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M. Jorf[1], **L. Oukhtite** (Department of Mathematics, Faculty of Science and Technology, Sidi Mohamed Ben Abdellah University, Fez, Morocco)

ON CENTER GRAPHS OF FINITE ASSOCIATIVE RINGS

ПРО ЦЕНТРАЛЬНІ ГРАФИ СКІНЧЕННИХ АСОЦІАТИВНИХ КІЛЕЦЬ

We consider a finite associative ring R, which may have or may not have a unit element. We also examine its center denoted by $Z(R)$. Our main focus is on the introduction of two distinct graphs associated with R, namely, the center graph denoted by $GC(R)$ and the strict center graph denoted by $\overline{GC(R)}$.

We present the properties of $GC(R)$ and explore its implications on the nature of $Z(R)$. Specifically, we demonstrate that if $GC(R)$ is complete, then $Z(R)$ is an ideal in R. Furthermore, in the case where R is a unital ring, the completeness of $GC(R)$ leads to the conclusion that R is a commutative ring.

As a specific application of our results, we provide an explicit construction of the graph $\overline{GC}(T_2(p))$, where $T_2(p)$ represents the ring of upper-triangular matrices with entries in the ring $\mathbb{Z}/p\mathbb{Z}$ and p is a prime integer.

In our investigations of the center graph and strict center graph, we aim to shed light on the properties of finite associative rings and their centers, thus providing valuable insights and applications in the ring theory.

Розглянуто скінченне асоціативне кільце R, яке може мати або не мати одиничний елемент, та досліджено його центр, позначений як $Z(R)$. Основну увагу приділено введенню двох різних графів, пов'язаних з R, а саме центрального графа, позначеного як $GC(R)$, і строго центрального графа, позначеного як $\overline{GC(R)}$.

Наведено властивості $GC(R)$ і досліджено їхні наслідки для природи $Z(R)$. Зокрема, показано, що у випадку, коли $GC(R)$ є повним, $Z(R)$ є ідеалом в R. Навіть більше, у випадку, коли R є унітарним кільцем, повнота $GC(R)$ приводить до висновку, що R є комутативним кільцем.

Як конкретне застосування одержаних результатів наведено явну конструкцію графа $\overline{GC}(T_2(p))$, де $T_2(p)$ — кільце верхньотрикутних матриць з елементами, що належать кільцю $\mathbb{Z}/p\mathbb{Z}$, а p — просте ціле число.

Досліджуючи центральний граф і строгий центральний граф, ми прагнемо пролити світло на властивості скінченних асоціативних кілець та їхніх центрів і отримати цінні висновки і застосування в теорії кілець.

1. Introduction. Let R be a finite associative ring (not necessarily unital) and $Z(R)$ be its center. The study of finite rings is crucial in coding theory and cryptography [15, 16]. In 1988, Beck [9] created connections between graphs and finite rings. This idea was very beneficial for both ring theory and graph theory.

We use [10] and [17] for the standard terminology of simple and algebraic graphs, respectively. Let G be a finite simple graph. We denote the vertex set and the edge set of G, $V(G)$ and $E(G)$, respectively. If $a \in V(G)$, then the degree of a is denoted by $\deg(a)$. If a and b are two adjacent vertices of G, then we write $a \sim b$. A graph G in which any two distinct vertices are adjacent is said to be complete. If any two vertices a and b in G are connected by a path $a = a_0 \sim a_1 \sim \ldots \sim a_n = b$, then G is called a connected graph. A path is a cycle if $a = b$. The length of a path or a cycle is the number of distinct edges. A cycle of length n is denoted by C_n. The number of edges of the shortest walk joining a and b is called the distance between a and b and is denoted by $d(a;b)$. The maximum value of the distance function in a connected graph G is called the diameter of G and is denoted by $\mathrm{diam}(G)$.

Actually, graphs allow us to show finite ring's properties geometrically. For this reason, many graphs have been associated with finite rings. Especially, we have:

[1] Corresponding author, e-mail: mohamedjorf1995@gmail.com.

Zero-divisor graphs, introduced by Anderson and Livingston [8] in 1999. In these graphs, two vertices $x, y \in R$ are adjacent if and only if $xy = 0$. Many properties of finite rings have been found using zero-divisor graphs in several works [3, 5, 6].

Commuting graphs, introduced by Akbari et al. [1] in 2004. In this graph, two vertices $x, y \in R$ are adjacent if and only if $xy = yx$. These graphs were also the subject of many papers [4, 14].

Total graphs, introduced by Anderson and Badawi [7] in 2008. In these graphs, two vertices $x, y \in R$ are adjacent if and only if the sum $x + y$ is a zero-divisor in R. As well, they have been widely studied [2, 11].
Recently, new graphs associated with rings have been introduced, we refer, for example, to clean graphs [12] introduced in 2021.

In this paper, we investigate when $ab \in Z(R)$ for two elements $a, b \in R$. So, we introduce the center graph $GC(R)$ of R such that its set of vertices is R and two vertices a and b are adjacent if and only if either $ab \in Z(R)$ or $ba \in Z(R)$. We show that $GC(R)$ is always connected with $\mathrm{diam}(GC(R)) \leq 2$. Afterwards, we notice that 0 is adjacent with every vertex in $GC(R)$, then we remove it from $GC(R)$ to obtain a subgraph of $GC(R)$, that we call the strict center graph $\overline{GC}(R)$ of R, where its set of vertices is $R \setminus \{0\}$. The graphs $GC(R)$ and $\overline{GC}(R)$ allow us to get a graphic characterization of some properties of R, like the commutativity of R and the algebraic structure of $Z(R)$. As an application, we give the explicit form of $\overline{GC}(T_2(p)$, where $T_2(p)$ is the ring of upper-triangular matrices with entries in $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ for a prime integer p.

In the second section, we give some important preliminaries that we will need to prove our results. In the third section, we give and prove our main results.

2. Preliminaries. In this paper, we consider a finite associative ring, denoted as R, which may or may not have a unit element (identity). The center of this ring is represented by $Z(R)$, and it consists of elements that commute with all other elements in the ring. Specifically, for any $x \in R$, we have $xy = yx$ for all $y \in R$.

We investigate the concept of zero-divisors in R, which are elements that multiply with a nonzero element to produce zero. Formally, an element $a \in R$ is a zero-divisor if there exists a nonzero element $b \in R$ such that $ab = 0$ or $ba = 0$. The set of all zero-divisors in R is denoted as $D(R)$.

Furthermore, we define unit elements in R as elements that have a multiplicative inverse, i.e., there exists an element b such that $ab = ba = 1$. The set of all unit elements in R is denoted as $U(R)$.

In the context of rings with identity, a division ring is characterized by the property that every nonzero element is a unit, meaning it has a multiplicative inverse $(ax = xa = 1)$.

In addition, we introduce the Lie product of two elements x and y in R, defined as $[x, y] = xy - yx$. This product plays an important role in the analysis of certain ring properties.

Throughout the paper, we explore the properties and interactions of these elements and concepts in finite associative rings, aiming to deepen our understanding of ring theory and its applications.

Let us recall some preliminaries in ring theory.

Lemma 1 [13, p. 5, Exercise I.8]**.** *Let A be a finite commutative unital ring. Then*

$$A = U(A) \cup D(A).$$

For any element x of A, x is either a zero divisor or a unit.

By this lemma, we get immediately the following result.

***Corollary* 1.** *Let R be a finite noncommutative unital ring. Then we have*

$$Z(R) = U(Z(R)) \cup D(Z(R)).$$

In addition, to the center graph $GC(R)$, we introduce two more graphs associated with the ring R. These graphs have their vertex set as the elements of the ring R and the set of nonzero in R, denoted as R^*. In these graphs, two distinct vertices a and b are connected by an edge if and only if the following conditions are met: (1) a is not equal to b and (2) either the product ab or ba belongs to the center of the ring R, denoted as $Z(R)$.

The first graph, known as the center graph, is denoted as $GC(R)$, and it consists of the vertex set R. The second graph is called the strict center graph, denoted as $\overline{GC}(R)$, and it has the vertex set as R^*. In both graphs, the edges are established based on the criteria involving central elements in the ring R.

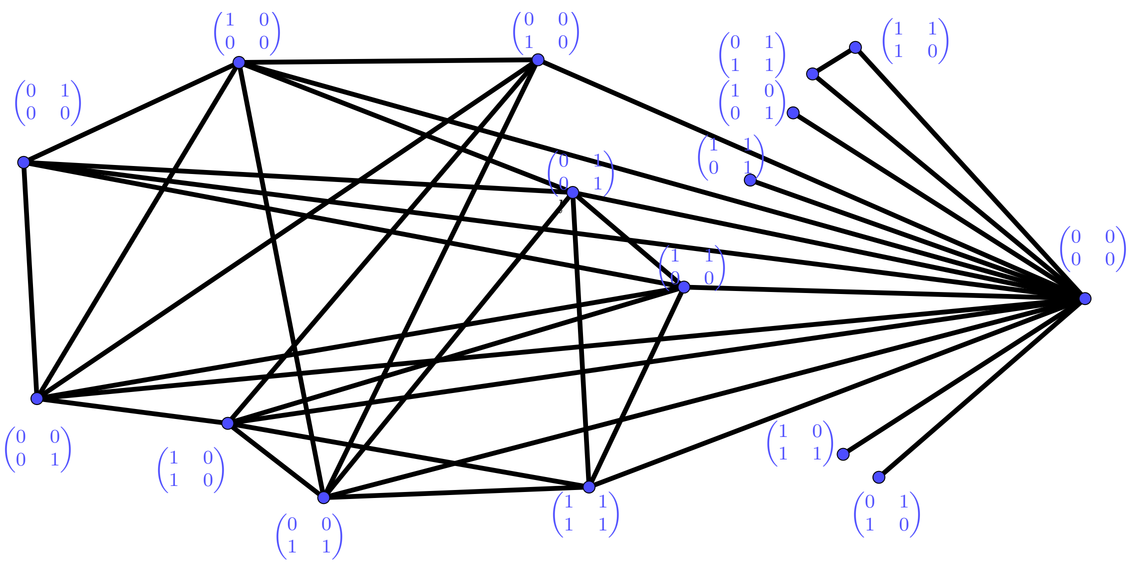
These graphs serve as valuable tools for analyzing the center of the ring R and provide a visual representation of the relationships between elements in R that exhibit commutativity properties with other elements.

Definition 1. *The center graph of a ring, denoted as $GC(R)$, is a type of undirected simple graph with its vertex set representing the elements of the ring R. Two distinct vertices a and b in R are connected by an edge in $GC(R)$ if and only if the following conditions are met*: (1) *a is not equal to b and* (2) *either the product ab or ba belongs to the center of the ring R, denoted as $Z(R)$.*

In other words, in $GC(R)$, vertices a and b are adjacent if and only if they are distinct and the result of their multiplication is a central element in the ring R. The center of the ring R, represented by $Z(R)$, comprises elements that commute with all other elements in the ring.

This graph-theoretic representation of the ring's structure provides valuable insights into the interactions and properties of elements in the ring's center, offering a graphical perspective to study the relationships between central elements and their role in the ring's algebraic properties.

***Example* 1:** $GC(M_2(\mathbb{Z}_2))$.

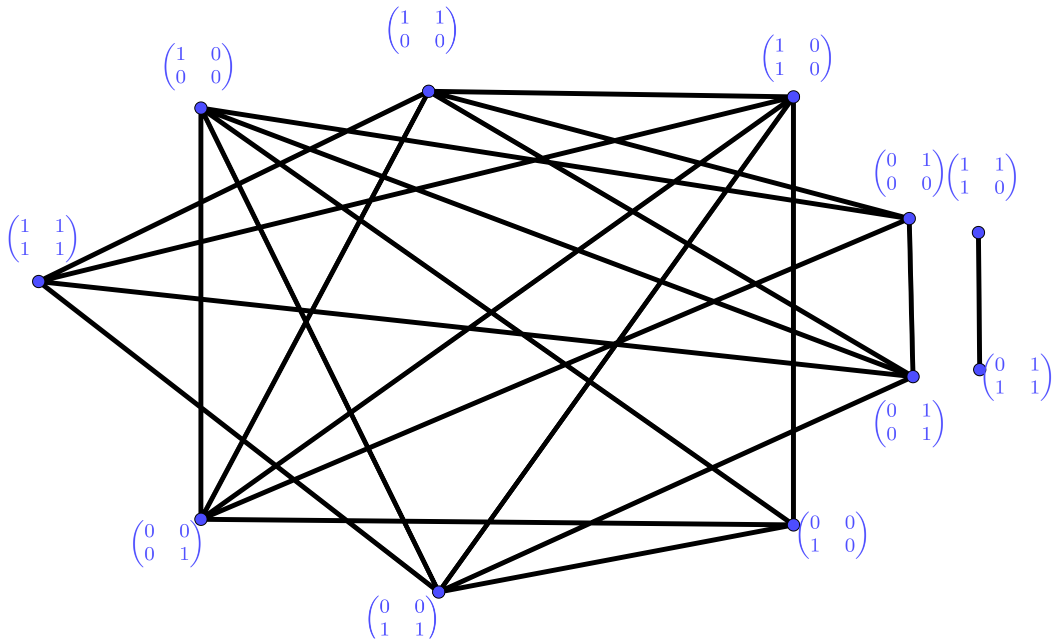


***Remark* 1.** If R is highly noncommutative in the sense that $Z(R) = \{0\}$, then $GC(R)$ is similar to $\Gamma(R)$, the well-known zero-divisor graph of R.

Note that every vertex is adjacent to 0, since $0x = 0 \in Z(R)$ for any $x \in R$. So, the graph $GC(R)$ is always connected. Therefore, we may remove the vertex 0 from the graph $GC(R)$, and we obtain a subgraph that we call *the strict center graph of* R.

Definition 2. *An undirected simple graph* $\overline{GC}(R)$ *is called the strict center graph of a ring whose vertex set is* $R - \{0\}$ *and two distinct vertices* a *and* b *in* R *are adjacent in* $\overline{GC}(R)$ *if and only if either* $ab \in Z(R)$ *or* $ba \in Z(R)$, *where* $Z(R)$ *is the center of* R.

***Example* 2:** $\overline{GC}(M_2(\mathbb{Z}_2))$.



***Remark* 2.** As illustrated in Example 2:, it is evident that $\overline{GC}(R)$ is not always connected. Specifically, for the ring $R = M_2(\mathbb{Z}_2)$, the strict center graph $\overline{GC}(R)$ has isolated vertices for all elements $x \in S(M_2(\mathbb{Z}_2))$.

In $\overline{GC}(R)$, two vertices a and b are considered adjacent if $a \neq b$ and either $ab \in Z(R)$ or $ba \in Z(R)$. For the ring $R = M_2(\mathbb{Z}_2)$, the center $Z(R)$ is composed of scalar matrices. Since scalar matrices commute with every element of the ring, the adjacency condition is not satisfied for any elements in $S(M_2(\mathbb{Z}_2))$. Hence, all elements in $S(M_2(\mathbb{Z}_2))$ are isolated vertices in $\overline{GC}(R)$.

As a result, $\overline{GC}(R)$ becomes disconnected, with the isolated vertices representing separate components in the graph. This example demonstrates that the strict center graph $\overline{GC}(R)$ can exhibit disconnected structures, depending on the specific properties of the ring R.

In a center graph, we distinguish two kinds of neighbours, the right and the left.

Definition 3. *For the center graph* $GC(R)$ *of the ring* R, *we define the following neighbor sets for a vertex* x:

the left neighbors set of x:

$$L_x = \{y \in R \setminus \{x\} \mid yx \in Z(R)\};$$

the right neighbors set of x:

$$R_x = \{y \in R \setminus \{x\} \mid xy \in Z(R)\};$$

the neighbors set of x:

$$V_x = L_x \cup R_x.$$

In the context of the center graph $GC(R)$, the left neighbors set of a vertex x contains all other vertices y (excluding x) such that the product yx belongs to the center $Z(R)$. Similarly, the right neighbor6s set of x consists of all other vertices y (excluding x) for which the product xy is an element of the center $Z(R)$. The neighbors set V_x for a vertex x is the union of its left and right neighbors sets, representing all vertices adjacent to x in the graph $GC(R)$.

It is easy to show that L_x and R_x are two additive subgroups of $(R, +)$.

3. Main results. In this section, our focus is on the properties of the center graph $GC(R)$ associated with the ring R. We will demonstrate that $GC(R)$ is always a connected graph, meaning that there is a path between every pair of vertices in $GC(R)$. In addition, we will explore the diameter of $GC(R)$, which refers to the longest distance between any two vertices in the graph. We aim to establish that the diameter of $GC(R)$ is relatively small, highlighting its compact nature.

Furthermore, we will identify specific instances where $GC(R)$ forms complete graphs. A complete graph is one where each vertex is connected to every other vertex, forming a fully connected structure. By determining the cases where $GC(R)$ is complete, we gain valuable insights into the commutative properties of the elements in the ring R and their relationships with the center of the ring.

Theorem 1. *The center graph $GC(R)$ associated with the ring R is a connected graph, which verifies the following properties*:

$\mathrm{diam}(GC(R)) \leq 2$,

$\deg(0) = |R| - 1$.

Proof. Since $0 \cdot x = 0$ for all $x (\neq 0) \in R$, there is an edge from 0 to x for all $x \in V(GC(R)) = R$. For any two nonzero elements $x, y \in R$ with $(x, y) \notin E(GC(R))$ or $(y, x) \notin E(GC(R))$, there are two edges one from x to 0, and another from 0 to y. This shows that the graph $GC(R)$ is connected. Moreover, $d(0, x) = 1$ and $d(x, y) \leq 2$ for any two nonzero elements $x, y \in R$. Hence, $\mathrm{diam}(GC(R)) \leq 2$. Moreover, we have $\deg(0) = |R - \{0\}| = |R| - 1$.

The theorem is proved.

Building upon the insights gained from Theorem 1, we now delve into Theorem 2, which presents some properties of R provided when $GC(R)$ is a complete graph.

Theorem 2. *If $GC(R)$ is a complete graph, then*:

(1) *$Z(R)$ is an ideal of R*;

(2) *if R is a unital ring, then R is commutative*;

(3) *if R is a prime ring with $Z(R) \neq \{0\}$, then R is commutative.*

Proof. 1. We know that $(Z(R), +)$ is an additive subgroup of $(R, +)$. Suppose that $GC(R)$ is complete, namely, for all $x, y \in R$, we have either $xy \in Z(R)$ or $yx \in Z(R)$. In particular, if $x \in Z(R)$ and $y \in R$, then we have $yx = xy$. It follows that $xy \in Z(R)$. Therefore, $Z(R)$ is an ideal.

2. If R is unital ring, then we have $1 \in Z(R)$. Since $Z(R)$ is an ideal, we get that $Z(R) = R$, namely, R is commutative.

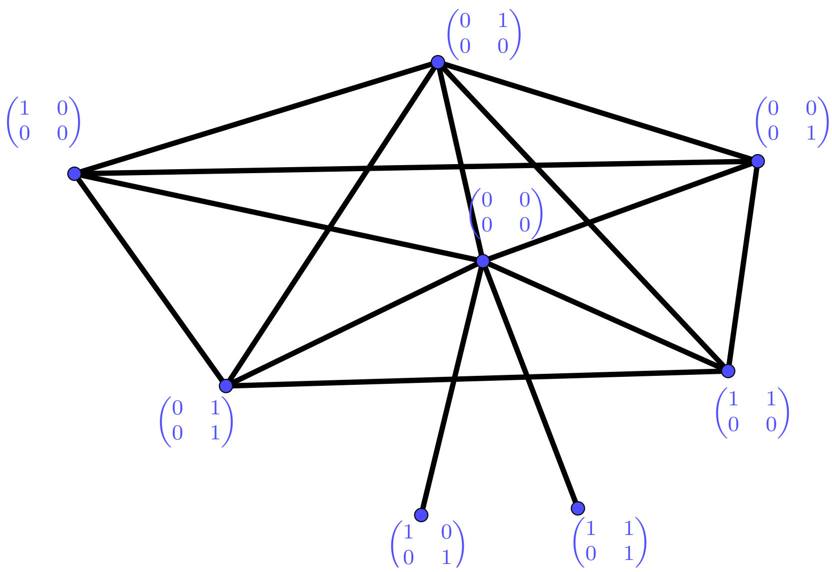
3. Let us fix $x \in R$. Since $GC(R)$ is a complete graph, we have $xy \in Z(R)$ or $yx \in Z(R)$. Since L_x and R_x are two subgroups of $(R, +)$ and $R = R_x \cup L_x$, we obtain that either $R = R_x$ or $R = L_x$. Let us suppose that $R = R_x$. Then, for any $y \in R$, we get $xy \in Z(R)$. In particular, for $y \in Z(R) \setminus \{0\}$, we have for every $r \in R$, $[xy, r] = 0$, namely, $xyr - rxy = 0$. Since $y \in Z(R)$,

we get that $xyr - rxy = y(xr - rx) = 0$. But, we obtain that R is prime and $y \neq 0$, it follows that $xr - rx = 0$, for any $r \in R$. Therefore, $x \in Z(R)$. Thus, R is commutative.

The theorem is proved.

Consider a positive integer n, and let $T_2(n)$ be the set of upper-triangular matrices with entries in the ring $\mathbb{Z}_n$. In this context, we define $D_2(n)$, $U_2(n)$, and $Z_2(n)$ as the sets of zero divisors, the inverse group, and the center, respectively, of the ring $T_2(n)$. The main focus of this study is to explore the properties of the center graph and the strict center graph associated with the ring $T_2(n)$.

***Example* 3:** $GC(T_2(2))$.



We describe a complete characterization of the sets $D_2(n)$ and $U_2(n)$.

Theorem 3. *Let* $n \geq 2$ *and* $A = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} \in T_2(n)$ *with* $A \neq 0$. *Then*:

(1) $A \in U_2(n)$ *if and only if* $\gcd(a, n) = 1$ *and* $\gcd(c, n) = 1$;

(2) $A \in D_2(n)$ *if and only if* $\gcd(a, n) \neq 1$ *or* $\gcd(c, n) \neq 1$;

(3) $|U_2(n)| = n\phi(n)^2$ *and* $|D_2(n)| = n^3 - n\phi(n)^2 - 1$, *where* ϕ *denotes Euler's totient function.*

Proof. 1. A is invertible if and only if $\det(A) = \overline{ac}$ is a unit in $\mathbb{Z}_n$. Equivalently, $\overline{a}$ and $\overline{c}$ are units in $\mathbb{Z}_n$. That is equivalent to $gcd(a, n) = 1$ and $gcd(c, n) = 1$.

2. By Lemma 1, $A \in D_2(n)$ if and only if $A \notin U_2(n)$. Therefore, $A \in D_2(n)$ if and only if either $gcd(a, n) \neq 1$ or $gcd(c, n) \neq 1$.

3. We already proved by 1 that

$$U_2(n) = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_n) \mid gcd(a, n) = gcd(c, n) = 1 \right\}.$$

By the definition of Euler's totient function $\phi(n)$, we have $\phi(n) = |\{\overline{x} \in \mathbb{Z}_n \mid gcd(x, n) = 1\}$. Then we get that $\phi(n)$ possible values for $\overline{a}$ and $\overline{c}$, and n possible value for $\overline{b}$. Thus, $|U_2(n)| = n.\phi(n)^2$. On the other hand, we obtain $|D_2(n)| = |\mathbb{Z}_n \setminus \{0\}| - |U_2(n)| = n^3 - 1 - n\phi(n)^2$.

The theorem is proved.

***Corollary* 2.** *Let* p *be a prime positive integer. Then* $|U_2(p)| = p(p-1)^2$ *and* $|D_2(p)| = 2p^2 - p - 1$.

Proof. It is enough to see that $\phi(p) = p - 1$ since p is prime. By Theorem 3, we get that $|U_2(p)| = p(p-1)^2$ and $|D_2(p)| = p^3 - p(p-1)^2 - 1 = 2p^2 - p - 1$.

Next, we determine the graph $\overline{GC}(T_2(p))$, where p is a prime positive integer. We need the following lemma.

Lemma 2. *Let p be a prime integer. Then* $Z_2(p) = \left\{ \begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix} \,\middle|\, \overline{a} \in \mathbb{Z}_p \right\}$.

Proof. Suppose that $A = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} \in Z_2(p)$. Then, for any $A' = \begin{pmatrix} \overline{a}' & \overline{b}' \\ \overline{0} & \overline{c}' \end{pmatrix}$, we have $A \times A' = A' \times A$, namely,

$$\begin{pmatrix} \overline{aa'} & \overline{ab' + bc'} \\ \overline{0} & \overline{cc'} \end{pmatrix} = \begin{pmatrix} \overline{a'a} & \overline{a'b + b'c} \\ \overline{0} & \overline{c'c} \end{pmatrix}.$$

Equivalently, $\overline{ab' + bc'} = \overline{a'b + b'c}$. In particular, for $\overline{b'} = \overline{0}$ in $\mathbb{Z}_p$, we get that $\overline{b(c' - a')} = \overline{0}$. If $\overline{c'} \neq \overline{a'}$, then $\overline{b} = \overline{0}$ since $\mathbb{Z}_p$ is integral. On the other hand, if $\overline{b'} \neq \overline{0}$, then we obtain that $\overline{(a-c)b'} = \overline{0}$, namely, $\overline{a} = \overline{c}$. Therefore, $A = \begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix}$.

Theorem 4. *Let p be a prime number. Then $\overline{GC}(T_2(p))$ is a disjoint union of two graphs*

$$\overline{GC}(T_2(p)) = \Gamma(D_2(p)) \cup \Gamma(U_2(p)),$$

where $\Gamma(D_2(p))$ is a connected graph containing eight bipartite subgraphs and one complete subgraph, and $\Gamma(U_2(p))$ is a union of disjoint bipartite graphs and complete graphs.

Proof. Let $A = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} \in T_2(p)$. Then we have either $A \in U_2(p)$ or $A \in D_2(p)$. We distinguish it into 2 cases:

Case 1. Suppose that $A \in D_2(p)$. By Theorem 3, we get that either $gcd(a,p) \neq 1$ or $gcd(c,p) \neq 1$. Since p is prime, we have either $\overline{a} = \overline{0}$ or $\overline{c} = \overline{0}$ in $\mathbb{Z}_p$. Set the following sets:

$$S_1 = \left\{ \begin{pmatrix} x & y \\ \overline{0} & \overline{0} \end{pmatrix} \,\middle|\, x, y \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\},$$

$$S_2 = \left\{ \begin{pmatrix} \overline{0} & x \\ \overline{0} & y \end{pmatrix} \,\middle|\, x, y \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\},$$

$$S_3 = \left\{ \begin{pmatrix} x & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \,\middle|\, x \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\},$$

$$S_4 = \left\{ \begin{pmatrix} \overline{0} & x \\ \overline{0} & \overline{0} \end{pmatrix} \,\middle|\, x \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\},$$

$$S_5 = \left\{ \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & x \end{pmatrix} \,\middle|\, x \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\}.$$

If A is nonzero, then $A \in S_i$ for some $i \in \{1,2,3,4,5\}$. We denote by $S_iS_j = \{xy \mid x \in S_i \text{ and } y \in S_j\}$. Then we have

$$S_2S_1 = S_4S_1 = S_5S_1 = S_2S_3 = S_2S_4 = S_5S_4 = S_5S_3 = S_4S_3 = S_4S_4 = \left\{\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}\right\}.$$

This yields a connected graph, which we denote $\Gamma(D_2(p))$. The graph $\Gamma(D_2(p))$ contains eight bipartite subgraphs and one complete subgraph.

Case 2. Suppose that $A \in U_2(p)$. Let us determine its neighbours set V_A. We know that $V_A = L_A \cup R_A$. As well, for any $B \in T_2(p)$, we have

$$\begin{aligned} B \in L_A &\Longleftrightarrow A \times B \in Z_2(p) \quad \text{and} \quad B \neq A, \\ &\Longleftrightarrow \exists C \in Z_2(p) \setminus \{A^2\}, \quad B = A^{-1} \times C = C \times A^{-1}, \\ &\Longleftrightarrow B \times A \in Z_2(p) \setminus \{A^2\}, \\ &\Longleftrightarrow B \in R_A. \end{aligned}$$

Then $L_A = R_A$. It follows that $V_A = L_A$. By the same method, we note that $V_A = A^{-1} \times Z_2(p) \setminus \{A^2\}$. By combining Lemma 2 and Theorem 3, we obtain that

$$V_A = \begin{cases} \{\overline{a}.A^{-1} \mid \overline{a} \in \mathbb{Z}_p \setminus \{\overline{0}\}\} & \text{if} \quad A^2 \notin Z_2(p), \\ \{\overline{a}.A^{-1} \mid \overline{a} \in \mathbb{Z}_p \setminus \{\overline{0}, \overline{x}\}\} & \text{if} \quad A^2 = \overline{x}I_2 \in Z_2(p). \end{cases}$$

So, we get a graph not necessarily connected, which we denote by $\Gamma(U_2(p))$. Moreover, note that if $A^2 \notin Z_2(p)$, then $\Gamma(U_2(p))$ contains a disjoint bipartite subgraph $\Gamma(A)$ with the vertex set of the form $V(\Gamma(A)) = U \cup V$, where $U = \{\overline{x}A \mid \overline{x} \in \mathbb{Z}_p \setminus \{\overline{0}\}\}$ and $V = \{\overline{x}A^{-1} \mid \overline{x} \in \mathbb{Z}_p \setminus \{\overline{0}\}\}$ and $u \sim v$ for any $(u, v) \in U \times V$. As well as, if $A^2 \in Z_2(p)$, then $\Gamma(U_2(p))$ contains a disjoint complete subgraph $\Gamma(A)$ with the vertex set $V(\Gamma(A)) = \{\overline{x}A \mid \overline{x} \in \mathbb{Z}_p \setminus \{\overline{0}\}\}$.

Since the vertices in $\Gamma(D_2(p))$ are zero divisors, and vertices in $\Gamma(U_2(p))$ are units, the graphs $\Gamma(D_2(p))$ and $\Gamma(U_2(p))$ are disjoint.

The theorem is proved.

We see in the proof how the condition $A^2 \in Z_2(p)$ was important, so let us characterize when this condition holds.

Proposition 1. *Let $A \in T_2(p)$, where p is a positive prime integer. Then $A^2 \in Z_2(p)$ if and only if either A is of the form $\begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix} = \overline{x}I_2$ or of the form $\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & -\overline{a} \end{pmatrix}$, where $\overline{a}, \overline{b} \in \mathbb{Z}_p$. Then we write*

$$\sqrt{T_2(p)} := \left\{\begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix}, \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & -\overline{a} \end{pmatrix} \;\middle|\; \overline{a}, \overline{b} \in \mathbb{Z}_p\right\}.$$

Proof. Let $A = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{pmatrix} \in T_2(p)$ such that $A^2 \in Z_2(p)$. We have

$$A^2 = \begin{pmatrix} \overline{a}^2 & \overline{b(a+c)} \\ \overline{0} & \overline{c}^2 \end{pmatrix}.$$

By Lemma 2, we get that $A^2 \in Z_2(p)$ if and only if the following equations hold:

$$\begin{cases} \overline{a}^2 = \overline{c}^2, \\ \overline{b(a+c)} = \overline{0} \end{cases} \iff \begin{cases} \overline{a} = \overline{c} \quad \text{or} \quad \overline{a} = -\overline{c}, \\ \overline{b} = \overline{0} \quad \text{or} \quad \overline{a} = -\overline{c}. \end{cases}$$

Equivalently, A is either of the form $\begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix} = \overline{x} I_2$ or of the form $\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & -\overline{a} \end{pmatrix}$.

So we obtain the following result.

***Corollary* 3.** *Let p be a positive prime integer and $A \in U_2(p)$. Then we have*:

in $GC(T_2(p))$

$$\deg(A) = \begin{cases} n & \textit{if} \quad A \notin \sqrt{T_2(p)}, \\ n-1 & \textit{if}\ A \in \sqrt{T_2(p)}, \end{cases}$$

and in $\overline{GC}(T_2(n))$

$$\deg(A) = \begin{cases} n-1 & \textit{if} \quad A \notin \sqrt{T_2(p)}, \\ n-2 & \textit{if}\ A \in \sqrt{T_2(p)}. \end{cases}$$

Proof. Let V_A be the neighbours set of A in $\overline{GC}(T_2(p))$. By the proof of Theorem 4, we have two cases:

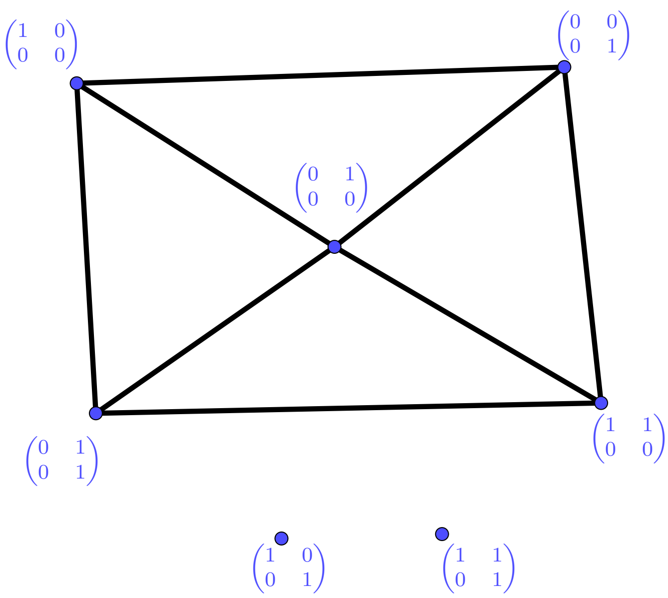
Case 1. If $A \notin \sqrt{T_2(p)}$, then $V_A = \{\overline{a}.A^{-1} \mid \overline{a} \in \mathbb{Z}_p \backslash \{\overline{0}\}\}$. It follows that $\deg(A) = |V_A| - 1 = p - 2$.

Case 2. If $A^2 = \overline{x} I_2 \in Z_2(p)$, then $V_A = \{\overline{a}.A^{-1} \mid \overline{a} \in \mathbb{Z}_p \setminus \{\overline{0}, \overline{x}\}\}$. It follows that $\deg(A) = |V_A| - 1 = p - 3$.

On the other hand, in $GC(T_2(p))$ it is sufficient to see that there is only one supplement edge, which joins any vertex A with the vertex 0.

The following example shows how if p a prime number $GC(\bar{T}_2(2))$ is a union of disjoint subgraphs.

***Example* 4:** $\overline{GC}(T_2(2))$.



We had by Theorem 4, that $\Gamma(U_2(p))$ is a union of disjoint bipartite subgraphs and complete subgraphs. Let us determine the number of bipartite subgraphs and the number of complete subgraphs in $\Gamma(U_2(p))$.

Theorem 5. *Let p be a prime number, α be the number of disjoint bipartite subgraphs of $\Gamma(U_2(p))$, and β be the number of disjoint complete subgraphs of $\Gamma(U_2(p))$. Then*

$$(\alpha, \beta) = \left(\frac{p^2 - 2p - 1}{2},\ p + 1\right) \quad \textit{if} \quad p > 2,$$

$$(\alpha, \beta) = (0, 2) \quad \textit{if} \quad p = 2.$$

Proof. We know that $(U_2(p), \times)$ is a group, and by Corollary 2, its order is $p(p-1)^2$. As well as, $(Z(U_2(p)), \times)$ is a normal subgroup of $(U_2(p), \times)$, and by combining Lemma 2 and Theorem 1, it is easy to see that its order is $p - 1$. Then the order of the quotient group $U_2(p)/Z(U_2(p))$ is $p(p-1)$. Note that, for any $A \in U_2(p)$, its class in $U_2(p)/Z(U_2(p))$ is the set $\{\overline{x}A \mid \overline{x} \in \mathbb{Z}_p \setminus \{\overline{0}\}\}$. We distinguish two cases:

Case 1. If $A \notin \sqrt{T_2(p)}$, then, by the same notations in the proof of Theorem 4, we have that the vertex set $V(\Gamma(A)) = U \cup V$, where U is the class of A in $U_2(p)/Z(U_2(p))$ and V is the class of A^{-1} in $U_2(p)/Z(U_2(p))$. Each class contains $p - 1$ elements. Then $\Gamma(A)$ is a disjoint bipartite subgraph containing $2(p-1)$ vertices.

Case 2. If $A \in \sqrt{T_2(p)}$, then $V(\Gamma(A))$ is the class of A in $U_2(p)/Z(U_2(p))$. It follows that $\Gamma(A)$ is a disjoint complete subgraph containing $p - 1$ elements.

Let $n = |U_2(p) \cap \sqrt{T_2(p)}|$ and $m = |U_2(p) \setminus \sqrt{T_2(p)}|$. Then we have

$$\text{the number of disjoint bipartite subgraphs of} \quad \Gamma(U_2(p)) \quad \text{is} \quad \frac{m}{2(p-1)},$$

$$\text{the number of disjoint complete subgraphs of} \quad \Gamma(U_2(p)) \quad \text{is} \quad \frac{n}{p-1}.$$

By Proposition 1 and Theorem 1, we obtain

$$U_2(p) \cap \sqrt{T_2(p)} = \left\{ \begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix} \middle| \ \overline{a} \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\} \dot{\bigcup} \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & -\overline{a} \end{pmatrix} \middle| \ \overline{a}, \overline{b} \in \mathbb{Z}_p \quad \text{and} \quad \overline{a} \neq \overline{0} \right\}.$$

As we have this union is disjoint if and only if $p \neq 2$. In this case we get that

$$\left| \left\{ \begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{a} \end{pmatrix} \middle| \ \overline{a} \in \mathbb{Z}_p \setminus \{\overline{0}\} \right\} \right| = p - 1,$$

$$\left| \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{0} & -\overline{a} \end{pmatrix} \middle| \ \overline{a}, \overline{b} \in \mathbb{Z}_p \ \text{and} \ \overline{a} \neq \overline{0} \right\} \right| = p(p-1),$$

it follows that $n = (p-1) + p(p-1) = (p+1)(p-1)$. Therefore, $m = |U_2(p)| - (p^2 - 1)$, namely, $m = p(p-1)^2 - (p+1)(p-1) = (p^2 - 2p - 1)(p-1)$. Thus,

$$\text{the number of disjoint bipartite subgraphs of} \quad \Gamma(U_2(p)) \quad \text{is} \quad \frac{p^2 - 2p - 1}{2},$$

$$\text{the number of disjoint complete subgraphs of} \quad \Gamma(U_2(p)) \quad \text{is} \quad p + 1.$$

However, if $p = 2$, then $\overline{a} = -\overline{a}$, and we have

$$U_2(p) \cap \sqrt{T_2(p)} = \left\{ \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix} \right\}.$$

Then $n = 2$ and $m = 0$. Thus, in this case, we obtain

$$\text{the number of disjoint bipartite subgraphs of} \quad \Gamma(U_2(2)) \quad \text{is} \quad 0,$$

$$\text{the number of disjoint complete subgraphs of} \quad \Gamma(U_2(p)) \quad \text{is} \quad 2.$$

The theorem is proved.

Conclusion. In this paper, we have delved into the properties of finite associative rings, both with and without a unit element, with a specific focus on their centers, denoted as $Z(R)$. We introduced two distinctive graph structures, namely, the center graph $GC(R)$ and the strict center graph $\overline{GC(R)}$, and explored their properties.

Our findings reveal that the completeness of $GC(R)$ has direct implications for $Z(R)$, highlighting that in cases where $GC(R)$ is complete, $Z(R)$ acts as an ideal within R. Furthermore, for unital rings, the completeness of $GC(R)$ leads to the conclusion that R is, indeed, a commutative ring.

We have also put theory into practice by explicitly constructing the graph $\overline{GC}(T_2(p))$, where $T_2(p)$ represents the ring of upper-triangular matrices with entries from the ring $\mathbb{Z}/p\mathbb{Z}$, with p as a prime integer. This practical application underscores the real-world relevance of our graph-theoretical concepts in the realm of ring theory.

In summary, our research significantly advances the understanding of finite associative rings and their centers. It provides not only theoretical insights but also tangible applications in the ring theory. This work paves the way for further exploration and research into the graph-theoretical aspects of ring theory, opening new avenues for discovery and development.

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