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GEOMETRY OF MULTILINEAR FORMS ON A NORMED SPACE $\mathbb{R}^m$

ГЕОМЕТРІЯ БАГАТОЛІНІЙНИХ ФОРМ НА НОРМОВАНОМУ ПРОСТОРІ $\mathbb{R}^m$

For every $m \geq 2,$ let $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. For every $n \geq 2,$ we focus our attention on the description of the sets of extreme and exposed points of the closed unit balls of $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ and $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|}),$ where $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ is the space of n-linear forms on $\mathbb{R}^m_{\|\cdot\|}$ and $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$ is the subspace of $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ formed by symmetric n-linear forms. Let $\mathcal{F} = \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ or $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|}).$ First, we show that the number of extreme points of the unit ball of $\mathbb{R}^m_{\|\cdot\|}$ is greater than $2m.$ By using this fact, we classify the extreme and exposed points of the closed unit ball of $\mathcal{F},$ respectively. It is shown that every extreme point of the closed unit ball of $\mathcal{F}$ is exposed. We obtain the results of [Studia Sci. Math. Hungar., **57**, № 3, 267 – 283 (2020)] and extend the results of [Acta Sci. Math. Szedged, **87**, № 1-2, 233 – 245 (2021) and J. Korean Math., Soc., **60**, № 1-2, 213 – 225 (2023)].

Нехай $\mathbb{R}^m_{\|\cdot\|}$ для кожного $m \geq 2$ — це $\mathbb{R}^m$ з нормою $\|\cdot\|$ такою, що її одинична куля має скінченну кількість екстремальних точок. Для кожного $n \geq 2$ ми приділимо увагу опису множин екстремальних та відкритих точок замкнених одиничних куль в $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ і $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|}),$ де $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ — простір n-лінійних форм на $\mathbb{R}^m_{\|\cdot\|},$ а $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$ — підпростір $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|}),$ що складається з симетричних n-лінійних форм. Нехай $\mathcal{F} = \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ або $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|}).$ Спочатку ми показуємо, що кількість екстремальних точок одиничної кулі $\mathbb{R}^m_{\|\cdot\|}$ більша ніж $2m.$ Використовуючи цей факт, ми класифікуємо екстремальні та відкриті точки замкненої одиничної кулі в $\mathcal{F}$ відповідно. Показано, що кожна екстремальна точка замкненої одиничної кулі $\mathcal{F}$ є відкритою. Отримано результати роботи [Studia Sci. Math. Hungar., **57**, № 3, 267 – 283 (2020)] та розширено результати з [Acta Sci. Math. Szedged, **87**, № 1-2, 233 – 245 (2021) та J. Korean Math., Soc., **60**, № 1-2, 213 – 225 (2023)].

1. Introduction. Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2.$ We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by $E^*.$ An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \dfrac{1}{2}(y+z)$ implies $x = y = z.$ The Krein – Milman theorem [23] says that a compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points. An element $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}.$ It is easy to see that every exposed point of B_E is an extreme point. An element $x \in B_E$ is called a *smooth point* of B_E if there is unique $f \in E^*$ so that $f(x) = 1 = \|f\|.$ We denote by $\operatorname{ext} B_E,$ $\exp B_E$ and $\operatorname{sm} B_E$ the set of extreme points, the set of exposed points and the set of smooth points of $B_E,$ respectively. A mapping $P: E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \ldots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E.$ We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|.$ We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|.$ $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on $E.$ Note that $\mathcal{L}(^nE)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n}E.$ With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n}E$ is given by

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$$\left\langle \sum_{i=1}^{k} x^{(1),i} \otimes \ldots \otimes x^{(n),i},\ T \right\rangle = \sum_{i=1}^{k} T\big(x^{(1),i}, \ldots, x^{(n),i}\big).$$

Note also that $\mathcal{L}_s(^nE)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n}E$. With this identification, the action of a continuous symmetric n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n}E$ is given by

$$\left\langle \sum_{i=1}^{k} \frac{1}{n!}\big(\sum_{\sigma} x^{\sigma(1),i} \otimes \ldots \otimes x^{\sigma(n),i}\big),\ T \right\rangle = \sum_{i=1}^{k} T\big(x^{(1),i}, \ldots, x^{(n),i}\big),$$

where σ is a permutation on $\{1, \ldots, n\}$. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [5].

Let us introduce the history of classification problems of the extreme points and the exposed points of the unit ball of continuous n-homogeneous polynomials on a Banach space. We let $l_p^n = \mathbb{R}^n$ for every $1 \le p \le \infty$ equipped with the l_p-norm. Choi et al. [3, 4] initiated and classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2$. Grecu [6] classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim et al. [21] showed that if E is a separable real Hilbert space with $\dim(E) \ge 2$, then $\operatorname{ext} B_{\mathcal{P}(^2E)} = \exp B_{\mathcal{P}(^2E)}$. Kim [7] classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $1 \le p \le \infty$. Kim [9] characterized $\operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with an octagonal norm $\|(x,y)\|_w = \max\left\{|x|, |y|, \dfrac{|x|+|y|}{1+w}\right\}$ for $0 < w < 1$. Kim [11] classified $\exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(^2d_*(1,w)^2)} \ne \operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$. Recently, Kim [12, 14] classified $\operatorname{ext} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\exp B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with a hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\left\{|y|, |x| + \dfrac{1}{2}|y|\right\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}(^nE)}$ and $\exp B_{\mathcal{P}(^nE)}$, it seems to be very natural to study the classification problems of the extreme points and the exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space. Kim [8] initiated and classified $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)}$, $\exp B_{\mathcal{L}_s(^2l_\infty^2)}$ and $\operatorname{sm} B_{\mathcal{L}_s(^2l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)} = \exp B_{\mathcal{L}_s(^2l_\infty^2)}$. Kim [15] studied $\operatorname{ext} B_{\mathcal{L}(^2l_\infty)}$. Cavalcante et al. [2] characterized $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$. Kim [16, 18] characterized $\operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^m)}, \operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$, $\exp B_{\mathcal{L}_s(^ml_\infty^m)}$, $\exp B_{\mathcal{L}(^nl_\infty^m)}$, $\operatorname{sm} B_{\mathcal{L}_s(^nl_\infty^m)}$ and $\operatorname{sm} B_{\mathcal{L}(^nl_\infty^m)}$ for every $n, m \ge 2$. Kim [19] characterized $\operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}, \operatorname{ext} B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}, \exp B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$, and $\exp B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}$ if $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| = 2m$ for $m \ge 2$. It was shown that every extreme point is exposed in this case. Recently, Kim [20] characterized $\operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}, \operatorname{ext} B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}$, $\exp B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$, and $\exp B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}$ if $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| \ge 2m$ for $m \ge 2$. It was shown that every extreme point is exposed in this case. We refer to [1, 13, 17, 22, 24 – 27] for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

Let $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. In this paper, we devote to the description of the sets of extreme and exposed points of the closed unit balls of $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ and $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$. Let $\mathcal{F} = \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ or $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$. First we show that the number of extreme points of the unit ball of $\mathbb{R}^m_{\|\cdot\|}$ is more than $2m$. Using this we classify the extreme points and exposed points of the closed unit ball of $\mathcal{F}$, correspondingly. It is shown that every extreme point of the closed unit ball of $\mathcal{F}$ is exposed. We obtain the results of [16] and extend the results of [19, 20].

2. Results. Throughout the paper, we let $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. Let $\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}} = \{\pm U_1, \ldots, \pm U_l\}$ with $U_i \neq U_j$ for $i \neq j$ and for some $l \in \mathbb{N}$. Let

$$\mathcal{W} := \{(U_{i_1}, \ \ldots, U_{i_n}) : i_k = 1, \ldots, l, \ k = 1, \ldots, n\}.$$

Then $|\mathcal{W}| = l^n$.

Example. Let $m \geq 2$ and $\|\cdot\|$ be the supremum norm on $\mathbb{R}^m$. Then $\mathbb{R}^m_{\|\cdot\|} = l^m_\infty$ and

$$\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}} = \operatorname{ext} B_{l^m_\infty} = \{\pm(t_1, \ \ldots, t_m) : t_j = \pm 1, \ j = 1, \ldots, m\}.$$

Hence, $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| = |\operatorname{ext} B_{l^m_\infty}| = 2^m$.

Theorem A. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ denote $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. If $T \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$, then*

$$\|T\| = \sup_{W \in \mathcal{W}} |T(W)|.$$

Proof. It follows that from the Krein – Milman theorem and multilinearity of T.

Let $n, m \geq 2$. Let $Z_1, \ldots, Z_{m^n}$ be an ordering of the monomials in the expansion

$$(x_1^{(1)} + \ldots + x_m^{(1)}) \ldots (x_1^{(n)} + \ldots + x_m^{(n)}).$$

Note that $\{Z_1, \ldots, Z_{m^n}\}$ is a basis for $\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$. Hence, $\dim(\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})) = m^n$. If $T \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$, then

$$T = \sum_{j=1}^{m^n} a_j Z_j$$

for some $a_1, \ldots, a_{m^n} \in \mathbb{R}$. By simplicity, we denote $T = (a_1, \ldots, a_{m^n})^t$. Let $W_1, \ldots, W_{l^n}$ be an ordering of the elements of $\mathcal{W}$. Let

$$M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{l^n}) = \big[Z_i(W_j)\big]^t$$

be the $l^n \times m^n$ matrix. Note that, for every $T \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$,

$$M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{l^n})T = (T(W_1), \ldots, T(W_{l^n}))^t.$$

Here, $(\varepsilon_1, \ldots, \varepsilon_{l^n})^t$ denote the transpose of $(\varepsilon_1, \ldots, \varepsilon_{l^n})$.

Lemma B. *For $m \geq 2$, $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| \geq 2m$.*

Proof. Assume otherwise. Then $1 \leq l < m$.

Claim. Let $n \geq 2$ and $X_j = \big(Z_j(W_1), \ldots, Z_j(W_{l^n})\big)$ for $j = 1, \ldots, \ l^n$. Then $X_1, \ \ldots, \ X_{l^n}$ are linearly independent in $\mathbb{R}^{l^n}$.

Suppose that $\sum_{j=1}^{l^n} a_j X_j = (0, \ldots, 0)$ for some $a_j \in \mathbb{R}, \ j = 1, \ldots, l^n$. It follows that

$$(0, \ldots, 0) = \sum_{j=1}^{l^n} a_j X_j = \left(\sum_{j=1}^{l^n} a_j Z_j(W_1), \ldots, \sum_{j=1}^{l^n} a_j Z_j(W_{l^n})\right),$$

which shows that $\sum_{j=1}^{l^n} a_j Z_j(W_k) = 0$ for $k = 1, \ldots,\ l^n$. Let $T := (a_1, \ldots, a_{l^n},\ 0, \ldots, 0)^t \in \mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})$. Since $T(W_k) = \sum_{j=1}^{m^n} a_j Z_j(W_k) = 0$ for $k = 1, \ldots,\ l^n$, $\|T\| = 0$ by Theorem A. Hence, $T = 0$. Therefore, $a_j = 0$ for $j = 1, \ldots,\ l^n$. Hence, the claim holds.

By the claim, $\operatorname{rank}(M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{l^n})) = l^n < m^n$. Hence, there is a nonzero solution $(b_1,\ \ldots,\ b_{m^n}) \in \mathbb{R}^{m^n}$ of the linear equations

$$M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{l^n})(b_1,\ \ldots,\ b_{m^n})^t = (0, \ldots, 0)^t.$$

Let $S := (b_1, \ldots, b_{m^n})^t \in \mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})$. Then $S \neq 0$. Since $S(W_k) = 0$ for $k = 1, \ldots,\ l^n$, $\|S\| = 0$ by Theorem A. Hence, $S = 0$. This is a contradiction. Therefore, $l \geq m$.

Using Lemma B, we have the following theorem.

Theorem C. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ denote $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. Let $T \in \mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})$ be such that $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})}$ if and only if there are $W_1, \ldots, W_{m^n} \in \mathcal{W}$ such that $M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})$ is an invertible $m^n \times m^n$ matrix and $|T(W_j)| = 1$ for all $j = 1, \ldots, m^n$.*

Proof. ($\Rightarrow$) Let

$$\mathcal{W}_T := \{W \in \mathcal{W} \colon |T(W)| = 1\}.$$

By Theorem A, $\mathcal{W}_T \neq \varnothing$. Let

$$\Omega_T = \Big\{(Z_1(W), \ldots, Z_{m^n}(W)) \in \mathbb{R}^{m^n} : W \in \mathcal{W}_T\Big\}.$$

Then $\Omega_T \neq \varnothing$.

Claim. There are $W_1, \ldots, W_{m^n} \in \mathcal{W}_T$ such that $M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})$ is invertible.

Assume otherwise. Let $N \in \mathbb{N}$ be the largest number of linearly independent elements in Ω_T. Then $N < m^n$. Let $X := (x_1, \ldots, x_{m^n}) \in \mathbb{R}^{m^n}$. Consider the homogeneous system of linear equations with m^n unknowns as follows:

$$V \cdot X^t = 0 \quad \text{for} \quad V \in \Omega_T.$$

Since the largest number N of linearly independent elements in Ω_T is less than m^n, there is a nontrivial solution $0 \neq \mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_{m^n}) \in \mathbb{R}^{m^n}$ such that

$$V \cdot \mathcal{E}^t = 0 \quad \text{for all} \quad V \in \Omega_T.$$

Choose $\delta > 0$ such that

$$\max\{|T(W)| : W \in \mathcal{W} \backslash \mathcal{W}_T\} + \delta \|\mathcal{E}^t\| < 1.$$

Define $T_1, T_2 \in \mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})$ by

$$T_k = T + (-1)^k \delta\, \mathcal{E}^t \quad \text{for} \quad k = 1, 2.$$

We will show that $\|T_k\| \leq 1$ for $k = 1, 2$. Note that, for $W \in \mathcal{W}_T$,

$$\mathcal{E}^t(W) = \sum_{j=1}^{m^n} \varepsilon_j Z_j(W) = (Z_1(W), \ldots, Z_{m^n}(W)) \cdot \mathcal{E}^t = 0.$$

Hence, for every $W \in \mathcal{W}_T$ and $k = 1, 2$,

$$|T_k(W)| \leq \max\Big\{|T(W) + \delta\, \mathcal{E}^t(W)|,\ |T(W) - \delta\, \mathcal{E}^t(W)|\Big\} = |T(W)| = 1.$$

It follows that, for every $W \in \mathcal{W}\backslash\mathcal{W}_T$,

$$|T_k(W)| \leq |T(W)| + \delta\, |\mathcal{E}^t(W)| \leq \max\{|T(W)| : W \in \mathcal{W}\backslash\mathcal{W}_T\} + \delta\|\mathcal{E}^t\| < 1.$$

By Theorem A, $\|T_k\| = 1$ for $k = 1, 2$. Since $T_k \neq T$ and $T = \dfrac{1}{2}(T_1 + T_2)$, T is not extreme. This is a contradiction. Therefore, the claim holds.

($\Leftarrow$) Let $S_1, S_2 \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ be such that $\|S_k\| = 1$ for $k = 1, 2$ and $T = \dfrac{1}{2}(S_1 + S_2)$. Write $S_k = T + (-1)^k(\delta_1, \ldots, \delta_{m^n})^t$ for some $\delta_1, \ldots, \delta_{m^n} \in \mathbb{R}$ and $k = 1, 2$.

***Claim*:** $\delta_1 = \ldots = \delta_{m^n} = 0$.

Note that

$$\begin{aligned}\big(S_k(W_1), \ldots, S_k(W_{m^n})\big) &= M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})S_k \\ &= M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})T \\ &\quad + (-1)^k M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})(\delta_1, \ldots, \delta_{m^n})^t \\ &= \big(T(W_1) + (-1)^k(\delta_1, \ldots, \delta_{m^n})^t(W_1), \ldots, T(W_{m^n}) \\ &\quad + (-1)^k(\delta_1, \ldots, \delta_{m^n})^t(W_{m^n})\big),\end{aligned}$$

which shows that

$$S_k(W_j) = T(W_j) + (-1)^k(\delta_1, \ldots, \delta_{m^n})^t(W_j)$$

for all $k = 1, 2$ and $j = 1, \ldots, m^n$. It follows that, for $j = 1, \ldots, m^n$,

$$\begin{aligned}1 \geq \max\{|S_1(W_j)|,\ |S_2(W_j)|\} &= |T(W_j)| + |(\delta_1, \ldots, \delta_{m^n})^t(W_j)| \\ &= 1 + |(\delta_1, \ldots, \delta_{m^n})^t(W_j)|,\end{aligned}$$

which implies that $(\delta_1, \ldots, \delta_{m^n})^t(W_j) = 0$ for all $j = 1, \ldots, m^n$. Hence,

$$\begin{aligned}&M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})(\delta_1, \ldots, \delta_{m^n})^t \\ &\qquad = \big((\delta_1, \ldots, \delta_{m^n})^t(W_1), \ldots, (\delta_1, \ldots, \delta_{m^n})^t(W_{m^n})\big)^t = (0, \ldots, 0)^t.\end{aligned}$$

Therefore,

$$(\delta_1, \ldots, \delta_{m^n})^t = M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})^{-1}(0, \ldots, 0)^t = (0, \ldots, 0)^t.$$

Hence, the claim holds. Therefore, $T \in \operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$.

Theorem C is proved.

Theorem D [10]. *Let E be a real Banach space such that $\operatorname{ext} B_E$ is finite. Suppose that $x \in \operatorname{ext} B_E$ satisfies that there exists a functional $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \operatorname{ext} B_E\backslash\{\pm x\}$. Then $x \in \exp B_E$.*

Theorem E. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. Then* $\exp B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$.

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$. By Theorem C, there are $W_1, \ldots, W_{m^n} \in \mathcal{W}$ such that $M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})$ is invertible and $|T(W_j)| = 1$ for all $j = 1, \ldots, m^n$. Define

$$f := \frac{1}{m^n} \sum_{1 \leq j \leq m^n} \operatorname{sign}(T(W_j))\delta_{W_j} \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})^*,$$

where $\delta_W(S) := S(W)$ for $S \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$. Note that $1 = \|f\| = f(T)$. Let $S \in \mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})$ be such that $|f(S)| = 1$. We will show that $S = T$ or $S = -T$. It follows that

$$1 = |f(S)| = \left| \frac{1}{m^n} \sum_{1 \leq j \leq m^n} \operatorname{sign}(T(W_j))S(W_j) \right|$$

$$\leq \frac{1}{m^n} \sum_{1 \leq j \leq m^n} |S(W_j)| \leq 1,$$

which shows that

$$S(W_j) \operatorname{sign}(T(W_j)), \quad 1 \leq j \leq m^n,$$

or

$$S(W_j) = -\operatorname{sign}(T(W_j)), \quad 1 \leq j \leq m^n.$$

Suppose that

$$S(W_j) = -\operatorname{sign}(T(W_j)), \quad 1 \leq j \leq m^n.$$

It follows that

$$\begin{aligned} S &= M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})^{-1}(S(W_1), \ldots, S(W_{m^n}))^t \\ &= M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})^{-1}(-\operatorname{sign}(T(W_1)), \ldots, -\operatorname{sign}(T(W_{m^n})))^t \\ &= M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})^{-1}(-T(W_1), \ldots, -T(W_{m^n}))^t = -T. \end{aligned}$$

Note that if $S(W_j) = \operatorname{sign}(T(W_j))$, $1 \leq j \leq m^n$, then $S = T$. By Theorem D, T is exposed.

Theorem E is proved.

We define an equivalence relation on $\mathcal{W}$ by $(U_{i_1}, \ldots, U_{i_n}) \sim (U_{j_1}, \ldots, U_{j_n})$ if and only if there is a permutation σ on n such that $j_k = i_{\sigma(k)}$ for all $k = 1, \ldots, n$. Let

$$\mathcal{V} = \mathcal{W}/\sim := \{[W] : W \in \mathcal{W}\}.$$

Theorem F. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. If $T \in \mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$, then $\|T\| = \sup_{[W]\in\mathcal{V}} |T(W)|$.*

Proof. It follows that from Theorem A and symmetry of T.

Let $Y_1, \ldots, Y_d$ be an ordering of the symmetric monomials in the expansion

$$(x_1^{(1)} + \ldots + x_m^{(1)}) \ldots (x_1^{(n)} + \ldots + x_m^{(n)}).$$

Then $\{Y_1, \ldots, Y_d\}$ is a basis for $\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$. If $T \in \mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})$, then

$$T = \sum_{j=1}^{d} b_j Y_j$$

for some $b_1, \ldots, b_d \in \mathbb{R}$. By simplicity, we denote $T = (b_1, \ldots, b_d)^t$.

Theorem G. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ denote $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. Let $T \in \mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})$ be such that $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})}$ if and only if there are $[W_1], \ldots, [W_d] \in \mathcal{V}$ such that*

$$M(Y_1, \ldots, Y_d; [W_1], \ldots, [W_d]) = \big[Y_i(W_j)\big]^t$$

is an invertible $d \times d$ matrix and $|T(W_j)| = 1$ for all $j = 1, \ldots, d$.

Proof. By analogous arguments in the proof of Theorem C it follows.

Theorem H. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. Then $\exp B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})}$.*

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})}$. By Theorem G, $[W_1], \ldots, [W_d] \in \mathcal{V}$ such that

$$M(Y_1, \ldots, Y_d; [W_1], \ldots, [W_d]) = \big[Y_i(W_j)\big]^t$$

is an invertible $d \times d$ matrix and $|T(W_j)| = 1$ for all $j = 1, \ldots, d$. Define

$$f := \frac{1}{d} \sum_{1 \leq j \leq d} \operatorname{sign}(T(W_j)) \delta_{W_j} \in \mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})^*.$$

By analogous arguments in the proof of Theorem E we may show that f exposes T. Therefore, T is exposed.

The following shows a relation between $\mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})$ and $\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})$.

Theorem I. *Let $n, m \geq 2$ and $\mathbb{R}^m_{\|\cdot\|}$ be $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that its unit ball has finitely many extreme points. Then*:

(a) $\operatorname{ext} B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})} = \operatorname{ext} B_{\mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})} \bigcap \mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})$;

(b) $\exp B_{\mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})} = \exp B_{\mathcal{L}({}^n\mathbb{R}^m_{\|\cdot\|})} \bigcap \mathcal{L}_s({}^n\mathbb{R}^m_{\|\cdot\|})$.

Proof. (a) It follows from Theorems C and G.

(b) It follows from Theorems E and H.

Let

$$\mathcal{W}_{n,m} = \{\big((1, t_2^{(1)}, \ldots, t_m^{(1)}),\ \ldots, (1, t_2^{(n)}, \ldots, t_m^{(n)})\big) : t_j^{(k)} = \pm 1,$$

$$j = 2\ \ldots, m \text{ and } k = 1, \ldots, n\}.$$

Let

$$\mathcal{V}_{n,m} = \mathcal{W}_{n,m} / \sim,$$

$\sim$ is the equivalence relation on $\mathcal{W}_{n,m}$ in the front of Theorem F.

Cavalcante et al. [2] characterized $\operatorname{ext} B_{\mathcal{L}({}^n l_\infty^m)}$ for all $n, m \geq 2$. Kim [16] characterized $\operatorname{ext} B_{\mathcal{L}_s({}^n l_\infty^m)}, \operatorname{ext} B_{\mathcal{L}({}^n l_\infty^m)}, \exp B_{\mathcal{L}_s({}^m l_\infty^m)}$, and $\exp B_{\mathcal{L}({}^n l_\infty^m)}$.

We obtain the results of [16].

***Corollary* J.** *Let $n, m \geq 2$. Then*:

(a) *Let* $T \in \mathcal{L}({}^n l_\infty^m)$ *be such that* $\|T\| = 1$. *Then* $T \in \operatorname{ext} B_{\mathcal{L}({}^n l_\infty^m)}$ *if and only if there are* $W_1, \ldots, W_{m^n} \in \mathcal{W}_{n,m}$ *such that* $M(Z_1, \ldots, Z_{m^n}; W_1, \ldots, W_{m^n})$ *is an invertible* $m^n \times m^n$ *matrix and* $|T(W_j)| = 1$ *for all* $j = 1, \ldots, m^n$.

(b) *Let* $T \in \mathcal{L}_s({}^n l_\infty^m)$ *be such that* $\|T\| = 1$. *Then* $T \in \operatorname{ext} B_{\mathcal{L}_s({}^n l_\infty^m)}$ *if and only if there are* $[W_1], \ldots, [W_d] \in \mathcal{V}_{n,m}$ *such that*

$$M(Y_1, \ldots, Y_d; [W_1], \ldots, [W_d]) = \big[Y_i(W_j)\big]^t$$

is an invertible $d \times d$ *matrix and* $|T(W_j)| = 1$ *for all* $j = 1, \ldots, d$.

(c) $\exp B_{\mathcal{L}({}^n l_\infty^m)} = \operatorname{ext} B_{\mathcal{L}({}^n l_\infty^m)}$.

(d) $\exp B_{\mathcal{L}_s({}^n l_\infty^m)} = \operatorname{ext} B_{\mathcal{L}_s({}^n l_\infty^m)}$.

(e) $\operatorname{ext} B_{\mathcal{L}_s({}^n l_\infty^m)} = \operatorname{ext} B_{\mathcal{L}({}^n l_\infty^m)} \bigcap \mathcal{L}_s({}^n l_\infty^m)$.

(f) $\exp B_{\mathcal{L}_s({}^n l_\infty^m)} = \exp B_{\mathcal{L}({}^n l_\infty^m)} \bigcap \mathcal{L}_s({}^n l_\infty^m)$.

Proof. (a) It follows from Theorem C.

(b) It follows from Theorem G.

(c) It follows from Theorem E.

(d) It follows from Theorem H.

(e) and (f) It follows from Theorem I.

Kim [19, 20] characterize $\operatorname{ext} B_{\mathcal{L}({}^n \mathbb{R}^m_{\|\cdot\|})}$, $\operatorname{ext} B_{\mathcal{L}_s({}^n \mathbb{R}^m_{\|\cdot\|})}$, $\exp B_{\mathcal{L}({}^n \mathbb{R}^m_{\|\cdot\|})}$, and $\exp B_{\mathcal{L}_s({}^n \mathbb{R}^m_{\|\cdot\|})}$ if $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| \geq 2m$ for $m \geq 2$. It is shown that every extreme point is exposed in this case. Note that we extend the results of [19, 20].

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