k=0}^m \binom{m}{k} \big|b(x)-b_{R_Q}\big|^{m-k} \Big\langle \big|f|b-b_{R_Q}|^k\big| \Big\rangle_{R_Q}\right]^2 \chi_{R_Q}(x).$$

Then we get (1.4).

Now we prove the claim (2.2). By (2.1), we know it suffices to prove (2.2) with $\mu_{\Omega,b}^m$ replaced by $\widetilde{\mu_{\Omega,b}^m}$. It suffices to prove the following recursive estimate: there exist pairwise disjoint cubes $P_l \in \mathcal{D}(Q_0)$ such that $\sum_l |P_l| \le \frac{1}{2}|Q_0|$ and

$$\Big[\widetilde{\mu_{\Omega,b}^{m}}(f\chi_{3Q_0})(x)\Big]^2\chi_{Q_0}(x) \le c_n C_\Omega^2\left[\sum_{k=0}^{m}\binom{m}{k}\big|b(x)-b_{R_{Q_0}}\big|^{m-k}\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}\right]^2\chi_{Q_0}(x)$$

$$+\sum_{l}\Big[\widetilde{\mu_{\Omega,b}^{m}}(f\chi_{3P_l})(x)\Big]^2\chi_{P_l}(x) \tag{2.3}$$

a.e. in Q_0. Iterating (2.3) we obtain (2.2) with $\mathcal{F}$ being the union of all the family $\{P_l^k\}$ where $\{P_l^0\}=\{Q_0\}$, $\{P_l^1\}=\{P_l\}$ and $\{P_l^k\}$ are the cubes obtained at the kth stage of the iterative process. Indeed, choosing $E_{P_l^k}=P_l^k\setminus\cup_l P_l^{k+1}$ for each P_l^k, we observe that $\mathcal{F}$ is a $\dfrac{1}{2}$-sparse family.

Now we consider the recursive estimate. Let $E=\bigcup_{k=0}^{m}E_k$ with

$$E_k=\Big\{x\in Q_0:\widetilde{\mu}_\Omega\Big(|b-b_{R_{Q_0}}|^k f\chi_{3Q_0}\Big)(x)>\alpha_n C_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}\Big\}$$

$$\cup\Big\{x\in Q_0:\mathscr{M}^*_{\mu_\Omega,Q_0}\Big(|b-b_{R_{Q_0}}|^k f\Big)(x)>\alpha_n C_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}\Big\}$$

for $k=0,1,2,\ldots,m$, where α_n is a positive constant. Using Lemma 2.2 and the weak type (1,1) boundedness of $\widetilde{\mu}_\Omega$, we have

$$|E_k|\le\frac{c_nC_\Omega}{\alpha_nC_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}}\Big\||b-b_{R_{Q_0}}|^kf\chi_{3Q_0}\Big\|_1$$

$$+\frac{c_nC_\Omega}{\alpha_nC_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}}\Big\||b-b_{R_{Q_0}}|^kf\chi_{3Q_0}\Big\|_1$$

$$\le\frac{3^nc_n\frac{1}{|3Q_0|}\int\limits_{3Q_0}\big|f|b-b_{R_{Q_0}}|^k\big|}{\alpha_n\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}}|Q_0|+\frac{3^nc_n}{\alpha_n}|Q_0|\frac{1}{|3Q_0|}\int\limits_{3Q_0}\frac{\big|f|b-b_{R_{Q_0}}|^k\big|}{\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}}dx$$

$$\le 2\cdot\frac{3^nc_n}{\alpha_n}|Q_0|.$$

Then, choosing α_n large enough, we get

$$|E|\le\frac{1}{2^{n+2}}|Q_0|.$$

Now on the cube Q_0, we apply the Calderón–Zygmund decomposition to the function χ_E at level $\dfrac{1}{2^{n+1}}$, then we obtain pairwise disjoint cubes $\{P_l\}\subset\mathcal{D}(Q_0)$, such that

(1) $\chi_E(x)\le\dfrac{1}{2^{n+1}}$ for almost every $x\notin\cup_lP_l$;

(2) $\sum_l|P_l|=|\cup_lP_l|\le 2^{n+1}|E|\le\dfrac{1}{2}|Q_0|$;

(3) $\dfrac{1}{2^{n+1}}\le\dfrac{1}{|P_l|}\displaystyle\int_{P_l}\chi_E(x)=\dfrac{|P_l\cap E|}{|P_l|}\le\dfrac{1}{2}$.

From part (1), it is obvious that $|E\setminus\cup_lP_l|=0$, and from part (3), we get, for each P_l, $|P_l\cap E^c|>0$, that is, $P_l\cap E^c\ne\varnothing$.

Write

$$\Big[\widetilde{\mu^m_{\Omega,b}}(f\chi_{3Q_0})(x)\Big]^2\chi_{Q_0}(x)=\Big[\widetilde{\mu^m_{\Omega,b}}(f\chi_{3Q_0})(x)\Big]^2\chi_{Q_0\setminus\cup_l P_l}(x)$$

$$+\sum_l\left(\int_1^2\sum_{j=J_{P_l}}^{\infty}|F^m_{j,b}(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{P_l}(x)$$

$$+\sum_l\left(\int_1^2\sum_{j=-\infty}^{J_{P_l}-1}|F^m_{j,b}(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{P_l}(x)$$

$$=\left(\int_1^2\sum_{j\in\mathbb{Z}}|F^m_{j,b}(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{Q_0\setminus\cup_l P_l}(x)$$

$$+\sum_l\left(\int_1^2\sum_{j=J_{P_l}}^{\infty}|F^m_{j,b}(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{P_l}(x)$$

$$+\sum_l\left(\int_1^2\sum_{j=-\infty}^{J_{P_l}-1}|F^m_{j,b}(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{P_l}(x). \tag{2.4}$$

For any $c\in\mathbb{R}^n$, we note that

$$F^m_{j,b}f=\frac{1}{2^j}\int\limits_{2^{j-1}t<|x-y|\le 2^jt}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]^mf(y)dy$$

$$=\frac{1}{2^j}\int\limits_{2^{j-1}t<|x-y|\le 2^jt}\frac{\Omega(x-y)}{|x-y|^{n-1}}[(b(x)-c)-(b(y)-c)]^mf(y)dy=F^m_{j,b-c}f$$

and

$$F^m_{j,b-c}f=\sum_{k=0}^m(-1)^k\tbinom{m}{k}F_j((b-c)^kf)(b-c)^{m-k}.$$

By using the Minkowski inequality, we obtain

$$\int_1^2\sum_{j\in\mathbb{Z}}\big|F^m_{j,b}(f\chi_{3Q_0})dy\big|^2dt\le\int_1^2\sum_{j\in\mathbb{Z}}\left|\sum_{k=0}^m\tbinom{m}{k}F_j((b-c)^kf\chi_{3Q_0})(b-c)^{m-k}\right|^2dt$$

$$\le\left[\sum_{k=0}^m\tbinom{m}{k}|b-c|^{m-k}\left(\int_1^2\sum_{j\in\mathbb{Z}}\Big|F_j((b-c)^kf\chi_{3Q_0})\Big|^2dt\right)^{\frac12}\right]^2$$

$$\le\left[\sum_{k=0}^m\tbinom{m}{k}|b-c|^{m-k}|\widetilde{\mu}_\Omega((b-c)^kf\chi_{3Q_0})|\right]^2.$$

Let $c = b_{R_{Q_0}}$, then we have

$$|\widetilde{\mu^m_{\Omega,b}}(f\chi_{3Q_0})|^2\chi_{Q_0\setminus\cup_j P_j} \le \left[\sum_{k=0}^{m}\binom{m}{k}|b-b_{R_{Q_0}}|^{m-k}|\widetilde{\mu}_\Omega((b-b_{R_{Q_0}})^k f\chi_{3Q_0})|\right]^2\chi_{Q_0\setminus\cup_j P_j}.$$

Do the same for (2.4), we obtain

$$\begin{aligned}
&\left[\mu^m_{\Omega,b}(f\chi_{3Q_0})(x)\right]^2\chi_{Q_0}(x)\\
&\le \left[\sum_{k=0}^{m}\binom{m}{k}|b-b_{R_{Q_0}}|^{m-k}\widetilde{\mu}_\Omega\Big(|b-b_{R_{Q_0}}|^k f\chi_{3Q_0}\Big)(x)\right]^2\chi_{Q_0\setminus\cup_l P_l}(x)\\
&\quad+\sum_l\left[\sum_{k=0}^{m}\binom{m}{k}|b-b_{R_{Q_0}}|^{m-k}\left(\int_1^2\sum_{j=J_{P_l}}^{\infty}|F_j\Big(|b-b_{R_{Q_0}}|^k f\chi_{3Q_0}\Big)(x,t)|^2dt\right)^{1/2}\right]^2\chi_{P_l}(x)\\
&\quad+\sum_l\left(\int_1^2\sum_{j=-\infty}^{J_{P_l}-1}|F_j(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{P_l}(x).
\end{aligned}$$

We observe that

$$\widetilde{\mu}_\Omega\Big(|b-b_{R_{Q_0}}|^k f\chi_{3Q_0}\Big)\chi_{Q_0\setminus\cup_l P_l}(x) \le \alpha_n C_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}\chi_{Q_0}(x).$$

Since $P_l\cap E^c\neq\varnothing$, for some $x\in P_l$,

$$\mathscr{M}^*_{\mu_\Omega,Q_0}\Big(|b-b_{R_{Q_0}}|^k f\Big)(x)\le\alpha_n C_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0},$$

then we get, for $\xi\in P_l$,

$$\begin{aligned}
\left(\int_1^2\sum_{j=J_{P_l}}^{\infty}|F_j\Big(|b-b_{R_{Q_0}}|^k f\chi_{3Q_0}\Big)(\xi,t)|^2dt\right)^{\frac12} &\le \inf_{y\in P_l}\Big(\mathscr{M}^*_{\mu_\Omega,Q_0}\Big(|b-b_{R_{Q_0}}|^k f(y)\Big)\Big)\\
&\le \alpha_n C_\Omega\Big\langle\big|f|b-b_{R_{Q_0}}|^k\big|\Big\rangle_{3Q_0}.
\end{aligned}$$

Now we note that, for $j\le J_{P_l}-1,\ t\in[1,2]$ and $x\in P_l$, if $\chi_{2^{j-1}t<|x-y|\le 2^jt}=1$, then

$$|x-y|\le 2^jt\le 2^{J_{P_l}}\le\ell(P_l),$$

but, for $y\in\chi_{3Q_0\setminus 3P_l}$, we have $|x-y|>\ell(P_l)$. Therefore, $F^m_{j,b}\big(f\chi_{3Q_0\setminus 3P_l}\big)(x,t)=0$ and

$$\left(\int_1^2\sum_{j=-\infty}^{J_{P_l}-1}|F^m_{j,b}(f\chi_{3Q_0})(x,t)|^2dt\right)\chi_{P_l}(x)\le\left[\widetilde{\mu^m_{\Omega,b}}(f\chi_{3P_l})(x)\right]^2\chi_{P_l}(x).$$

Consequently, for almost everywhere $x\in Q_0$,

$$\left[\widetilde{\mu^m_{\Omega,b}}(f\chi_{3Q_0})(x)\right]^2\chi_{Q_0}(x) \le c_n C_\Omega^2 \left[\sum_{k=0}^{m}\binom{m}{k}|b-b_{R_{Q_0}}|^{m-k}\Big\langle |f|b-b_{R_{Q_0}}|^k| \Big\rangle_{3Q_0}\right]^2 \chi_{3Q_0}(x)$$
$$+\sum_{l}\left[\widetilde{\mu^m_{\Omega,b}}(f\chi_{3P_l})(x)\right]^2\chi_{P_l}(x).$$

Theorem 1.1 is proved.

3. Proof of Theorem 1.2. To prove part (a) of Theorem 1.2, we take some ideas in [17].

Lemma 3.1 [15]. *Given a dyadic lattice $\mathcal{D}$ there exist 3^n dyadic lattices $\mathcal{D}_j$ such that*

$$\{3Q : Q\in\mathcal{D}\} = \bigcup_{j=1}^{3^n}\mathcal{D}_j,$$

and, for every cube $Q\in\mathcal{D}$, we can find a cube R_Q in each $\mathcal{D}_j$ such that $Q\subset R_Q$ and $3\ell(Q)=\ell(R_Q)$.

Fix a lattice $\mathcal{D}$. For an arbitrary cube $Q\subset\mathbb{R}^n$, there is a cube $Q'\in\mathcal{D}$ such that $\dfrac{\ell_Q}{2}<\ell_{Q'}<\ell_Q$ and $Q\subset 3Q'$. By Lemma 3.1, there is $j\in\{1,\ldots,3^n\}$ such that $3Q'=P\in\mathcal{D}_j$. Therefore, for every cube $Q\subset\mathbb{R}^n$, there exists $P\in\mathcal{D}_j$ such that $Q\subset P$ and $\ell_P\le 3\ell_Q$. At this point, we observe that $|Q|\le|P|\le 3^n|Q|$. A similar statement can be found in [11].

From (1.4), it is straightforward that

$$[\mu^m_{\Omega,b}(f)(x)]^2 \le c_n C_\Omega^2\left[\sum_{Q\in\mathcal{S}}\sum_{k=0}^{m}\binom{m}{k}|b(x)-b_Q|^{m-k}\langle|f|b-b_Q|^k|\rangle_Q\right]^2\chi_Q(x).$$

There exist 3^n dyadic lattices $\mathcal{D}_j$ and sparse families $\mathcal{S}_j\subset\mathcal{D}_j$ such that

$$|\mu^m_{\Omega,b}(f)(x)| \le c_n C_\Omega\sum_{j=1}^{3^n}\sum_{k=0}^{m}\binom{m}{k}\sum_{Q\in\mathcal{S}_j}|b(x)-b_Q|^{m-k}\langle|f|b-b_Q|^k|\rangle_Q\chi_Q(x).$$

We recall that given a sparse family $\mathcal{S}$ from $\mathcal{D}$, we define the sparse operator with respect to BMO

$$\mathcal{A}^{m,k}_{\mathcal{S}}(b,f)(x)=\sum_{Q\in\mathcal{S}}|b(x)-b_Q|^{m-k}\langle|f|b-b_Q|^k|\rangle_Q\chi_Q(x).$$

From the proof process of Theorem 1.1 in [17], we have

$$\|\mathcal{A}^{m,k}_{\mathcal{S}}(b,f)\|_{L^p(\lambda)}\le c\|b\|^m_{BMO_{v^{1/m}}}([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(u)},$$

so

$$\|\mu^m_{\Omega,b}(f)\|_{L^p(\lambda)}\le c_{n,m}C_\Omega\|\mathcal{A}^{m,k}_{\mathcal{S}}(b,f)\|_{L^p(\lambda)}$$
$$\le c_{n,m}C_\Omega\|b\|^m_{BMO_{v^{1/m}}}([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(u)}.$$

Therefore, the proof of part (a) is completed.

Recall that the local mean oscillation of f on Q is represented by

$$\omega_\lambda(f;Q)=\inf_{a\in\mathbb{R}}((f-a)\chi_Q)^*(\lambda|Q|),\quad 0<\lambda<1,$$

where $((f-a)\chi_Q)^*$ denotes the nonincreasing rearrangement of $(f-a)\chi_Q$. Let us prove then part (b) of Theorem 1.2. We rely upon the following lemmas.

Lemma 3.2 [17]. *Let $\eta\in A_\infty$. Then*

$$\|f\|_{BMO_\eta}\le c\sup_Q\omega_\lambda(f;Q)\frac{|Q|}{\eta(Q)}\quad\left(0<\lambda\le\frac{1}{2^{n+2}}\right),$$

where c depends only on η.

Lemma 3.3 [17]. *For $\lambda=\dfrac{1}{2^{n+2}}$, there exist $0<\varepsilon_0,\xi_0<1$ and $k_0>1$ depending only on Ω and n such that the following holds. For every cube $Q\subset\mathbb{R}^n$, there exist measuralbe sets $E\subset Q, F\subset k_0Q$ and $|G|\subset E\times F$ with $G\ge\xi_0|Q|^2, |E|=\dfrac{1}{2^{n+3}}|Q|$ such that*

(i) $\omega_\lambda(b;Q)\le|b(x)-b(y)|$ *for all* $(x,y)\in E\times F$;

(ii) $\Omega\left(\dfrac{x-y}{|x-y|}\right)$ *and* $b(x)-b(y)$ *do not change sign in* $E\times F$;

(iii) $\Omega\left(\dfrac{x-y}{|x-y|}\right)\ge\varepsilon_0$ *for all* $(x,y)\in G$.

Combining properties (i) and (iii) of Lemma 3.3, we obtain

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m|G|\le\frac{1}{\varepsilon_0}\iint\limits_G|b(x)-b(y)|^m\left|\Omega\left(\frac{x-y}{|x-y|}\right)\right|dxdy.$$

Then taking into account $|x-y|\le\dfrac{k_0+1}{2}\sqrt{n}\ell(Q)$, for all $(x,y)\in G$, we have

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m|G|\le\frac{1}{\varepsilon_0}\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)^{n-1}\iint\limits_G|b(x)-b(y)|^m\left|\Omega\left(\frac{x-y}{|x-y|}\right)\right|\frac{dxdy}{|x-y|^{n-1}}.$$

By property (ii), we know that $|b(x)-b(y)|^m\Omega\left(\dfrac{x-y}{|x-y|}\right)$ does not change sign in $E\times F$. Hence, using $|G|\ge\xi_0|Q|^2$, we get

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m\le\frac{1}{\varepsilon_0\xi_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n\frac{1}{|Q|}\frac{1}{\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)}$$

$$\times\int\limits_E\int\limits_F|b(x)-b(y)|^m\left|\Omega\left(\frac{x-y}{|x-y|}\right)\right|\frac{dydx}{|x-y|^{n-1}}$$

$$=\frac{1}{\varepsilon_0\xi_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n\frac{1}{|Q|}\frac{1}{\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)}$$

$$\times \int_E \left| \int_F (b(x)-b(y))^m \Omega\left(\frac{x-y}{|x-y|}\right) \frac{dy}{|x-y|^{n-1}} \right| dx.$$

Taking into account $|x-y| \leq \frac{k_0+1}{2}\sqrt{n}\ell(Q)$ for all $(x,y) \in G$, we observe that $\{y\colon |x-y| \leq t\} \cap F = F$ when $t \geq (k_0+1)\sqrt{n}\ell(Q)$. Then we have

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m \leq \frac{c}{|Q|} \frac{1}{\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)} \int_E \left| \int_F (b(x)-b(y))^m \Omega\left(\frac{x-y}{|x-y|}\right) \frac{dy}{|x-y|^{n-1}} \right| dx$$

$$\leq \frac{c}{|Q|} \int_E \left(\int_{(k_0+1)\sqrt{n}\ell(Q)}^{\infty} \left| \int_{\{y\colon |x-y|\leq t\}\cap F} (b(x)-b(y))^m \Omega\left(\frac{x-y}{|x-y|}\right) \frac{dy}{|x-y|^{n-1}} \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} dx$$

$$\leq \frac{c}{|Q|} \int_E \left| \mu_{\Omega,b}^m(\chi_F) \right| dx,$$

where c depends only on Ω and n. By using Hölder inequality and the assumption $\|\mu_{\Omega,b}^m(\chi_F)\|_{L^p(\lambda)} \leq cu(F)^{\frac{1}{p}}$, we obtain

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m \leq \frac{c}{|Q|} \left(\int_E \left| \mu_{\Omega,b}^m(\chi_F) \right|^p \lambda dx \right)^{\frac{1}{p}} \left(\int_Q \lambda^{-\frac{1}{p-1}} dx \right)^{\frac{1}{p'}}$$

$$\leq \frac{c}{|Q|} u(F)^{\frac{1}{p}} \left(\int_Q \lambda^{-\frac{1}{p-1}} dx \right)^{\frac{1}{p'}} \leq c \left(\frac{1}{|Q|} \int_Q u \right)^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q \lambda^{-\frac{1}{p-1}} dx \right)^{\frac{1}{p'}}.$$

Since $u \in A_p$, we have that u is doubling,

$$u(F) \leq u(k_0 Q) \leq cu(Q),$$

and

$$\frac{1}{|Q|} \int_Q u \leq c \left(\frac{1}{|Q|} \int_Q u^{\frac{1}{r}} \right)^r, \quad r > 1.$$

Then

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m \leq c \left(\frac{1}{|Q|} \int_Q u^{\frac{1}{r}} \right)^{\frac{r}{p}} \left(\frac{1}{|Q|} \int_Q \lambda^{-\frac{1}{p-1}} \right)^{\frac{1}{p'}}.$$

Taking $r = mp+1$, using Hölder inequality with $\frac{mp}{r} + \frac{r-mp}{r} = 1$, and $u^{\frac{1}{r}} = \lambda^{\frac{1}{r}} \nu^{\frac{p}{r}}$, we get

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m \le c\left(\frac{1}{|Q|}\int\limits_Q \lambda^{\frac{1}{r}}\nu^{\frac{p}{r}}\right)^{\frac{r}{p}}\left(\frac{1}{|Q|}\int\limits_Q \lambda^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}$$

$$\le c\left(\frac{1}{|Q|}\int\limits_Q \nu^{\frac{1}{m}}\right)^{m}\left(\frac{1}{|Q|}\int\limits_Q \lambda\right)^{\frac{1}{p}}\left(\frac{1}{|Q|}\int\limits_Q \lambda^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\le c\left(\frac{1}{|Q|}\nu^{1/m}(Q)\right)^{m},$$

that is, $\omega_{\frac{1}{2^{n+2}}}(b;Q) \le \dfrac{c}{|Q|}\nu^{1/m}(Q)$. On the other hand, since $u,\lambda \in A_p$, we have $\nu^{1/m} \in A_2$ by Hölder's inequality. Therefore, taking into account Lemma 3.2, we obtain

$$\|b\|_{BMO_{\nu^{1/m}}} \le c\sup_Q \omega_{\frac{1}{2^{n+2}}}(b;Q)\frac{|Q|}{\nu^{1/m}(Q)} \le c\sup_Q \frac{c}{|Q|}\nu^{1/m}(Q)\frac{|Q|}{\nu^{1/m}(Q)} \le c$$

for all Q. Finally, we prove that $b \in BMO_{\nu^{1/m}}$.

Theorem 1.2 is proved.

4. Proof of Theorems 1.3 and 1.4. To prove Theorems 1.3 and 1.4, we need to give the following lemma.

Lemma 4.1 [7]. *For any nonnegative function $g(x)$, we have*

$$\int\limits_{\mathbb{R}^n} (\mu^{*,m}_{\Omega,\alpha,b}(f)(x))^2 g(x)dx \le C_\alpha \int\limits_{\mathbb{R}^n} (\mu^{m}_{\Omega,b}(f)(x))^2 Mg(x)dx,$$

where M is the Hardy – Littlewood maximal operator.

For part (a) of Theorems 1.3 and 1.4, since

$$\mu^m_{\Omega,S,b} \le \left(\iint\limits_{\mathbb{R}^{n+1}_+}\left(\frac{2t}{t+|x-y|}\right)^{n\alpha}\left|\int\limits_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+3}}\right)^{\frac{1}{2}}$$

$$\le 2^{n\alpha}\mu^{*,m}_{\Omega,\alpha,b}(f)(x),$$

we need only consider $\mu^{*,m}_{\Omega,\alpha,b}$. For $p>2$, setting $r = \dfrac{p}{2}$, then, by duality, Lemma 4.1 and Hölder's inequality, we get

$$\|\mu^{*,m}_{\Omega,\alpha,b}(f)\|_{L^p(\lambda)} = \left(\int\limits_{\mathbb{R}^n}(\mu^{*,m}_{\Omega,\alpha,b}(f)(x))^p\lambda(x)dx\right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{\mathbb{R}^n}[(\mu^{*,m}_{\Omega,\alpha,b}(f)(x))^2\lambda^{r'-1}]^r\lambda^{1-r'}dx\right)^{\frac{1}{2r}}$$

$$= \left(\sup_{\|g\|_{L^{r'}(\lambda^{1-r'})}\leq 1} \int_{\mathbb{R}^n} (\mu^{*,m}_{\Omega,\alpha,b}(f)(x))^2 \lambda^{r'-1} g(x)\lambda^{1-r'} dx \right)^{\frac{1}{2}}$$

$$= \left(\sup_{\|g\|_{L^{r'}(\lambda^{1-r'})}\leq 1} \int_{\mathbb{R}^n} (\mu^{*,m}_{\Omega,\alpha,b}(f)(x))^2 g(x) dx \right)^{\frac{1}{2}}$$

$$\leq C_\alpha \left(\sup_{\|g\|_{L^{r'}(\lambda^{1-r'})}\leq 1} \int_{\mathbb{R}^n} (\mu^{m}_{\Omega,b}(f)(x))^2 Mg(x) dx \right)^{\frac{1}{2}}$$

$$\leq C_\alpha \|\mu^{m}_{\Omega,b}(f)\|_{L^p(\lambda)} \left(\sup_{\|g\|_{L^{r'}(\lambda^{1-r'})}\leq 1} \|Mg\|_{L^{r'}(\lambda^{1-r'})} \right)^{\frac{1}{2}} \leq C_\alpha \|\mu^{m}_{\Omega,b}(f)\|_{L^p(\lambda)}.$$

Here, we have used the fact that M is weak (1,1) in the last inequality. Using the conclusion of Theorem 1.2, namely,

$$\|\mu^{m}_{\Omega,b}(f)\|_{L^p(\lambda)} \leq c_{n,m} C_\Omega \|b\|^m_{BMO_{v^{1/m}}} ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(u)},$$

we can directly obtain

$$\|\mu^{*,m}_{\Omega,\alpha,b}(f)\|_{L^p(\lambda)} \leq c_{n,m,\alpha} C_\Omega \|b\|^m_{BMO_{v^{1/m}}} ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(u)}$$

and

$$\|\mu^{m}_{\Omega,S,b}(f)\|_{L^p(\lambda)} \leq c_{n,m,\alpha} C_\Omega \|b\|^m_{BMO_{v^{1/m}}} ([\lambda]_{A_p}[u]_{A_p})^{\frac{m+1}{2}\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(u)}.$$

For part (b) of Theorems 1.3 and 1.4, we need only consider the operator $\mu^m_{\Omega,S,b}$. In fact, since $\mu^m_{\Omega,S,b} \leq 2^{n\alpha}\mu^{*,m}_{\Omega,\alpha,b}(f)(x)$, part (b) of Theorem 1.4 is indeed a direct consequence of part (b) of Theorem 1.3. Let E_x and E_y denote the range of x and the range of y, respectively, and $E_x = E_y = E$. Combining properties (i) and (iii) of Lemma 3.3, for $x \in E_x$, we obtain

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m |G| \leq \frac{1}{\varepsilon_0} \iint_G |b(x)-b(z)|^m \left| \Omega\left(\frac{y-z}{|y-z|}\right)\right| dydz.$$

Then taking into account $|y-z| \leq \dfrac{k_0+1}{2}\sqrt{n}\ell(Q)$ for all $(y,z)\in G$, we have

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m |G| \leq \frac{1}{\varepsilon_0} \left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)^{n-1} \iint_G |b(x)-b(z)|^m \left| \Omega\left(\frac{y-z}{|y-z|}\right)\right| \frac{dzdy}{|y-z|^{n-1}}.$$

By property (ii), we know that $|b(x)-b(z)|^m \Omega\left(\dfrac{y-z}{|y-z|}\right)$ does not change sign in $E_y \times F$. Hence, using $|G| \geq \xi_0 |Q|^2, |E| = \dfrac{1}{2^{n+3}}|Q|$, integrating with respect to x yield that

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m \leq \frac{1}{\varepsilon_0\xi_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n \frac{1}{|Q||E|}\frac{1}{\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)}$$

$$\times \int\limits_{E_x}\int\limits_{E_y}\int\limits_{F}|b(x)-b(z)|^m\left|\Omega\left(\frac{y-z}{|y-z|}\right)\right|\frac{dzdy}{|y-z|^{n-1}}dx$$

$$\leq \frac{2^{n+3}}{\varepsilon_0\xi_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n \frac{1}{|Q|^2}\frac{1}{\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)}$$

$$\times \int\limits_{E_x}\int\limits_{E_y}\left|\int\limits_{F}(b(x)-b(z))^m\Omega\left(\frac{y-z}{|y-z|}\right)\frac{dz}{|y-z|^{n-1}}\right|dydx.$$

Taking into account $|y-z| \leq \dfrac{k_0+1}{2}\sqrt{n}\ell(Q)$ for all $(y,z)\in G$, we observe that $\{z\colon |y-z|\leq t\}\cap F = F$ when $t \geq (k_0+1)\sqrt{n}\ell(Q)$. Then we have

$$\omega_{\frac{1}{2^{n+2}}}(b;Q)^m$$

$$\leq \frac{c}{|Q|^2}\frac{1}{\left(\frac{k_0+1}{2}\sqrt{n}\ell(Q)\right)}\int\limits_{E_x}\int\limits_{E_y}\left|\int\limits_{F}(b(x)-b(z))^m\Omega\left(\frac{y-z}{|y-z|}\right)\frac{dz}{|y-z|^{n-1}}\right|dydx$$

$$\leq \frac{c}{|Q|^{\frac{3}{2}}}\int\limits_{E_x}\int\limits_{E_y}\left(\int\limits_{(k_0+1)\sqrt{n}\ell(Q)}^{\infty}\left|\int\limits_{\{z\colon |y-z|\leq t\}\cap F}(b(x)-b(z))^m\Omega\left(\frac{y-z}{|y-z|}\right)\frac{dz}{|y-z|^{n-1}}\right|^2\frac{dt}{t^{n+3}}\right)^{\frac{1}{2}}dydx$$

$$\leq \frac{c}{|Q|^{\frac{3}{2}}}\int\limits_{E_x}\left(\int\limits_{E_y}\int\limits_{(k_0+1)\sqrt{n}\ell(Q)}^{\infty}\left|\int\limits_{\{z\colon |y-z|\leq t\}}(b(x)-b(z))^m\Omega\left(\frac{y-z}{|y-z|}\right)\frac{\chi_F dz}{|y-z|^{n-1}}\right|^2\frac{dtdy}{t^{n+3}}\right)^{\frac{1}{2}}|E_y|^{\frac{1}{2}}dx$$

$$\leq \frac{c}{|Q|}\int\limits_{E_x}\left(\int\limits_{(k_0+1)\sqrt{n}\ell(Q)}^{\infty}\int\limits_{E_y}\left|\int\limits_{\{z\,:\,|y-z|\leq t\}}(b(x)-b(z))^m\Omega\left(\frac{y-z}{|y-z|}\right)\frac{\chi_F dz}{|y-z|^{n-1}}\right|^2\frac{dydt}{t^{n+3}}\right)^{\frac{1}{2}}dx$$

$$\leq \frac{c}{|Q|}\int\limits_{E_x}|\mu^m_{\Omega,S,b}(\chi_F)|dx,$$

where c depends only on Ω and n. By the same way as part (b) of Theorem 1.2, we obtain that $b\in BMO_{\nu^{1/m}}$.

Theorems 1.3 and 1.4 are proved.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

Funding. Y. Chen was partially supported by the National Natural Science Foundation of China (No. 11871096).

Author contributions. All authors have contributed equally to the work.

References

1. A. Al-Salman, H. Al-Qassem, L. C. Cheng, Y. Pan, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett., **9**, 697 – 700 (2002).
2. A. Banedek, A. P. Calderón, R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Natl. Acad. Sci. USA., **48**, 356 – 365 (1962).
3. Y. Chen, Y. Ding, *Commutators of Littlewood – Paley operators*, Sci. China Ser. A, **52**, 2493 – 2505 (2009).
4. Y. Chen, Y. Ding, *L^p boundedness of the commutators of Marcinkiewicz integrals with rough kernels*, Forum Math., **27**, 2087 – 2111 (2015).
5. Y. Ding, D. Fan, Y. Pan, *Weighted boundedness for a class of rough Marcinkiewicz integrals*, Indiana Univ. Math. J., **48**, 1037 – 1055 (1999).
6. Y. Ding, D. Fan, Y. Pan, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernels*, Acta Math. Sin. (Engl. Ser.), **16**, 593 – 600 (2000).
7. Y. Ding, S. Lu, K. Yabuta, *On commutators of Marcinkiewicz integrals with rough kernel*, J. Math. Anal. and Appl., **275**, 60 – 68 (2002).
8. X. Duong, R. Gong, M. S. Kuffner, J. Li, B. D. Wick, D. Yang, *Two weight commutators on spaces of homogeneous type and applications*, J. Geom. Anal., **31**, 980 – 1038 (2021).
9. D. Fan, S. Sato, *Weak type (1,1) estimates for Marcinkiewicz integrals with rough kernels*, Tohoku Math. J., **53**, 265 – 284 (2001).
10. G. Hu, M. Qu, *Quantitative weighted L^p bounds for the Marcinkiewicz integral*, Math. Inequal. Appl., **22**, 885 – 899 (2019).
11. T. P. Hytönen, M. T. Lacey, C. Pérez, *Sharp weighted bounds for the q-variation of singular integrals*, Bull. London Math. Soc., **45**, 529 – 540 (2013).
12. G. H. Ibañez-Firnkorn, I. P. Rivera-Ríos, *Sparse and weighted estimates for generalized Hörmander operators and commutators*, Monatsh. Math., **191**, 125 – 173 (2020).
13. A. K. Lerner, *A simple proof of the A_2 conjecture*, Int. Math. Res. Not. IMRN, **14**, 3159 – 3170 (2013).
14. A. K. Lerner, *On pointwise estimates involving sparse operators*, New York J. Math., **22**, 341 – 349 (2016).
15. A. K. Lerner, F. Nazarov, *Intuitive dyadic calculus: the basics*, Expo. Math., **37**, 225 – 265 (2019).
16. A. K. Lerner, S. Ombrosi, I. P. Rivera-Ríos, *On pointwise and weighted estimates for commutators of Calderón – Zygmund operators*, Adv. Math., **319**, 153 – 181 (2017).
17. A. K. Lerner, S. Ombrosi, I. P. Rivera-Ríos, *Commutators of singular integrals revisited*, Bull. London Math. Soc., **51**, 107 – 119 (2019).
18. K. Li, H. Martikainen, E. Vuorinen, *Bloom-type inequality for bi-parameter singular integrals: efficient proof and iterated commutators*, Int. Math. Res. Not. IMRN, 8153 – 8187 (2021).
19. K. Li, W. Sun, *Weak and strong type weighted estimates for multilinear Calderón – Zygmund operators*, Adv. Math., **254**, 736 – 771 (2014).
20. E. M. Stein, *On the functions of Littlewood – Paley, Lusin, Marcinkiewicz*, Trans. Amer. Math. Soc., **88**, 430 – 466 (1958).
21. X. Tao, G. Hu, *A sparse domination for the Marcinkiewicz integral with rough kernel and applications*, Publ. Math. Debrecen, **96**, 377 – 399 (2020).
22. A. Torchinsky, S. Wang, *A note on the Marcinkiewicz integral,* Colloq. Math., **60/61**, 235 – 243 (1990).
23. T. Walsh, *On the function of Marcinkiewicz,* Stud. Math., **44**, 203 – 217 (1972).

Received 24.01.23