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Alper Erdem, Orhan Dişkaya[1]**, Hamza Menken** (Department of Mathematics, Mersin University, Turkey)

ON THE F-BERNSTEIN POLYNOMIALS

ПРО F-ПОЛІНОМИ БЕРНШТЕЙНА

We construct a new Bernstein operator, which is called the F-Bernstein operator obtained by using the F-factorial (Fibonacci factorial) and the Fibonomial (Fibonacci binomial). Then we examine the F-Bernstein basis polynomials and some of their properties. Moreover, we acquire certain connection between the F-Bernstein polynomials and the Fibonacci numbers.

Побудовано новий оператор Бернштейна, що називається F-оператором Бернштейна. Цей оператор отримано за допомогою F-факторіала (факторіала Фібоначчі) та фібонома (бінома Фібоначчі). Крім того, розглянуто базисні F-поліноми Бернштейна та деякі їхні властивості. Навіть більше, отримано певний зв'язок між F-поліномами Бернштейна та числами Фібоначчі.

1. Introduction. Sergey Bernstein defined a polynomial in the form of a sum to prove the Weierstrass approximation theorem which laid the foundation of the approximation theory in 1912.

The Bernstein polynomials have useful applications in various areas namely probability theory, numerical analysis, approximation theory and special functions theory. Many generalizations of the Bernstein polynomials have been considered in [13, 17]. One of them is the q-analogue version of the Bernstein polynomials. George M. Phillips defined the q-Bernstein polynomials grounded on the q-analysis in [18]. Many studies have been done on the q-Bernstein polynomials (see [5, 11, 12, 14]). Various studies on (p, q)-Bernstein polynomials can be found in [2, 10].

In present paper, inspired by the Fibonacci calculus introduced by E. Krot [8] and related works Ozvatan [15] and Pashaev [16], we define the F-Bernstein polynomials, then we examine some properties and relations with the Fibonacci numbers.

2. Preliminaries. For the reader's convenience, we give a summary of mathematical notations and some basis.

The Fibonacci sequence which is denoted by $\{F_n\}$ is given by the reccurence relation

$$F_n = \begin{cases} 0, & \text{if} \quad n = 0, \\ 1, & \text{if} \quad n = 1, \\ F_{n-1} + F_{n-2}, & \text{if} \quad n > 1. \end{cases}$$

The Fibonacci numbers are the famousest among all integer sequences, as they have surprising properties (see [3, 4, 6, 7]).

The Fibonacci calculus is based on combinatorial interpretation of the Fibonacci numbers. The F-factorial is definied as $F_n! = F_n F_{n-1} \ldots F_2 F_1$ with $F_0! = 1$.

The Fibonomial coefficients are definied for $n \geq m \geq 0$ as

$$\binom{n}{m}_F = \frac{F_n!}{F_{n-m}! F_m!}$$

[1] Corresponding author, e-mail: orhandiskaya@mersin.edu.tr.

with $\binom{n}{0}_F = 1$ and $\binom{n}{m}_F = 0$ for $n < m$. It is suprisingly bizarre that all Fibonomial coefficients are always integers. This fact can be verifed via mathematical induction among the formula

$$F_n = F_m F_{n-m-1} + F_{m+1} F_{n-m}.$$

The Fibonomial coefficients have the following properties:

$$\binom{n}{r}_F = \binom{n}{n-r}_F, \qquad \binom{n}{r}_F \binom{r}{v}_F = \binom{n}{v}_F \binom{n-v}{r-v}_F,$$

$$\binom{n}{r}_F \binom{n-r}{v}_F = \binom{n}{r+v}_F \binom{r+v}{r}_F, \qquad \binom{n}{r}_F = \frac{F_{n-r+1}}{F_r} \binom{n}{r-1}_F \quad (r \neq 0),$$

$$\binom{n}{r}_F = \frac{F_n}{F_{n-r}} \binom{n-1}{r}_F \quad (r \neq n), \qquad \binom{n}{r}_F = F_{r-1} \binom{n-1}{r}_F + F_{n-r} \binom{n-1}{r-1}_F.$$

The Fibonomial triangle with general the term $\binom{n}{k}_F$ has interesting properties. By using this triangle, we have shown the Fibonomial matrix

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 6 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 5 & 15 & 15 & 5 & 1 & 0 & \ldots \\ 1 & 8 & 40 & 60 & 40 & 8 & 1 & \ldots \\ 1 & 13 & 104 & 260 & 260 & 104 & 13 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $\{a_n^{(j)}\}$ is the sequence of the numbers which are placed through the pth column in the Fibonomial matrix. The first few columns of the Fibonomial matrix are given:

$$\{a_n^{(1)}\} = (1, 1, 1, 1, 1, \ldots),$$

$$\{a_n^{(2)}\} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots),$$

$$\{a_n^{(3)}\} = (0, 0, 1, 2, 6, 15, 40, 104, \ldots),$$

$$\{a_n^{(4)}\} = (0, 0, 0, 1, 3, 15, 60, 260, \ldots).$$

It is easly seen that the general term of $\{a_n^{(j)}\}$ can expressed by

$$a_n^{(j)} = \begin{cases} 0, & \text{if} \quad n \leq j-1, \\ \dfrac{F_n F_{n+1} \ldots F_{n+j-2}}{F_{j-1}!}, & \text{if} \quad n > j-1, \end{cases} \tag{2.1}$$

where $j \geq 2$. Thus, we have the following results.

Proposition 2.1. *The sequence* $\{a_n^{(j)}\}$ *has a recursive relation as follows*:

$$a_{n+1}^{(j)} = \frac{F_{n+j-1}}{F_n} a_n^{(j)},$$

where $p \geq 1$

Proof. From (2.1), we have

$$\begin{aligned} a_{n+1}^{(j)} &= \frac{F_{n+1}F_{n+2}\ldots F_{n+j-2}F_{n+j-1}}{F_{j-1}!} \\ &= \frac{F_nF_{n+1}F_{n+2}\ldots F_{n+j-2}F_{n+j-1}}{F_nF_{j-1}!} = \frac{F_{n+j-1}}{F_n} a_n^{(j)}. \end{aligned}$$

An operator $L(f;t)$ is called linear operator if any the functions $f(x)$ and $g(x)$ which are in its domain, the function $af(x)+bg(x)$ belongs its domain and

$$L(\sigma f + \omega g; t) = \sigma L(f;t) + \omega L(g;t),$$

where σ and ω are constants.

Let $\mathcal{P}$ represent the polynomial algebra over the field **K** of characteristic zero. So, the linear operator defined by $\partial_F : \mathcal{P} \to \mathcal{P}$ is the F-derivative operator:

$$\partial_F t^m = F_m t^{m-1} \tag{2.2}$$

for $m \geq 0$.

The binomial theorem for the F-analogue is given by

$$(x +_F y)^\sigma = \sum_{0 \leq m \leq \sigma} \binom{\sigma}{m}_F x^{\sigma-m} y^m \tag{2.3}$$

and the F-exponential function e_F^μ is defined by

$$e_F^\mu = \sum_{0 \leq n} \frac{\mu^n}{F_n!} \tag{2.4}$$

(for details see [8, 9]).

3. *F*-Bernstein basis polynomials and some properties.

Definition 3.1. *Let* $\sigma, \omega \in \mathbb{R}$, $r, n \in \mathbb{Z}^+$, *where* $r \leq n$. *Then the* n*th degree* F*-Bernstein basis polynomials defined by*

$$B_{r,n}^{F[\sigma,\omega]}(t) = \binom{n}{r}_F \frac{(t-\sigma)^r (\omega - t)^{n-r}}{(\omega - \sigma)^n}$$

for $r = 0, 1, \ldots, n$.

If we take $\sigma = 0$ and $\omega = 1$, then

$$B_{r,n}^{F[0,1]}(t) = \binom{n}{r}_F t^r (1-t)^{n-r}, \tag{3.1}$$

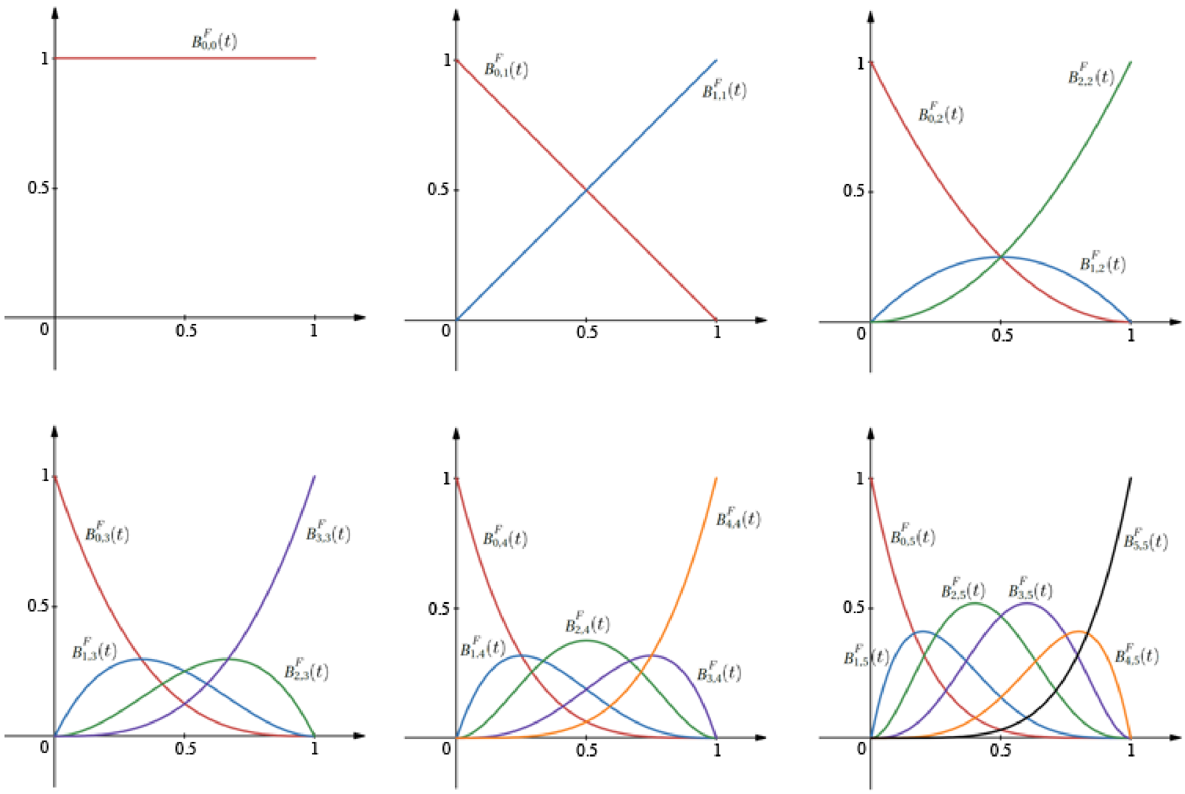

Fig. 1. Graphs of $B^F_{n,k}$ for $0 \le k \le n \le 5$.

where $r = 0, 1, \ldots, n$. For simplicity we write $B^F_{r,n}$ instead of $B^{F[0,1]}_{r,n}$.

The first few polynomials are listed and shown in graph (see Fig. 1):

$$B^F_{0,0}(t) = 1,$$

$$B^F_{0,1}(t) = 1 - t, \quad B_{1,1}(t) = t,$$

$$B^F_{0,2}(t) = 1 - 2t + t^2, \quad B^F_{1,2}(t) = t - t^2, \quad B^F_{2,2}(t) = t^2,$$

$$B^F_{0,3}(t) = 1 - 3t + 3t^2 - t^3, \quad B^F_{1,3}(t) = 2t - 4t^2 + 2t^3, \quad B^F_{2,3}(t) = 2t^2 - 2t^3, \quad B^F_{3,3}(t) = t^3,$$

$$B^F_{0,4}(t) = 1 - 4t + 6t^2 - 4t^3 + t^4, \quad B^F_{1,4}(t) = 3t - 9t^2 + 9t^3 - 3t^4,$$

$$B^F_{2,4}(t) = 6t^2 - 12t^3 + 6t^4, \quad B^F_{3,4}(t) = 3t^3 - 3t^4, \quad B^F_{4,4}(t) = t^4,$$

$$B^F_{0,5}(t) = 1 - 5t + 10t^2 - 10t^3 + 5t^4 - t^5, \quad B^F_{1,5}(t) = 5t - 20t^2 + 30t^3 - 20t^4 + 5t^5,$$

$$B^F_{2,5}(t) = 15t^2 - 45t^3 + 45t^4 - 15t^5, \quad B^F_{3,5}(t) = 15t^3 - 30t^4 + 15t^5,$$

$$B^F_{4,5}(t) = 5t^4 - 5t^5, \quad B^F_{5,5}(t) = t^5.$$

Proposition 3.1. *The F-Bernstein basis polynomials are symmetric.*

Proof. From (3.1), we have

$$B_{r,n}^F(t) = \binom{n}{r}_F t^r (1-t)^{n-r} = \binom{n}{n-r}_F (1-t)^{n-r} t^r = B_{n,n-r}^F(1-t).$$

Proposition 3.2. *The F-Bernstein basis polynomials have the following recursive relation*:

$$B_{r,n}^F(t) = \frac{F_{n-r+1}}{F_r}\; B_{r-1,n}^F(t)\, \frac{t}{1-t},$$

where $t \in [0,1)$.

Proof. For the proof, we use (3.1). Then

$$\begin{aligned} B_{r,n}^F(t) &= \binom{n}{r}_F t^r (1-t)^{n-r} \\ &= \frac{F_{n-r+1}}{F_r} \binom{n}{r-1}_F t^r (1-t)^{n-r} \\ &= \frac{F_{n-r+1}}{F_r} \binom{n}{r-1}_F t^{r-1} (1-t)^{n-(r-1)} \frac{t}{1-t} \\ &= \frac{F_{n-r+1}}{F_r} \frac{t}{1-t} B_{r-1,n}^F(t). \end{aligned}$$

Theorem 3.1. *Any nth degree F-Bernstein basis polynomial can be expressed using the power basis which is expressed by* $\{1, t, t^2, \ldots\}$, *i.e.,*

$$B_{r,n}^F(t) = \sum_{r \le i \le n} \binom{n}{r}_F \binom{n-k}{i-r} (-1)^{i-r} t^i.$$

Proof. By virtue of the definition of the F-Bernstein basis polynomials, we get

$$\begin{aligned} B_{r,n}^F(t) &= \binom{n}{r}_F t^r (1-t)^{n-r} \\ &= \binom{n}{r}_F t^r \sum_{0 \le i \le n-r} (-1)^i \binom{n-r}{i} t^i \\ &= \sum_{0 \le i \le n-r} \binom{n}{r}_F \binom{n-r}{i} (-1)^i t^{i+r} \\ &= \sum_{r \le i \le n} \binom{n}{r}_F \binom{n-r}{i-r} (-1)^{i-r} t^i. \end{aligned}$$

Theorem 3.2. *The nth degree F-Bernstein basis polynomial can be expressed by combining two F-Bernstein basis polynomials with the degree $n-1$ such as*

$$B_{r,n}^F(t) = (1-t)\, F_{r-1}\, B_{r,n-1}^F(t) + t\, F_{n-r}\, B_{r-1,n-1}^F(t),$$

where $r = 0, 1, \ldots, n$ and $t \in [0,1]$.

Proof. From (3.1) and properties of the Fibonomials it follows that

$$\begin{aligned}
B^F_{r,n}(t) &= \binom{n}{r}_F t^r(1-t)^{n-r} \\
&= \left[F_{r-1}\binom{n-1}{r}_F + F_{n-r}\binom{n-1}{r-1}_F\right] t^r(1-t)^{n-r} \\
&= F_{r-1}\binom{n-1}{r}_F t^r(1-t)^{n-r} + F_{n-r}\binom{n-1}{r-1}_F t^r(1-t)^{n-r} \\
&= (1-t)F_{r-1}\binom{n-1}{r}_F t^r(1-t)^{n-r-1} + tF_{n-r}\binom{n-1}{r-1}_F t^{r-1}(1-t)^{n-r} \\
&= (1-t)\, F_{r-1}\, B^F_{r,n-1}(t) + t\, F_{n-r}\, B^F_{r-1,n-1}(t).
\end{aligned}$$

Theorem 3.3. *Any of lower-degree F-Bernstein basis polynomials of degree less then n can be expressed as a linear combination of nth degree F-Bernstein basis polynomials such as*

$$B^F_{r,n-1}(t) = \frac{F_{n-r}}{F_n} B^F_{r,n}(t) + \frac{F_{r+1}}{F_n} B^F_{r+1,n}(t).$$

Proof. With smooth calculation

$$\begin{aligned}
\frac{F_{n-r}}{F_n} B^F_{r,n}(x) &= \frac{F_{n-r}}{F_n} \sum_{0\le r\le n} \binom{n}{r}_F t^r(1-t)^{n-r} \\
&= \sum_{0\le r\le n} \binom{n-1}{r}_F t^r(1-t)^{n-r} \\
&= (1-t)B^F_{r,n-1}(t)
\end{aligned}$$

and

$$\begin{aligned}
\frac{F_{r+1}}{F_n} B^F_{r+1,n}(t) &= \frac{F_{r+1}}{F_n} \sum_{0\le r\le n} \binom{n}{r+1}_F t^{r+1}(1-t)^{n-r-1} \\
&= \sum_{0\le r\le n} \binom{n-1}{r}_F t^{r+2}(1-t)^{n-r-1} \\
&= xB^F_{r,n-1}(t).
\end{aligned}$$

Thus,

$$B^F_{r,n-1}(t) = \frac{F_{n-r}}{F_n} B^F_{r,n}(t) + \frac{F_{r+1}}{F_n} B^F_{r+1,n}(t),$$

which express a $n-1$ degree F-Bernstein basis polynomial regarding the linear combination of F-Bernstein polynomials with degree n.

Theorem 3.4. *The polynomials with degree $n-1$ are the F-derivative of the F-Bernstein basis polynomials with degree n. In addition, this derivative can be explicit as a linear combination of the F-Bernstein basis polynomials such as*

$$\partial_F B^F_{r,n}(t) = F_n\big(B^F_{r-1,n-1}(t) - B^F_{r,n-1}(t)\big).$$

Proof. By virtue of the definition of the F-derivative (2.2), we get

$$\begin{aligned}
\partial_F B^F_{r,n}(t) &= \partial_F\left[\binom{n}{r}_F t^r(1-t)^{n-r}\right]\\
&= \frac{F_n!}{F_{n-r}!F_r!}\big[F_r t^{r-1}(1-t)^{n-r} - F_{n-r}t^r(1-t)^{n-r-1}\big]\\
&= \frac{F_n!}{F_{n-r}!F_r!}F_r t^{r-1}(1-t)^{n-r} - \frac{F_n!}{F_{n-r}!F_r!}F_{n-r}t^r(1-t)^{n-r-1}\\
&= F_n\left(\frac{F_{n-1}!}{F_{n-r}!F_{r-1}!}t^{r-1}(1-t)^{n-r} - \frac{F_{n-1}!}{F_{n-r-1}!F_r!}t^r(1-t)^{n-r-1}\right)\\
&= F_n\big(B^F_{r-1,n-1}(t) - B^F_{r,n-1}(t)\big).
\end{aligned}$$

Theorem 3.5. *The generating function for the F-Bernstein polynomials is*

$$\sum_{0\le r\le n} B^F_{r,n}(t)\mu^r = ((1-x) +_F \mu x)^n.$$

Proof. By utilizing (2.3), we obtain

$$((1-t) +_F \mu t)^n = \sum_{0\le r\le n}\binom{n}{r}_F(\mu t)^r(1-t)^{n-r} = \sum_{0\le r\le n}\binom{n}{r}_F t^r(1-t)^{n-r}\mu^r = \sum_{0\le r\le n} B^F_{r,n}(t)\mu^r.$$

The following theorem has proved by a similar method which was given in [1].

Theorem 3.6. *The exponential generating function for the F-Bernstein polynomials is*

$$\sum_{r\le n} B^F_{r,n}(t)\frac{\mu^n}{F_n!} = \frac{t^r\mu^r}{F_r!}e_F^{(1-t)\mu}.$$

Proof. By using the F-exponential function e^t_F (2.4), we have

$$\begin{aligned}
\frac{t^r\mu^r}{F_r!}e_F^{(1-t)_F\mu} &= \frac{t^r\mu^r}{F_r!}\sum_{0\le n}\frac{(1-t)^n\mu^n}{F_n!}\\
&= \frac{1}{F_r!}\sum_{0\le n}\frac{t^r(1-t)^n\mu^{n+r}}{F_n!}\\
&= \sum_{0\le n}\frac{F_{n+r}!}{F_n!F_r!}\frac{t^r(1-t)^n\mu^{n+r}}{F_{n+r}!}
\end{aligned}$$

$$= \sum_{0\le n} \binom{n+r}{r} \frac{t^r(1-t)^n \mu^{n+r}}{F_{n+r}!}$$

$$= \sum_{r\le n} \binom{n}{r}_F \frac{t^r(1-t)^{n-r} \mu^{n}}{F_{n}!}$$

$$= \sum_{r\le n} B^F_{r,n}(t) \frac{\mu^n}{F_n!}.$$

4. F-Bernstein polynomials and some properties related with the Fibonacci numbers .

Definition 4.1. *Let $\sigma, \omega \in \mathbb{R}$, $f\colon [\sigma,\omega] \to \mathbb{R}$ a function, $r, n \in \mathbb{Z}^+$ and $r \le n$. Then the nth degree F-Bernstein polynomial with respect to the function f is defined by*

$$\mathfrak{B}_n^{F[\sigma,\omega]}(f;t) = \sum_{0\le r\le n} f\left(\frac{F_r}{F_n}\right) B^{F[\sigma,\omega]}_{r,n}(t)$$

for $r = 0, 1, \ldots, n$.

If we take $\sigma = 0$ and $\omega = 1$, then

$$\mathfrak{B}_n^{F[0,1]}(f;t) = \sum_{r=0}^{n} f\left(\frac{F_r}{F_n}\right) B^{F[0,1]}_{r,n}(t).$$

For simplicity we write $\mathfrak{B}_n^F$ instead of $\mathfrak{B}_n^{F[0,1]}$.

The first few F-Bernstein polynomials with respect to the functions $f(x) = 1$ and $f(x) = x$ are listed with below:

$$\begin{array}{ll}
\mathfrak{B}_0^F(1;t) = 1, & \mathfrak{B}_0^F(x;t) = t, \\
\mathfrak{B}_1^F(1;t) = 1, & \mathfrak{B}_1^F(x;t) = t, \\
\mathfrak{B}_2^F(1;t) = 1 - t + t^2, & \mathfrak{B}_2^F(x;t) = t, \\
\mathfrak{B}_3^F(1;t) = 1 - t + t^2, & \mathfrak{B}_3^F(x;t) = t - t^2 + t^3, \\
\mathfrak{B}_4^F(1;t) = 1 - t + 3t^2 - 4t^3 + 2t^4, & \mathfrak{B}_4^F(x;t) = t - t^2 + t^3, \\
\mathfrak{B}_5^F(1;t) = 1 + 5t^2 - 10t^3 + 5t^4, & \mathfrak{B}_5^F(x;t) = t - t^2 + 3t^3 - 4t^4 + 2t^5, \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

Proposition 4.1. *The F-Bernstein polynomials with respect to $f(x) = 1$ and $f(x) = t$ has the realation*

$$\mathfrak{B}_n^F(x;t) = t \cdot \mathfrak{B}_{n-1}^F(1;t).$$

Proof. With smooth calculation, we have

$$\mathfrak{B}_n^F(x;t) = \sum_{0\le r\le n} \frac{F_r}{F_n} \binom{n}{r}_F t^r(1-t)^{n-r}$$

$$= \sum_{0\leq r\leq n} \binom{n-1}{r-1}_F t^r (1-t)^{n-r}$$

$$= \sum_{-1\leq r\leq n-1} \binom{n-1}{r}_F t^{r+1} (1-t)^{n-1-r}$$

$$= t \sum_{0\leq r\leq n-1} \binom{n-1}{r}_F t^r (1-t)^{n-1-r}$$

$$= t \cdot \mathfrak{B}_{n-1}^F(1;t).$$

Theorem 4.1. *The F-Bernstein polynomials with respect to the function $f(x) = 1$ have following property*:

$$[t^\beta]\mathfrak{B}_n^F(1;t) = \sum_{0\leq j\leq\beta} (-1)^{\beta-j} \binom{n}{j}_F \binom{n-j}{n-\beta}, \tag{4.1}$$

where $[t^\beta]\mathfrak{B}_n^F(1;t)$ denotes the coefficient of t^β in $B_n^F(1;t)$.

Proof. By using the F-Bernstein polynomials to define the following, we obtain

$$\mathfrak{B}_n^F(1;t) = \sum_{0\leq r\leq n} \binom{n}{r}_F t^r (1-t)^{n-r}$$

$$= \sum_{0\leq r\leq n} \left[\binom{n}{r}_F t^r \sum_{0\leq i\leq n-r} \left(\binom{n-r}{i} (-1)^{n-r-i} t^{n-r-i} \right) \right]$$

$$= \sum_{0\leq r\leq n} \left[\binom{n}{r}_F \sum_{0\leq i\leq n-r} \left(\binom{n-r}{i} (-1)^{n-r-i} t^{n-i} \right) \right].$$

From here, we get

$$[t^\beta]\mathfrak{B}_n^F(1;t) = \binom{n}{0}_F \binom{n}{n-\beta} (-1)^\beta + \binom{n}{1}_F \binom{n-1}{n-\beta} (-1)^{\beta-1} + \ldots + \binom{n}{\beta}_F \binom{n-\beta}{n-\beta} (-1)^0$$

$$= \sum_{0\leq j\leq\beta} (-1)^{\beta-j} \binom{n}{j}_F \binom{n-j}{n-\beta},$$

which completes the proof.

***Corollary* 4.1.** *The F-Bernstein polynomial with respect to the function $f(x) = 1$ has following relation with respect to Fibonacci numbers such as*

$$[t]\mathfrak{B}_n^F(1;t) = F_n - n,$$

where $[t]\mathfrak{B}_n^F(1;t)$ denotes the coefficients of t in $B_n^F(1;t)$.

Proof. From equality (4.1) for $\beta = 1$, we have

$$[t]\mathfrak{B}_n^F(1;t) = F_n - n.$$

Theorem 4.2. *For the F-Bernstein polynomials with respect to $f(x) = x^\beta$ where $\beta \geq 2$ has following relation with respect to Fibonacci numbers*:

$$[t]\mathfrak{B}_n^F(x^\beta; t) = \frac{1}{(F_n)^{\beta-1}},$$

where $[t]\mathfrak{B}_n^F(x^\beta; t)$ denotes the coefficients of t in $B_n^F(x^\beta; t)$.

Proof. By virtue of the definition of the F-Bernstein polynomials, we get

$$\begin{aligned}
\mathfrak{B}_n^F(x^\beta; t) &= \sum_{0\leq r\leq n} \left(\frac{F_r}{F_n}\right)^\beta \binom{n}{r}_F t^r (1-t)^{n-r} \\
&= \sum_{0\leq r\leq n} \left[\left(\frac{F_r}{F_n}\right)^\beta \binom{n}{r}_F t^r \sum_{0\leq i\leq n-r} \binom{n-r}{i} (-t)^{n-r-i}\right] \\
&= \sum_{0\leq r\leq n} \left[\left(\frac{F_r}{F_n}\right)^\beta \binom{n}{r}_F \sum_{0\leq i\leq n-r} \binom{n-r}{i} (-1)^{n-r-i} t^{n-i}\right] \\
&= \sum_{0\leq r\leq n} \left[\left(\frac{F_r}{F_n}\right)^\beta \binom{n}{r}_F \left(\binom{n}{0}(-1)^{n-r}t^n + \binom{n-r}{1}(-1)^{n+r-1}t^{n-1} + \ldots + \binom{n-r}{n-r} t^r\right)\right].
\end{aligned}$$

From here, we obtain

$$\begin{aligned}
[t]\mathfrak{B}_n^F(x^\beta; t) &= \left(\frac{F_0}{F_n}\right)^\beta \binom{n}{0}_F \binom{n}{n-1}_F (-1) + \left(\frac{F_1}{F_n}\right)^\beta \binom{n}{1}_F \binom{n-1}{n-1} \\
&= \frac{1}{(F_n)^{\beta-1}}.
\end{aligned}$$

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