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EXTENDED TOTAL GRAPH ASSOCIATED TO FINITE COMMUTATIVE RINGS

РОЗШИРЕНИЙ ТОТАЛЬНИЙ ГРАФ, АСОЦІЙОВАНИЙ ЗІ СКІНЧЕННИМИ КОМУТАТИВНИМИ КІЛЬЦЯМИ

For a commutative ring R with nonzero identity $1 \neq 0$, let $Z(R)$ denote the set of zero divisors. The total graph of R denoted by $T_\Gamma(R)$ is a simple graph in which all elements of R are vertices and any two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. In this paper, we define an extension of the total graph denoted by $T(\Gamma^e(R))$ with vertex set as $Z(R)$, and two distinct vertices x and y are adjacent if and only if $x + y \in Z^*(R)$, where $Z^*(R)$ is the set of nonzero zero divisors of R. Our main aim is to characterize the finite commutative rings whose $T(\Gamma^e(R))$ has clique numbers $1, 2,$ and 3. In addition, we characterize finite commutative nonlocal rings R for which the corresponding graph $T(\Gamma^e(R))$ has the clique number 4.

Нехай $Z(R)$ позначає множину дільників нуля для комутативного кільця R з відмінною від нуля тотожністю $1 \neq 0$. Повний граф кільця R, який ми позначаємо $T_\Gamma(R)$, є простим графом, в якому всі елементи R є вершинами, а дві різні вершини x і y суміжні тоді і тільки тоді, коли $x+y \in Z(R)$. У цій статті ми визначаємо розширення тотального графа, який ми позначаємо $T(\Gamma^e(R))$ і який має вершину $Z(R)$, а дві різні вершини x і y суміжні тоді і тільки тоді, коли $x + y \in Z^*(R)$, де $Z^*(R)$ — набір ненульових дільників нуля в R. Основною метою статті є характеристика скінченних комутативних кілець, у яких $T(\Gamma^e(R))$ має клікові числа 1, 2 і 3. Крім того, охарактеризовано скінченні комутативні нелокальні кільця R, відповідний граф $T(\Gamma^e(R))$ яких має клікове число 4.

1. Introduction. All graphs considered in this article are connected, simple and finite. A graph is denoted by $G = G(V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. The order and the size of G are the cardinalities of $V(G)$ and $E(G)$, respectively. A complete graph on n vertices is denoted by K_n and a complete bipartite graph is denoted by K_{n_1,n_2}, where n_1 and n_2 are the cardinalities of partite subsets. The adjacency relation between two vertices x and y is denoted by $x \sim y$. For $S \subset V$, the graph $G[S]$ is an induced subgraph of G, with vertex set S and whose edge set consists of those edges of E whose end vertices are in S. A clique of a graph G is defined as the complete subgraph of G. The cardinality of the largest clique is called the clique number and is denoted by $\omega(G)$. In a graph G, the set of nonadjacent vertices is called an independent set and the cardinality of the maximum independent set is called the independence number of G and is denoted by $i(G)$. Also, by $\operatorname{comp}(G)$, we denote the number of connected components of a graph G. For more on graphs, we refer [12].

For notations and results about commutative rings, we use [8, 11, 16, 17] as basic references. A ring R is assumed to be a finite commutative ring with identity $1 \neq 0$. A ring R is said to be local, if it has a unique maximal ideal. The finite field with n elements is denoted by $\mathbb{F}_n$ and the ring $\mathbb{Z}_n$ denotes

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the ring of integers modulo n. The cardinality of a ring R is denoted by $|R|$. The direct product of two rings R_1 and R_2, denoted by $R_1 \times R_2$, consists of all the ordered pairs (a,b) with $a \in R_1$ and $b \in R_2$. The addition rule for such pairs is $(a,b)+(c,d)=(a+c,b+d)$ and multiplication rule for such pairs is $(a,b).(c,d)=(ac,bd)$. The structure theorem for Artinian rings states that any Artinian ring R is isomorphic to the direct product of local rings, that is, $R \cong R_1 \times R_2 \times \ldots \times R_n$ for some positive integer n, where R_i, $i=1,2,\ldots,n$, are local rings.

Let R be a commutative ring and $Z(R)$ be its set of all zero-divisors and $Z^*(R)=Z(R)\setminus\{0\}$ be the set of nonzero zero-divisors. The concept of the zero-divisor graph $\Gamma(R)$ associated to a ring R was first defined and introduced by Beck [4] and later modified by Anderson and Livingston [2]. A new extension of the zero-divisor graph can be seen in Cherrabi et al. [6, 7] and Bennis et al. [5]. Anderson and Badawi [1] introduced the total graph of R, denoted by $T_\Gamma(R)$, as a simple graph with all elements of R as vertices and two distinct vertices x and y are adjacent if and only if $x+y \in Z(R)$. We define a new extension of the total graph, denoted by $T(\Gamma^e(R))$, corresponding to a commutative ring R, whose vertex set is the set of nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $x+y \in Z^*(R)$. The justification can be seen from the fact that the number of edges in $T(\Gamma^e(R))$ is relatively less than the number of edges in $T_\Gamma(R)$. The extended total graphs $T(\Gamma^e(\mathbb{Z}_{16}))$ and $T(\Gamma^e(\mathbb{Z}_{27}))$ associated to $\mathbb{Z}_{16}$ and $\mathbb{Z}_{27}$ are shown in Fig. 1.

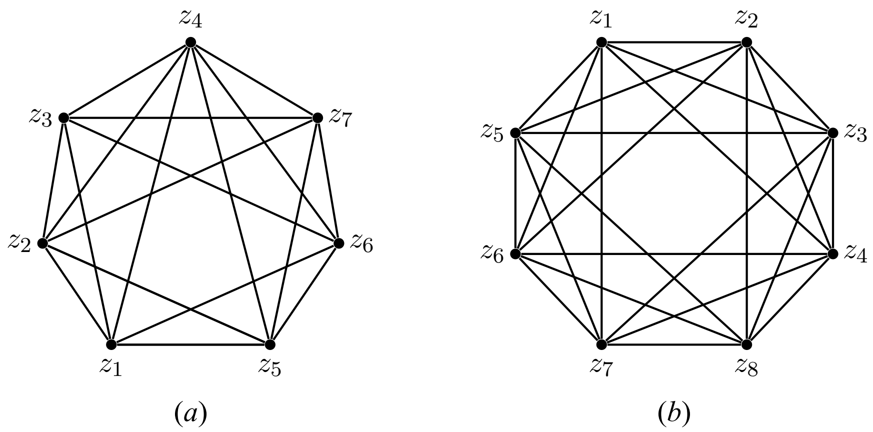

Fig. 1. The extended total graphs $T(\Gamma^e(\mathbb{Z}_{16}))$ (*a*) and $T(\Gamma^e(\mathbb{Z}_{27}))$ (*b*).

Liu et al. [9] completely determined the structure of all finite commutative rings whose zero-divisor graphs have clique number 1, 2, or 3. Also, Liu et al. [10] investigated the algebraic structure of rings R whose zero-divisor graph $G(R)$ has clique number 4. They obtained complete characterizations of all the finite commutative local rings with clique number 4. Tian and Li [18] proposed a constructive method for calculating the clique number of the zero-divisor graph of the ring Z_n. We observe that finding the clique of some particular order in $T(\Gamma^e(R))$ is relatively difficult than in $T_\Gamma(R)$. Thus, it is interesting to investigate the clique number of $T(\Gamma^e(R))$. Further, it will be interesting to investigate the graph invariants of $T(\Gamma^e(R))$. From the definition, we observe that $T(\Gamma^e(R))$ is an empty graph if and only if R is an integral domain. Some recent results can be seen in [13 – 15].

This paper is organized as follows. In Section 2, we discuss some basic properties of the graph $T(\Gamma^e(R))$, especially when R is the ring of integers modulo n. In Section 3, we characterize all finite commutative rings for which $\omega(T(\Gamma^e(R)) = 1, 2, 3$. Also, we characterize the finite commutative nonlocal rings R corresponding to $T(\Gamma^e(R))$, when $\omega(T(\Gamma^e(R))) = 4$.

2. Basic properties of $T(\Gamma^e(R))$ when $R \cong \mathbb{Z}_n$. For any ring R, we observe that $T(\Gamma^e(R))$ is connected. Further, we observe that $T(\Gamma^e(R))$ is connected for some specific families of rings. For example, consider the family of zero-rings $R \cong \dfrac{m\mathbb{Z}}{n\mathbb{Z}}$, the graph $T(\Gamma^e(R))$ is always connected, since all the elements of $\dfrac{m\mathbb{Z}}{n\mathbb{Z}}$ are zero-divisors and is closed with respect to addition. In case we consider the family of quotient rings, the graph $T(\Gamma^e(R))$ is not always connected. For example, if $R \cong \dfrac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, the graph $T(\Gamma^e(R))$ is connected and is isomorphic to K_3, while as for $R \cong \dfrac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, the graph $T(\Gamma^e(R))$ is disconnected with two isolated vertices. Further, if R is an Artinain ring, then $R \cong R_1 \times R_2 \times R_3 \times \ldots \times R_n$, where each R_i, $1 \le i \le n$, are local rings. Therefore, for $n \ge 2$, we observe that the graph $T(\Gamma^e(R))$ is connected. The above observations suggest to characterize those rings whose $T(\Gamma^e(R))$ is connected. In this direction, we have investigated the connectedness of $T(\Gamma^e(R))$, for the family of the ring of integers modulo n, for different values of n.

We start with the following observation.

Observation 2.1. *If $n = p$, where p is prime, then the graph $T(\Gamma^e(\mathbb{Z}_p))$ is an empty graph.*

Lemma 2.1. *If $n = 2^m$, $m \ge 4$, is an integer, then, for any vertex $v \in V\big(T(\Gamma^e(\mathbb{Z}_n))\big)$, we have*

$$\deg v = \begin{cases} 2^{m-1} - 2, & \text{if} \quad v = \dfrac{n}{2}, \\ 2^{m-1} - 3, & \text{otherwise}. \end{cases}$$

Further, $i\big(T(\Gamma^e(\mathbb{Z}_n))\big) = 2$.

Proof. Let $G = T(\Gamma^e(\mathbb{Z}_n))$. For $n = 2^m$, note that $Z^*(\mathbb{Z}_n) = \{2k (\operatorname{mod} n) \mid k \in \mathbb{Z}^+\}$. Therefore, $V(G) = \{2, 4, 6, \ldots, 2^m - 2\} = \{z_1, z_2 \ldots, z_{2^{m-1}-1}\}$, where $z_k = 2k$ for all $k = 1, 2, \ldots, 2^{m-1} - 1$. Hence, $|V(G)| = 2^{m-1} - 2$. Observe that, corresponding to every z_r, $r \ne 2^{m-2}$, there exists a unique z_s, $1 \le r \ne s \le 2^{m-1} - 1$, such that $z_r + z_s \equiv 0 \ (\operatorname{mod} n)$. However, if $r = 2^{m-2}$, there is no z_s, $1 \le r \ne s \le 2^{m-1} - 1$, such that $z_r + z_s \equiv 0 \ (\operatorname{mod} n)$. Since $z_r + z_s \in Z^*(\mathbb{Z}_n)$ for all $1 \le r \ne s \le 2^{m-1} - 1$ such that $z_r + z_s \not\equiv 0 \ (\operatorname{mod} n)$, therefore it follows that $\{z_r, z_s\} \in E(G)$ such that $z_r + z_s \not\equiv 0 \ (\operatorname{mod} n)$. In other words, for $k \ne 2^{m-2}$, $\deg z_k = |V(G)| - 2 = 2^{m-1} - 3$ and $\deg z_{2^{m-2}} = |V(G)| - 1 = 2^{m-1} - 2$.

Moreover, $i(G) = 2$ follows from the fact that $\{z_r, z_s\}$ is the maximal independent set if and only if $z_r + z_s \equiv 0 \ (\operatorname{mod} n)$.

One can consider the example of extended total graph of $\mathbb{Z}_{16}$ as shown in Fig. 1(*a*) to see for $k \ne 4$, $\deg z_k = 5$, and $\deg z_4 = 6$.

Lemma 2.2. *If $n = p^m$, where $m \ge 3$ and p is a prime greater or equal to 3, then the graph $T(\Gamma^e(\mathbb{Z}_n))$ is $p^{m-1} - 1$ regular. Moreover, $i(T(\Gamma^e(\mathbb{Z}_n)) = 2$.*

Proof. Let $G = T(\Gamma^e(\mathbb{Z}_n))$, where $n = p^m$, $m \ge 3$, is an integer and $p \ge 3$ is prime. Note that $Z^*(\mathbb{Z}_n) = \{kp \mid k = 1, 2, \ldots, p^{m-1} - 1\}$. Hence, $V(G) = \{p, 2p, \ldots, (p^{m-1} - 1)p\} = \{z_1, z_2, \ldots, z_{p^{m-1}-1}\}$, where $z_k = pk$ for all $k = 1, 2, \ldots, p^{m-1} - 1$. Now, for any $rp, sp \in Z^*(\mathbb{Z}_n)$, where $1 \le r \ne s \le p^{m-1} - 1$, we see that $rp + sp = (r + s)p \in Z^*(\mathbb{Z}_n)$ provided $r + s \ne p^{m-1}$. Therefore, $\{z_r, z_s\} \in E(G)$ provided $r + s \ne p^{m-1}$. Since for a given z_r, there exists a unique z_s such that $r + s = p^{m-1}$, we see that every vertex z_k of G is adjacent to every other vertex except the vertex $z_{p^{m-1}-k}$. Therefore, we conclude that G is $|V(G|) - 1 = p^{m-1} - 1$ regular.

Moreover, $i(G) = 2$ follows from the fact that $\{z_r, z_s\}$ is the maximal independent set if and only if $z_r + z_s \equiv 0 \pmod{n}$.

Furthermore, one can observe that the set $\{z_r, z_s\}$ is maximal independent set if and only if $r + s \neq p^{m-1}$. Hence, $i(G) = 2$.

Lemma 2.3. *Let $n = pq$, where are p, q be distinct primes with $p < q$. Then the graph $T(\Gamma^e(\mathbb{Z}_n))$ is disconnected and*

$$\operatorname{comp}(G) = \begin{cases} 2, & \text{if} \quad p, q \neq 3, \\ 3, & \text{otherwise}. \end{cases}$$

Proof. Let $G = T(\Gamma^e(\mathbb{Z}_n))$, where $n = pq$. Note that $Z^*(\mathbb{Z}_n) = \{rp, sq \mid 1 \leq r < q,\ 1 \leq s < p\}$. Hence, $V(G) = \{p, 2p, \ldots, (q-1)p, q, 2q, \ldots, (p-1)q\} = V_1 \sqcup V_2$, where $V_1 = \{p, 2p, \ldots, (q-1)p\} = \{u_1, u_2, \ldots, u_{q-1}\}$ and $V_2 = \{q, 2q, \ldots, (p-1)q\} = \{w_1, w_2, \ldots, w_{p-1}\}$. First note that if $u_i \in V_1$ and $w_j \in V_2$, then $u_i + w_j = ip + jq$ for some $1 \leq i < q$ and $1 \leq j < p$, and hence $u, v \in V_1$, since $\gcd(ip + jq, pq) = 1$. Thus, there is no edge in G whose one end vertex is in V_1 and other in V_2. This shows that $\operatorname{comp}(G) \geq 2$. On the other hand, suppose $u, v \in V_i$, $i = 1$ or 2. Without loss of generality, we may assume that $u, v \in V_1$. Then $u + v = rp + sp$ for some $1 \leq r, s < q$. Hence, $u + v \in Z^*(\mathbb{Z}_n)$ if and only if $r + s \neq q$. Similarly, one can observe that if $u = lq$, $v = mq \in V_2$, then $u + v \in Z^*(\mathbb{Z}_n)$ if and only if $l + m \neq p$, where $1 \leq l, m < p$. We have the following cases to consider.

Case 1. If $p, q \neq 3$, then clearly $q \geq 5$. Consider the induced subgraph $G[V_i]$ on $|V_i|$, $i = 1, 2$. We claim that $G[V_i]$ is connected. First suppose that $p = 2$ so that $q > 3$. Then $G[V_2]$ is a vertex, since $|V_2| = 1$. Hence, $G[V_2]$ is connected. On the other hand, $G[V_1]$ consists of more than 4 vertices. Since for any $u \in V_1$, there exists exactly one element $v \in V_1$ such that $u + v \notin Z^*(\mathbb{Z}_n)$. Therefore, we conclude that $G[V_1]$ is connected. However, if $p \neq 2$, then both $p, r \geq 5$ and by the same argument as above, we conclude that $G[V_i]$ is connected for $i = 1, 2$. Hence, $\operatorname{comp}(G) = 2$.

Case 2. If either p or q is equal to 3, then one of the subsets $|V_i|$ of $V(G)$ has size 2, where $i = 1$ or 2. Without loss of generality, suppose that $|V_1| = 2$. Then by the same fact as discussed above, for any $u \in V_1$, there exists exactly one element $v \in V_1$ such that $u + v \notin Z^*(\mathbb{Z}_n)$. This shows that $\operatorname{comp}(G[V_1]) = 2$. Hence, $\operatorname{comp}(G) = 3$.

Lemma 2.4. *If $n = 2m$, where $m \in \mathbb{Z}^+$ and $V\big(T(\Gamma^e(\mathbb{Z}_n))\big) = Z^*(\mathbb{Z}_n)$, then the graph $T(\Gamma^e(\mathbb{Z}_n))$ is connected or disconnected according as $n = 2^k$ or $n = 2^m.p^t$, where p is an odd prime and $t, k \in \mathbb{Z}^+$.*

Proof. The proof follows from Lemmas 2.1 and 2.3.

Theorem 2.1. *Let $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_k^{n_k}$, where $k \geq 3$, p_i's are distinct primes for all $1 \leq i \leq k$, and n_j, $1 \leq j \leq k$, are positive integers. Let $V\big(T(\Gamma^e(\mathbb{Z}_n))\big) = Z^*(\mathbb{Z}_n)$. Then the graph $T(\Gamma^e(\mathbb{Z}_n))$ is connected.*

Proof. Let $G = T(\Gamma^e(\mathbb{Z}_n))$. Consider the following subsets of the vertex set $V(G)$:

$$V_i^j = \big\{v \in V(G) : p_i^j + v \in Z^*(R),\ \ 1 \leq i \leq k,\ \ 1 \leq j \leq n_i\big\}.$$

Clearly, every element of the vertex set $V(G)$ will be present in at least one of the V_i^j, $1 \leq i \leq k$, $1 \leq j \leq n_i$, as defined above. Note that, by Lemma 2.2, the induced subgraph $G[V_i^j]$ on the vertex subsets V_i^j are connected.

Again, let for any $v \in V_i^j$ there exists at least one element $w \in V_l^m$, $i \neq l$ and $j \neq m$, such that from every V_j, $1 \leq j \leq n_k$, $j \neq t$, such that $v + w \in Z^*(\mathbb{Z}_n)$. This would imply that the graph $T(G)$ is connected.

3. Cliques in $T(\Gamma^e(R))$ associated to finite commutative rings. We start with the following facts.

Observation 3.1. *For a finite integral domain R, we observe that $\omega(T(\Gamma^e(R))) = 1$. Further, if in case R is a finite commutative ring, then $\omega(T(\Gamma^e(R))) = 2$ if and only if R is one of the following rings*:

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle},\ \mathbb{Z}_6 \qquad \textit{or} \qquad \mathbb{Z}_2 \times \mathbb{Z}_3,\ \frac{\mathbb{Z}_4[x,y]}{(2,y)^2},\ \mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{\langle f(x) \rangle},$$

where $f(x)$ is a reducible polynomial of degree two over $\mathbb{Z}_3$.

Observation 3.2. *Let $R \cong R_1 \times R_2 \times \ldots \times R_n$, $n \geq 2$, and $|R_i| \geq 2$, where R_i, $i = 1, 2, \ldots, n$, are local rings. Then $\omega(T(\Gamma^e(R))) \geq n + 1$, as we can choose a vertex subset*

$$V_1 = \big\{(0,0,0,\ldots,0),\ (1,0,0,\ldots,0),\ \ldots,\ (0,0,0,\ldots,1,0),\ (0,0,0,\ldots,0,1)\big\},$$

such that the induced subgraph by V_1 forms K_{n+1}, which implies that $\omega(T(\Gamma^e(R))) \geq n + 1$.

Now, we determine the finite commutative rings whose clique number is 3.

Theorem 3.1. *Let R be a finite commutative ring. Then $\omega(T(\Gamma^e(R))) = 3$ if and only if R is one of the following rings*:

$$\mathbb{Z}_8, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_5 \quad \textit{or} \quad \mathbb{Z}_{10}.$$

Proof. First assume that R is a finite nonlocal commutative ring. Then by the structure theorem, $R \cong R_1 \times R_2 \times \ldots \times R_n$, where R_i, $i = 1, 2, \ldots, n$, are local rings. We have the following cases to consider.

Case 1. For $n \geq 3$, we have $R \cong R_1 \times R_2 \times \ldots \times R_n$, where R_i, $i = 1, 2, \ldots, n$, are local rings. Then, by Observation 3.2, $\omega(T(\Gamma^e(R))) \geq 4$. Now, for $n = 2$, we get $R \cong R_1 \times R_2$, where R_i, $i = 1, 2$, are local rings. First assume that $|R_1| \geq 4$ and $|R_2| \geq 6$. Then the vertex subset $V_2 = \{(0,0), (0,y_1), (0,y_2), (0,y_3)\}$, where $y_i \in R_2$ for $i = 1, 2, 3$ and $y_i + y_j \neq |R_2|$, $i, j = 1, 2, 3$, when $R_2 \cong \mathbb{Z}_n$, $n \geq 6$, induces the subgraph K_4. Thus, $\omega(T(\Gamma^e(R))) \geq 4$. Next, when $|R_1| \leq 3$ and $|R_2| \geq 6$, then $R \cong \mathbb{Z}_2 \times R_2$ or $\mathbb{Z}_3 \times R_2$. Taking the similar vertex subset as above, we have $\omega(T(\Gamma^e(R))) \geq 4$. Finally, when $|R_1| \leq 3$ and $|R_2| \leq 5$, then R is one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$. Clearly, $\omega(T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_3))) = 2$ as seen in Observation 3.1. The graphs for rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ are shown in Fig. 2.

Clearly, the graph of $T\left(\Gamma^e\left(\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}\right)\right)$ contains K_4 as a maximal complete subgraph on the vertex subset $\{(0,0), (0,1), (0,x), (0,1+x)\}$, implies that $\omega\left(T(\Gamma)^e\left(\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}\right)\right) = 4$. Now, $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_4))$ contains K_3 as a maximal complete subgraph, implies that $\omega(T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 3$ as required. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $\omega(T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 3$. The graph associated to the ring $\mathbb{Z}_2 \times \mathbb{Z}_5$ is shown in Fig. 3.

Clearly, $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_5))$ contains K_3 as a maximal complete subgraph and hence $\omega(T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 3$.

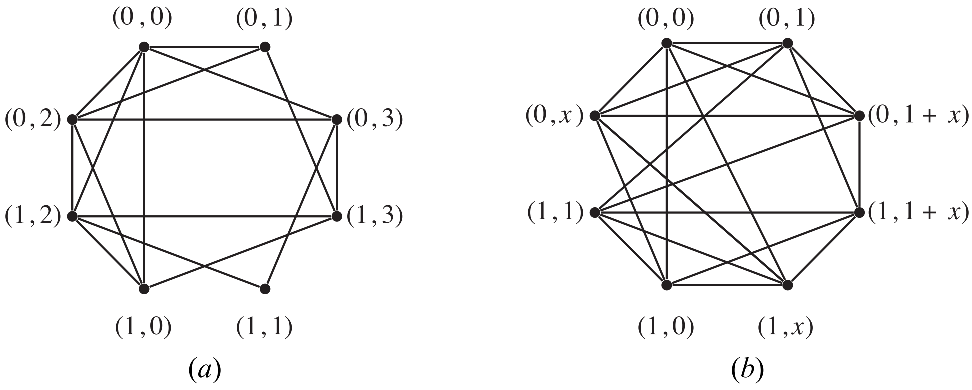

Fig. 2. The extended total graphs of $\mathbb{Z}_2 \times \mathbb{Z}_4$ (*a*) and $\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$ (*b*).

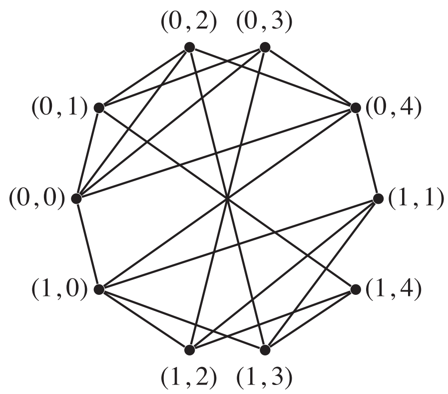

Fig. 3. The extended total graph of $\mathbb{Z}_2 \times \mathbb{Z}_5$.

Case 2. Let R be nonlocal. Then $|R| = p^n$, where p is prime and n is a nonnegative integer. First assume that R is a field. Then $\omega(T(\Gamma^e(R)) = 1$. Now, let R be not a field. First, let $p \geq 3$ and $n \geq 2$. Then $|Z(R)| = p^r$, where $r < n$ as $Z(R)$ is an ideal of R. Then the minimum cardinality of R is 9, for which $\omega(T(\Gamma^e(R)) = 2$. For $|R| > 9$, there exist at least three nonzero zero-divisors say $\alpha, \beta, \gamma \in Z^*(R)$ such that $\alpha + \beta,\ \alpha + \gamma,\ \beta + \gamma \in Z^*(R)$. Now, choose a vertex subset $V_3 = \{0, \alpha, \beta, \gamma\}$, the induced subgraph of V_3 forms $K_4 \subseteq T(\Gamma^e(R))$, which implies that $\omega(T(\Gamma^e(R)) \geq 4$. Now, for $p = 2$ and $n \geq 3$, we get $|R| \geq 8$. First, let $|R| > 8$. Then $2\mu \in Z^*(R), \mu \in R$ and choose the subset $V_4 = \{0, 2\mu_1, 2\mu_2, 2\mu_3\}$. Then the induced subgraph by V_4 is $K_4 \subseteq T(\Gamma^e(R))$, which implies that $\omega(T(\Gamma)^e(R)) \geq 4$. Now, for $|R| = 8$, we have the following rings [16]: $\mathbb{Z}_8$, $\dfrac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\dfrac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$, $\dfrac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\dfrac{\mathbb{Z}_4[x,y]}{(2,y)^2}$. For $R \cong \dfrac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\dfrac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$, $\dfrac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, choose subsets as $\{0, x, x^2, x+x^2\}$, $\{0, 2, x, 2{+}x\}$, $\{0, x, y, x{+}y\} \subseteq Z(R)$, respectively. The induced subgraphs formed by these subsets form $K_4 \subseteq T(\Gamma^e(R))$, which implies that $\omega(T(\Gamma^e(R))) \geq 4$. Also, $\omega\left(T\left(\Gamma^e\left(\dfrac{\mathbb{Z}_4[x,y]}{(2,y)^2}\right)\right)\right) = 2$, whereas for the graph $T(\Gamma^e(\mathbb{Z}_8)$, $\omega(T(\Gamma^e(\mathbb{Z}_8)) = 2$, as shown in Fig. 4.

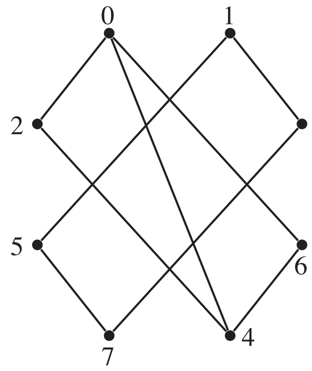

Fig. 4. The extended total graph of $\mathbb{Z}_8$.

Finally, if $p = 2$ and $n = 2$, then $|R| = 4$. In this case, $R \cong \mathbb{Z}_4$ or $\dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ and their associated graphs have clique number 2. Now, we characterize the finite commutative nonlocal rings whose clique is 4.

Theorem 3.2. *Let R be a finite nonlocal commutative ring. Then $\omega(T(\Gamma^e(R)) = 4$ if and only if R is one of the following rings*:

$$\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}, \quad \mathbb{Z}_2 \times \mathbb{F}_4, \quad \mathbb{Z}_{12}, \quad \mathbb{Z}_3 \times \mathbb{Z}_4, \quad \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}, \quad \mathbb{Z}_3 \times \mathbb{F}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_7, \quad \mathbb{Z}_{14},$$

$$\mathbb{Z}_3 \times \mathbb{Z}_5, \quad \mathbb{Z}_{15}, \; \mathbb{Z}_3 \times \mathbb{Z}_7, \quad \mathbb{Z}_{21}, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

Proof. Let R be a finite nonlocal ring. Then $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $R_i,\ i = 1, 2, \ldots, n$, are local rings. First, let $n \geq 4$. Then, by Observation 3.2, $\omega(T(\Gamma^e(R)) \geq 5$. Next, let $n \leq 3$. The following cases arise.

Case 1. Let $|R_i| \geq 4$ for each $i = 1, 2, 3$. If $|R_i| = 4$ for each $i = 1, 2, 3$, then we choose a vertex subset $V_5 = \{(0, 0, 0), (\alpha, 1, \alpha), (\alpha, \alpha, \alpha), (\alpha, \alpha, 1), (0, \alpha, \alpha)\}$, where $\alpha \in Z^*(R_i),\ i = 1, 2, 3$. Then the induced subgraph of V_5 forms $K_5 \subseteq T(\Gamma^e(R))$, implying that $\omega(T(\Gamma^e(R)) \geq 5$. If $|R_i| \geq 5$, $i = 1, 2, 3$, then choose the vertex subset $V_6 = \{(0, 0, 0), (0, 1, 1), (0, 1, \alpha), (0, 1, \beta), (0, 1, \gamma)\}$, where $\alpha, \beta, \gamma \in R_3$. A similar vertex subset will be taken if one or two of the rings have cardinality less than or equal to 4. Therefore, the induced subgraph of V_6 forms K_5 and hence $\omega(T(\Gamma^e(R)) \geq 5$.

Case 2. Let $n = 3$ and $|R_i| \leq 3,\ i = 1, 2, 3$. Then R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. The graph for the ring $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ contains one complete subgraph on five vertices with vertex subset $V_7 = \{(1, 1, 0), (0, 2, 2), (1, 0, 1), (1, 1, 2), (1, 1, 1)\}$. This implies that $\omega(T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) \geq 5$. Similarly, for the ring $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, we can choose a vertex subset on five vertices say $V_8 = \{(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 0, 2), (2, 0, 2)\}$ and the induced subgraph formed by V_8 is K_5, which gives $\omega\big(T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3))\big) \geq 5$. The graphs for rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ are shown in Fig. 5.

Clearly, from the graph in Fig. 5(*a*), we see that $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ contains K_4 as a maximal complete subgraph on the vertex subset $V_9 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, whereas the remaining vertices are adjacent to all the vertices in V_9 except one vertex. So the induced subgraph by V_9 forms K_4, which implies that $\omega(T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 4$ as desired. Similarly, for the graph in Fig. 5(*b*), $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3))$ contains K_4 as a maximal complete subgraph on the vertex subset $V_{10} = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. As none of the remaining eight vertices are adjacent to

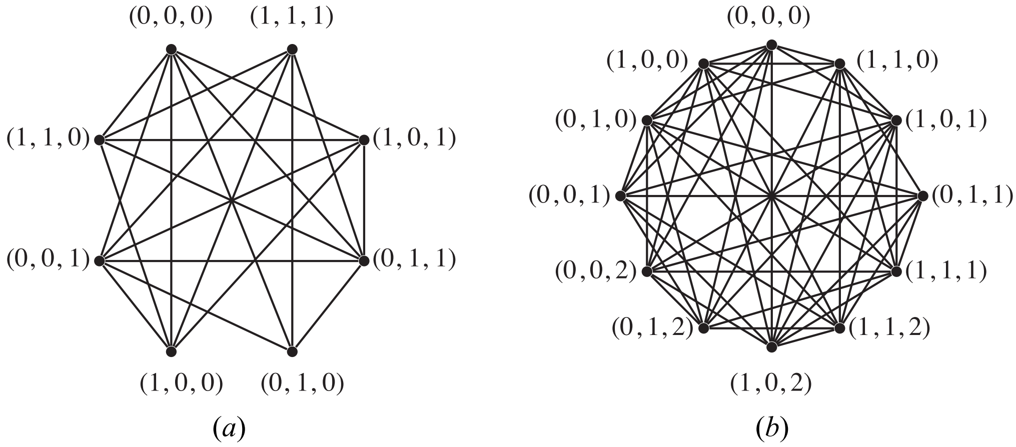

Fig. 5. The extended total graphs $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ (*a*) and $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3))$ (*b*).

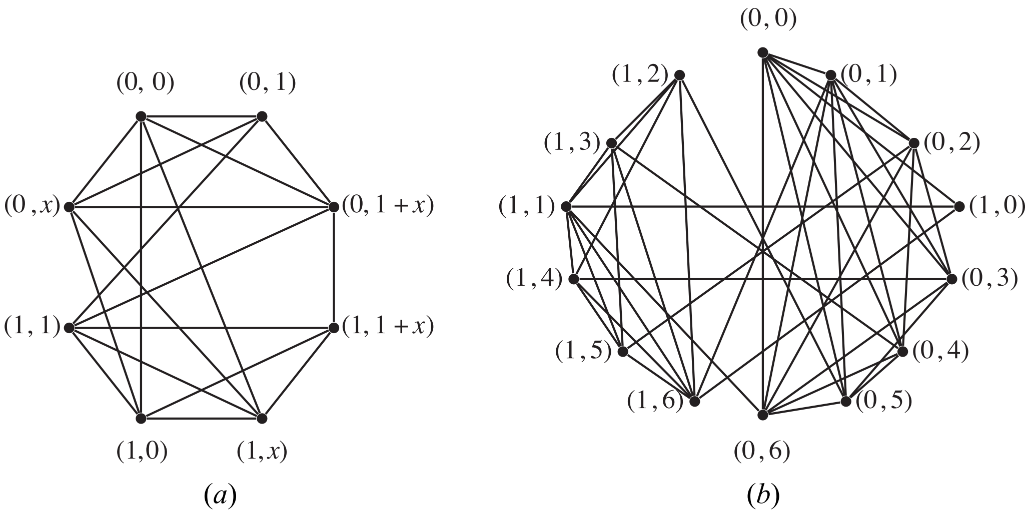

Fig. 6. The extended total graphs $T\left(\Gamma^e\left(\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\right)\right)$ (*a*) and $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_7))$ (*b*).

any one of the vertices in V_{10} and do not form a complete subgraph with five or more than five vertices, so the induced subgraph by V_{10} is K_4. Hence, $\omega(T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = 4$.

Case 3. If $n = 2$, then $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings. First, let $|R_1| \geq 4$ and $|R_2| \geq 8$. Construct a vertex subset as $V_{11} = \{(0,0),(0,1),(0,y_1),(0,y_2),(0,y_3)\}$, $y_i \in R_2$, $i = 1,2,3$, where $y_i + y_j \neq |R_2|$, $i,j = 1,2,3$, when $R_2 \cong \mathbb{Z}_n$, $n \geq 8$. Then the induced subgraph formed by V_{11} is K_5, implies that $\omega(T(\Gamma^e(R)) \geq 5$. If $|R_1| \leq 3$ and $|R_2| \leq 7$, then R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_7$. The rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ are discussed in Observation 3.1 and the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$ are discussed in Theorem 3.1. The graphs for rings $\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$ and $\mathbb{Z}_2 \times \mathbb{Z}_7$ are shown in Fig. 6.

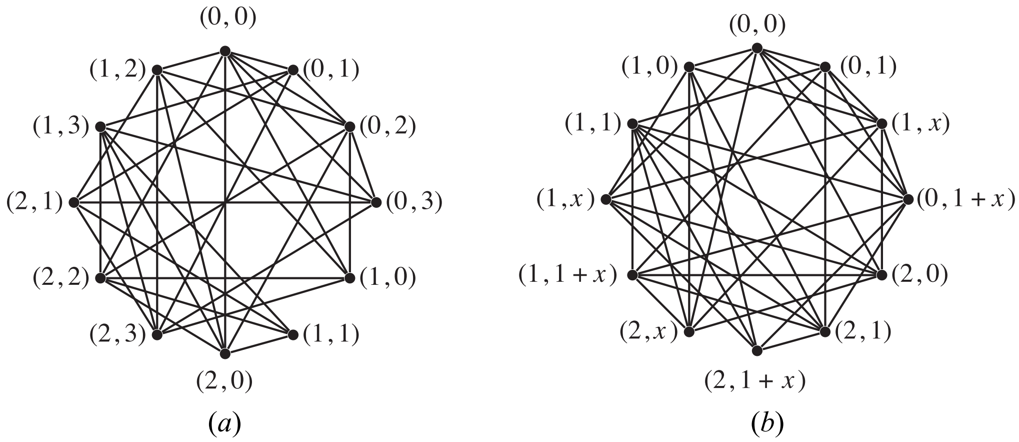

Fig. 7. The extended total graphs $T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_4))$ (a) and $T\left(\Gamma^e\left(\mathbb{Z}_3 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}\right)\right)$ (b).

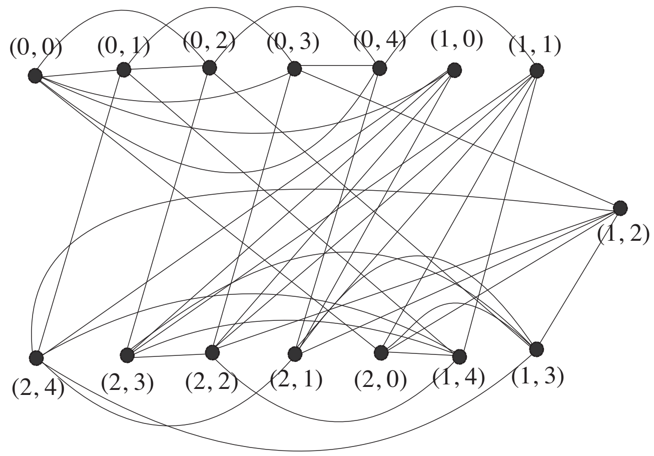

Fig. 8. $T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_5))$.

Clearly, $T\left(\Gamma^e\left(\mathbb{Z}_2 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}\right)\right)$ contains two complete subgraphs on maximal vertex subsets $V_{12} = \{(0,0),(0,1),(0,x),(0,1+x)\}$ and $V_{12} = \{(1,0),(1,x),(1,1),(1,1+x)\}$ and both form K_4. So $\omega(T(\Gamma^e(R)) = 4$. Similarly, $T(\Gamma^e(\mathbb{Z}_2 \times \mathbb{Z}_7))$ contains two different maximal vertex subsets $V_{13} = \{(0,0),(0,1),(0,2),(0,3)\}$ and $V_{13} = \{(1,1),(1,2),(1,0),(1,3)\}$ and both form K_4. So $\omega(T(\Gamma^e(R)) = 4$. The graphs for rings $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ are shown in Fig. 7.

Clearly, in Fig. 7(a), the graph has maximal complete subgraph on the vertex subset $V_{14} = \{(0,0),(2,0),(1,2),(0,2)\}$. This implies that $K_4 \subseteq T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_4))$, which gives $\omega(T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_4)) = 4$. Also, in Fig. 7($b$), the graph has a maximal complete subgraph on the vertex subset $V_{15} = \{(0,0),(0,1),(0,x),(0,1+x)\}$. This implies that $K_4 \subseteq T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_4))$, which gives $\omega\left(T(\Gamma^e\left(\mathbb{Z}_3 \times \dfrac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}\right)\right) = 4$. Also, the graph associated to the ring $\mathbb{Z}_3 \times \mathbb{F}_4$ contains a maximal complete subgraph K_4 on the vertex subset $V_{16} = \{(0,0),(0,1),(0,x),(0,1+x)\}$, which implies that $\omega\big(T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{F}_4))\big) = 4$. Finally, the graphs associated to rings $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_3 \times \mathbb{Z}_7$ are shown in Figs. 8 and 9, respectively.

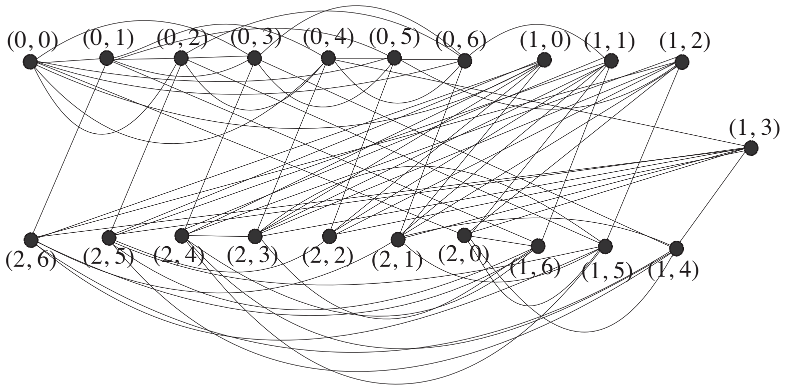

Fig. 9. $T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_7))$.

Clearly, from Fig. 8, the subgraph on the vertex subset $V_{17} = \{(2,2),(1,1),(2,3),(1,4)\}$ forms a complete subgraph $K_4 \subseteq T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_5))$ and there does not exist any other maximal complete subgraph larger than induced by V_{17}. This implies that $\omega(T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_7)) = 4$.

From Fig. 9, the subgraph on the vertex subset $V_{18} = \{(0,0),(0,1),(0,2),(0,3)\}$ forms a complete subgraph $K_4 \subseteq T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{F}_7))$, which implies that $\omega(T(\Gamma^e(\mathbb{Z}_3 \times \mathbb{Z}_7)) = 4$.

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