DOI: 10.3842/umzh.v76i7.7507

UDC 517.5

Sefa Anıl Sezer (Department of Mathematics, Istanbul Medeniyet University, Turkey),
İbrahim Çanak[1] (Department of Mathematics, Ege University, Izmir, Turkey)

CONDITIONS UNDER WHICH THE CONVERGENCE OF A SEQUENCE OR ITS CERTAIN SUBSEQUENCES FOLLOWS FROM THE SUMMABILITY BY DEFERRED WEIGHTED MEANS

УМОВИ, ЗА ЯКИХ ЗБІЖНІСТЬ ПОСЛІДОВНОСТІ АБО ДЕЯКИХ ЇЇ ПІДПОСЛІДОВНОСТЕЙ ВИПЛИВАЄ З СУМОВНОСТІ ЗА ВІДКЛАДЕНИМИ ВАГОВИМИ СЕРЕДНІМИ

Let (u_k) be a sequence of real or complex numbers. First, we consider a real sequence (u_k) and formulate one-sided Tauberian conditions, which are necessary and sufficient for the convergence of certain subsequences of (u_k) to follow from its deferred weighted summability. These conditions are satisfied if (u_k) is deferred slowly decreasing or if (u_k) obeys a Landau-type Tauberian condition. Second, we consider a complex sequence (u_k) and present a two-sided Tauberian condition which is necessary and sufficient in order that the convergence of certain subsequences of (u_k) follow from its deferred weighted summability. This condition is satisfied either if (u_k) is deferred slowly oscillating or if (u_k) obeys a Hardy-type Tauberian condition. Finally, we extend these results to sequences in ordered linear spaces over the real numbers.

Нехай (u_k) — послідовність дійсних або комплексних чисел. По-перше, ми розглядаємо дійсну послідовність (u_k) і наводимо односторонні тауберові умови, що є необхідними і достатніми для того, щоб збіжність певних підпослідовностей послідовності (u_k) випливала з її відкладеної вагової сумовності. Ці умови виконуються, якщо (u_k) є відкладено повільно спадною або якщо (u_k) задовольняє тауберові умови типу Ландау. По-друге, ми розглядаємо комплексну послідовність (u_k) і наводимо двосторонню тауберову умову, необхідну і достатню для того, щоб збіжність певних підпослідовностей послідовності (u_k) випливала з її відкладеної вагової сумовності. Ця умова виконується, якщо (u_k) є відкладено повільно осцилюючою або якщо (u_k) задовольняє тауберову умову типу Гарді. Насамкінець ми поширюємо ці результати на послідовності у впорядкованих лінійних просторах над дійсними числами.

1. Introduction. Let (t_k) be a sequence of nonnegative real numbers such that $\liminf_{k\to\infty} t_k > 0$ and

$$T_k^{\alpha\beta} := \sum_{j\in(\alpha_k,\beta_k]} t_j \to \infty, \quad k\to\infty, \tag{1}$$

where (α_k) and (β_k) are sequences of nonnegative integers satisfying

$$\alpha_k < \beta_k, \quad k = 1,2,\ldots, \tag{2}$$

and

$$\lim_{k\to\infty} \beta_k = \infty. \tag{3}$$

Given any sequence (u_k), its deferred weighted means $\sigma_k^{\alpha\beta}(u)$ are defined by

$$\sigma_k^{\alpha\beta}(u) = \frac{1}{T_k^{\alpha\beta}} \sum_{j\in(\alpha_k,\beta_k]} t_j u_j, \quad k = 1,2,\ldots . \tag{4}$$

[1] Corresponding author, e-mail: ibrahim.canak@ege.edu.tr.

A sequence (u_k) is said to be deferred weighted summable to ℓ if

$$\lim_{k\to\infty} \sigma_k^{\alpha\beta}(u) = \ell. \tag{5}$$

Obviously, in the case of $t_k = 1$ for all k, (4) reduces to the deferred Cesàro mean introduced by Agnew [2].

In recent years, deferred Cesàro and deferred weighted means have been extensively studied due to their applications in several areas, such as summability and approximation theory. Considering the concepts of deferred Cesàro means and statistical convergence, Küçükaslan and Yılmaztürk [14] introduced the deferred statistical convergence of sequences. Savaş [24] defined the concept of asymptotically deferred statistical equivalence of sequences, by combining the deferred density and the asymptotically statistical equivalence of sequences. Taking different kinds of statistical convergence into account, such as pointwise, uniform, and equistatistical convergence, for a sequence of real-valued functions, Saini et al. [23] obtained various deductive conclusions and proved an approximation theorem based on the deferred Cesàro and the deferred Euler statistical convergence. Later, Jena and Paikray [7] established some fundamental theorems associated with the concepts of statistical Riemann integrability, statistical Cauchy – Riemann integrability, deferred Cesàro statistical Riemann integrability, and statistical deferred Cesàro – Riemann summability for a sequence of real-valued functions. Besides, Jena et al. [6] introduced the statistical deferred Cesàro summability of sequences and utilized it to obtain Korovkin-type approximation theorems. Later, Srivastava et al. [27] presented the deferred weighted statistical convergence and the deferred weighted statistical summability methods and demonstrated similar approximation theorems. Recently, Sezer et al. [25, 26] proved some Tauberian theorems for the deferred Cesàro summability and the statistical deferred Cesàro summability of sequences, respectively. Furthermore, researchers investigated deferred means in various fields of mathematics, including Fourier series [9], sequence spaces of random variables [28], time scales [3], paranormed spaces [4], and sequence spaces of complex uncertain variables [12]. For more recent studies in this direction, see [8, 11, 21].

The deferred weighted summability method is regular under the conditions (1) – (3). That is, if (1) – (3) are fulfilled, then the convergence of a sequence (u_k) implies the convergence of deferred weighted means to the same limit.

In this paper, we are interested in the inverse implication. We investigate supplementary conditions on (t_k) and (u_k) under which the convergence of $\left(\sigma_k^{\alpha\beta}(u)\right)$ implies that of (u_k) or its certain subsequences. Such kind of conditions are called Tauberian conditions and results in this direction are known as Tauberian theorems. We introduce one- and two-sided conditions for real and complex sequences, respectively. The techniques used in this study also allow us to extend our Tauberian results to sequences in ordered linear spaces.

The reader concerned with the theory of summability and Tauberian theorems is referred to the monographs [1, 13, 19].

2. Auxiliary results. This section includes some lemmas that are required in the proofs of the main results.

In the sequel, let (β_k) be strictly increasing and define $\gamma_k := [\gamma k]$ for $\gamma > 0$, where $[.]$ means the integer part.

Lemma 1. (i) *If* $\gamma > 1$ *and* k *is large enough such that* $T_k^{\alpha\beta_\gamma} > T_k^{\alpha\beta}$, *then*

$$u_{\beta_k} - \sigma_k^{\alpha\beta}(u) = \frac{T_k^{\alpha\beta_\gamma}}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \big(\sigma_k^{\alpha\beta_\gamma}(u) - \sigma_k^{\alpha\beta}(u)\big) - \frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_k}]} t_j(u_j - u_{\beta_k}). \tag{6}$$

(ii) *If* $0 < \gamma < 1$ *and* k *is large enough such that* $T_k^{\alpha\beta} > T_k^{\alpha\beta_\gamma}$, *then*

$$u_{\beta_k} - \sigma_k^{\alpha_\gamma\beta_\gamma}(u) = \frac{T_k^{\alpha_\gamma\beta}}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \big(\sigma_k^{\alpha_\gamma\beta}(u) - \sigma_k^{\alpha_\gamma\beta_\gamma}(u)\big) + \frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j\in(\beta_{\gamma_k},\beta_k]} t_j(u_{\beta_k} - u_j). \tag{7}$$

Proof. (i) From the definition in (4), we have

$$\sigma_k^{\alpha\beta_\gamma}(u) - \sigma_k^{\alpha\beta}(u) = \frac{1}{T_k^{\alpha\beta_\gamma}} \sum_{j\in(\alpha_k,\beta_{\gamma_k}]} t_j u_j - \frac{1}{T_k^{\alpha\beta}} \sum_{j\in(\alpha_k,\beta_k]} t_j u_j$$

$$= \frac{1}{T_k^{\alpha\beta_\gamma}} \sum_{j\in(\beta_k,\beta_{\gamma_k}]} t_j u_j - \frac{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}}{T_k^{\alpha\beta_\gamma}} \frac{1}{T_k^{\alpha\beta}} \sum_{j\in(\alpha_k,\beta_k]} t_j u_j.$$

Hence,

$$\frac{T_k^{\alpha\beta_\gamma}}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \big(\sigma_k^{\alpha\beta_\gamma}(u) - \sigma_k^{\alpha\beta}(u)\big) - \frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_k}]} t_j u_j = -\sigma_k^{\alpha\beta}(u),$$

accordingly (6) follows.

(ii) The proof of (7) is similar to that of (6).

***Remark* 1.** It is worth noting that the sequence (P_k) is defined as follows:

$$P_k := \sum_{j\in(0,k]} t_j.$$

Clearly, (P_k) is a nondecreasing positive sequence, and further $T_k^{\alpha\beta}$ can be written as

$$T_k^{\alpha\beta} = P_{\beta_k} - P_{\alpha_k}.$$

Lemma 2. *For the sequence* (P_k), *the assertion*

$$\liminf_{k\to\infty} \frac{P_{\beta_{\gamma_k}}}{P_{\beta_k}} > 1 \quad \textit{for all} \quad \gamma > 1 \tag{8}$$

is equivalent with

$$\liminf_{k\to\infty} \frac{P_{\beta_k}}{P_{\beta_{\gamma_k}}} > 1 \quad \textit{for all} \quad 0 < \gamma < 1. \tag{9}$$

Proof. Let (8) holds and $0 < \gamma < 1$ be given. Take $r := \gamma_k = [\gamma k]$. Certainly, $\dfrac{1}{\gamma} > 1$ and $\dfrac{r}{\gamma} = \dfrac{[\gamma k]}{\gamma} \le k$, accordingly it follows

$$\beta_{[\frac{r}{\gamma}]} \leq \beta_k,$$

and so

$$\frac{P_{\beta_k}}{P_{\beta_{\gamma_k}}} \geq \frac{P_{\beta_{[\frac{r}{\gamma}]}}}{P_{\beta_r}}.$$

Hence, we have

$$\liminf_{k\to\infty} \frac{P_{\beta_k}}{P_{\beta_{\gamma_k}}} \geq \liminf_{k\to\infty} \frac{P_{\beta_{[\frac{r}{\gamma}]}}}{P_{\beta_r}} > 1.$$

Conversely, let (9) holds and $\gamma > 1$ be given. Take ρ such that $1 < \rho < \gamma$ and $r := \gamma_k = [\gamma k]$. Since $0 < \dfrac{1}{\gamma} < \dfrac{1}{\rho} < 1$, we obtain

$$k \leq \frac{\gamma k - 1}{\rho} < \frac{[\gamma k]}{\rho} = \frac{r}{\rho}$$

provided that $\rho \leq \gamma - \dfrac{1}{k}$, whenever k is necessarily large. For such k, it follows

$$\beta_k \leq \beta_{[\frac{r}{\rho}]},$$

and then

$$\frac{P_{\beta_{\gamma_k}}}{P_{\beta_k}} \geq \frac{P_{\beta_r}}{P_{[\frac{r}{\rho}]}}.$$

Therefore, we conclude

$$\liminf_{k\to\infty} \frac{P_{\beta_{\gamma_k}}}{P_{\beta_k}} \geq \liminf_{k\to\infty} \frac{P_{\beta_r}}{P_{[\frac{r}{\rho}]}} > 1.$$

3. Main results. First, we consider the case where (u_k) is a sequence of real numbers and prove the following Tauberian theorem under one-sided conditions.

Theorem 1. *Assume that* (8) *holds. If a real sequence* (u_k) *is deferred weighted summable to a finite number* ℓ, *then*

$$\lim_{k\to\infty} u_{\beta_k} = \ell \tag{10}$$

if and only if

$$\limsup_{\gamma\downarrow 1} \liminf_{k\to\infty} \frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_k}]} t_j(u_j - u_{\beta_k}) \geq 0 \tag{11}$$

and

$$\limsup_{\gamma\uparrow 1} \liminf_{k\to\infty} \frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j\in(\beta_{\gamma_k},\beta_k]} t_j(u_{\beta_k} - u_j) \geq 0. \tag{12}$$

In that case we necessarily have

$$\lim_{k\to\infty} \frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_k}]} t_j(u_j - u_{\beta_k}) = 0 \tag{13}$$

for every $\gamma > 1$ *and*

$$\lim_{k\to\infty} \frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j\in(\beta_{\gamma_k},\beta_k]} t_j(u_{\beta_k} - u_j) = 0 \tag{14}$$

for every $0 < \gamma < 1$.

At this point, a few remarks are in order.

***Remark* 2.** A real sequence (u_k) is said to be deferred slowly decreasing [25] if

$$\lim_{\gamma \downarrow 1} \liminf_{k\to\infty} \min_{\beta_k < j \le \beta_{\gamma_k}} (u_j - u_{\beta_k}) \ge 0. \tag{15}$$

To put it another way, (15) means that, for any $\epsilon > 0$, there exist $\gamma = \gamma(\epsilon) > 1$ and $k_1 = k_1(\epsilon) > 0$ satisfying $u_j - u_{\beta_k} \ge -\epsilon$, whenever $k_1 \le k$ and $\beta_k < j \le \beta_{\gamma_k}$.

The notion of “slow decrease” was first introduced by Schmidt [22] (see also [16]).

(15) can be reformulated equivalently as follows:

$$\lim_{\gamma \uparrow 1} \liminf_{k\to\infty} \min_{\beta_{\gamma_k} < j \le \beta_k} (u_{\beta_k} - u_j) \ge 0. \tag{16}$$

It is clear that if a real sequence (u_n) is deferred slowly decreasing, then (11) and (12) are satisfied. Then the next result is an obvious consequence of Theorem 1.

Theorem 2. *Assume that* (8) *holds and a real sequence* (u_k) *is deferred slowly decreasing. If* (u_k) *is deferred weighted summable to a finite number* ℓ, *then* (u_k) *is convergent to* ℓ.

***Remark* 3.** A positive sequence (β_k) is said to be regularly varying of index $\rho \in \mathbb{R}$ in the sense of Karamata [10] if

$$\lim_{k\to\infty} \frac{\beta_{\gamma_k}}{\beta_k} = \gamma^\rho \quad \text{for each} \quad \gamma > 0.$$

Clearly, if the one-sided Tauberian condition

$$k(u_k - u_{k-1}) \ge -C$$

of Landau [15] is satisfied for some $C > 0$ and every $k = 1, 2, \ldots,$ then (15) is definitely satisfied, as long as (β_k) is regularly varying of index $\rho > 0$.

Second, we consider the general case where (u_k) is a sequence of complex numbers and prove the following Tauberian theorem under two-sided conditions.

Theorem 3. *Assume that* (8) *holds. If a complex sequence* (u_k) *is deferred weighted summable to a finite number* ℓ, *then* (u_{β_k}) *is convergent to* ℓ *if and only if either*

$$\liminf_{\gamma \downarrow 1} \limsup_{k\to\infty} \left| \frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j \in (\beta_k, \beta_{\gamma_k}]} t_j (u_j - u_{\beta_k}) \right| = 0 \tag{17}$$

or

$$\liminf_{\gamma \uparrow 1} \limsup_{k\to\infty} \left| \frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j \in (\beta_{\gamma_k}, \beta_k]} t_j (u_{\beta_k} - u_j) \right| = 0. \tag{18}$$

In that case we necessarily have (13) *for every* $\gamma > 1$ *and* (14) *for every* $0 < \gamma < 1$.

Now, we can make similar discussions as in the real case above.

***Remark* 4.** A complex sequence (u_k) is said to be deferred slowly oscillating [25] if

$$\lim_{\gamma \downarrow 1} \limsup_{k\to\infty} \max_{\beta_k < j \le \beta_{\gamma_k}} |u_j - u_{\beta_k}| = 0. \tag{19}$$

In other saying, (19) means that, for any $\epsilon > 0$, there exist $\gamma = \gamma(\epsilon) > 1$ and $k_1 = k_1(\epsilon) > 0$ satisfying $|u_j - u_{\beta_k}| \le \epsilon$, whenever $k_1 \le k$ and $\beta_k < j \le \beta_{\gamma_k}$.

The notion of "slow oscillation" dates back to Schmidt [22] (see also [16]).

(19) can be reformulated equivalently as follows:

$$\lim_{\gamma\uparrow 1}\limsup_{k\to\infty}\min_{\beta_{\gamma_k}<j\le\beta_k}|u_{\beta_k}-u_j|=0. \tag{20}$$

It is easy to check that if a complex sequence (u_n) is deferred slowly oscillating, then (17) and (18) are satisfied. Hence, the next result is an immediate consequence of Theorem 3.

Theorem 4. *Assume that* (8) *holds and a complex sequence* (u_k) *is deferred slowly oscillating. If* (u_k) *is deferred weighted summable to a finite number* ℓ, *then* (u_k) *is convergent to* ℓ.

Remark **5.** Apparently, if the two-sided Tauberian condition

$$k|u_k-u_{k-1}|\le C$$

of Hardy [5] is satisfied for some $C>0$ and every $k=1,2,\ldots,$ then (19) is certainly satisfied, as long as (β_k) is regularly varying of index $\rho>0$.

Remark **6.** Special cases.

(i) In the case $\alpha_k=1,\ \beta_k=k$ for each $k\in\mathbb{N}$, then we have Tauberian theorems for the weighted mean summability given in [17].

(ii) In the case $t_k=1,\ \alpha_k=1,\ \beta_k=k$ for each $k\in\mathbb{N}$, then we obtain Tauberian theorems for the $(C,1)$ summability given in [16].

(iii) In the case $t_k=1/k,\ \alpha_k=1,\ \beta_k=k$ for each $k\in\mathbb{N}$, then we get Tauberian theorems for the $(L,1)$ summability given in [18].

(iv) In the case $t_k=1$ for each $k\in\mathbb{N}$, then we have Tauberian theorems for the deferred Cesàro summability given in [25].

Moreover, we characterize the convergence of deferred weighted moving averages via the following theorem. It states that if (5) and (8) are satisfied, then the so-called deferred weighted moving averages of a complex sequence (u_k) also converges to its deferred weighted sum.

Theorem 5. *Assume that* (8) *holds. If a complex sequence* (u_k) *is deferred weighted summable to a finite number* ℓ, *then, for any* $\gamma>1$,

$$\lim_{k\to\infty}\frac{1}{T_k^{\alpha\beta_\gamma}-T_k^{\alpha\beta}}\sum_{j\in(\beta_k,\beta_{\gamma_k}]}t_ju_j=\ell \tag{21}$$

and, for any $0<\gamma<1$,

$$\lim_{k\to\infty}\frac{1}{T_k^{\alpha\beta}-T_k^{\alpha\beta_\gamma}}\sum_{j\in(\beta_{\gamma_k},\beta_k]}t_ju_j=\ell. \tag{22}$$

4. Proofs. ***Proof of Theorem*** **1.** *Necessity*. By the fulfillment of (5) and (10), we obtain

$$\lim_{k\to\infty}\left(u_{\beta_k}-\sigma_k^{\alpha\beta}(u)\right)=0. \tag{23}$$

Considering (8), we have

$$\limsup_{k\to\infty}\frac{T_k^{\alpha\beta_\gamma}}{T_k^{\alpha\beta_\gamma}-T_k^{\alpha\beta}}\le\limsup_{k\to\infty}\frac{P_{\beta_{\gamma_k}}}{P_{\beta_{\gamma_k}}-P_{\beta_k}}$$

$$= \left\{ \liminf_{k\to\infty} \left(1 - \frac{P_{\beta_k}}{P_{\beta_{\gamma_k}}} \right) \right\}^{-1} = \left\{ 1 - \left(\liminf_{k\to\infty} \frac{P_{\beta_{\gamma_k}}}{P_{\beta_k}} \right)^{-1} \right\}^{-1} < \infty. \tag{24}$$

It follows from (5) and (24) that

$$\lim_{k\to\infty} \frac{T_k^{\alpha\beta_\gamma}}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \big(\sigma_k^{\alpha\beta_\gamma}(u) - \sigma_k^{\alpha\beta}(u) \big) = 0 \tag{25}$$

for all $\gamma > 1$. The same is valid for any fixed $0 < \gamma < 1$. Then, (13) (respectively, (14)) follows from (6) (respectively, (7)), (23) and (25).

Sufficiency. Suppose that the fulfillment of (5), (11) and (12). By (11), there exists a sequence $\gamma_s \downarrow 1$ such that

$$\lim_{s\to\infty} \liminf_{k\to\infty} \frac{1}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_{sk}}]} t_j(u_j - u_{\beta_k}) \geq 0, \tag{26}$$

where $\gamma_{sk} := [\gamma_s k]$.

We observe via (6) that

$$\limsup_{k\to\infty} \big(u_{\beta_k} - \sigma_k^{\alpha\beta}(u) \big) \leq \lim_{s\to\infty} \limsup_{k\to\infty} \frac{T_k^{\alpha\beta_{\gamma_s}}}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \big(\sigma_k^{\alpha\beta_{\gamma_s}}(u) - \sigma_k^{\alpha\beta}(u) \big)$$

$$+ \lim_{s\to\infty} \limsup_{k\to\infty} \left(- \frac{1}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_{sk}}]} t_j(u_j - u_{\beta_k}) \right).$$

Taking (5), (25) and (26) into account, we find

$$\limsup_{k\to\infty} \big(u_{\beta_k} - \sigma_k^{\alpha\beta}(u) \big) \leq - \lim_{s\to\infty} \liminf_{k\to\infty} \left(\frac{1}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k,\beta_{\gamma_{sk}}]} t_j(u_j - u_{\beta_k}) \right) \leq 0. \tag{27}$$

It follows from (12) that, for any sequence $\gamma_s \uparrow 1$,

$$\lim_{s\to\infty} \liminf_{k\to\infty} \frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j\in(\beta_{\gamma_{sk}},\beta_k]} t_j(u_{\beta_k} - u_j) \geq 0.$$

Following similar steps as above, we conclude

$$\liminf_{k\to\infty} \big(u_{\beta_k} - \sigma_k^{\alpha_{\gamma_s}\beta_{\gamma_s}}(u) \big) \geq \lim_{s\to\infty} \liminf_{k\to\infty} \frac{T_k^{\alpha_{\gamma_s}\beta}}{T_k^{\alpha\beta} - T_k^{\alpha\beta_{\gamma_s}}} \big(\sigma_k^{\alpha_{\gamma_s}\beta}(u) - \sigma_k^{\alpha_{\gamma_s}\beta_{\gamma_s}}(u) \big)$$

$$+ \lim_{s\to\infty} \liminf_{k\to\infty} \frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j\in(\beta_{\gamma_{sk}},\beta_k]} t_j(u_{\beta_k} - u_j) \geq 0. \tag{28}$$

Combining (27) and (28) gives (23), which is equivalent to (10) due to (5).

***Proof of Theorem* 2.** Let (15) holds, then so does (16). It is plain that (15) and (16) imply (11) and (12), respectively. Hence, from Theorem 1, we have $u_{\beta_k} \to \ell$. That is, for any $\epsilon > 0$, there exists $K = K(\epsilon) > 0$ such that, for all $k \geq K$,

$$-\frac{\epsilon}{2} \leq u_{\beta_k} - \ell \leq \frac{\epsilon}{2}. \tag{29}$$

It is obtained from (15) that, for any $\epsilon > 0$, there exist $k_0 = k_0(\epsilon) > 0$ and $\gamma = \gamma(\epsilon) > 1$ such that, for all $k \geq k_0$ and $\beta_k < j \leq \beta_{\gamma_k}$,

$$u_j - u_{\beta_k} \geq -\frac{\epsilon}{2}. \tag{30}$$

Besides, it follows from (16) that, for given $\epsilon > 0$, there exist $k_1 = k_1(\epsilon) > 0$ and $0 < \gamma = \gamma(\epsilon) < 1$ such that, for all $k \geq k_1$ and $\beta_{\gamma_k} < j \leq \beta_k$,

$$u_{\beta_k} - u_j \geq -\frac{\epsilon}{2}. \tag{31}$$

Considering (29) and (30), we obtain

$$u_j - \ell = u_j - u_{\beta_k} + u_{\beta_k} - \ell \geq -\epsilon \tag{32}$$

for all $j \geq k \geq K_1 = \max\{k_0, K\}$.

On the other hand, considering (29) and (31), we have

$$u_j - \ell = u_j - u_{\beta_k} + u_{\beta_k} - \ell \leq \epsilon \tag{33}$$

for all $j \geq k \geq K_2 = \max\{k_1, K\}$.

Therefore, using (32) and (33), we conclude

$$-\epsilon \leq u_j - \ell \leq \epsilon$$

for all $j \geq k \geq K_3 = \max\{K_1, K_2\}$, which completes the proof.

***Proof of Theorem* 3.** *Necessity.* Let (5) and (10) be satisfied. Analogously to the proof of the necessity part of Theorem 1, we derive (17) if $\gamma > 1$ and (18) if $0 < \gamma < 1$.

Sufficiency. Suppose that the fulfillment of (5) and (17). From (17), there exists a sequence $\gamma_s \downarrow 1$ such that

$$\lim_{s\to\infty} \limsup_{k\to\infty} \left| \frac{1}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k, \beta_{\gamma_{sk}}]} t_j(u_j - u_{\beta_k}) \right| = 0. \tag{34}$$

Since (6), we obtain

$$\limsup_{k\to\infty} \left| u_{\beta_k} - \sigma_k^{\alpha\beta}(u) \right| \leq \lim_{s\to\infty} \limsup_{k\to\infty} \frac{T_k^{\alpha\beta_{\gamma_s}}}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \left| \sigma_k^{\alpha\beta_{\gamma_s}}(u) - \sigma_k^{\alpha\beta}(u) \right|$$

$$+ \lim_{s\to\infty} \limsup_{k\to\infty} \left| \frac{1}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k, \beta_{\gamma_{sk}}]} t_j(u_j - u_{\beta_k}) \right|.$$

Taking (5), (25) and (34) into account, we have

$$\limsup_{k\to\infty} \left| u_{\beta_k} - \sigma_k^{\alpha\beta}(u) \right| \leq \lim_{s\to\infty} \limsup_{k\to\infty} \left| \frac{1}{T_k^{\alpha\beta_{\gamma_s}} - T_k^{\alpha\beta}} \sum_{j\in(\beta_k, \beta_{\gamma_{sk}}]} t_j(u_j - u_{\beta_k}) \right| = 0,$$

which implies $u_{\beta_k} \to \ell$.

A similar proof may be performed in the case (18) is provided.

***Proof of Theorem* 4.** Considering (19) together with Theorem 2, we have $u_{\beta_k} \to \ell$. That is, for any $\epsilon > 0$, there exists $K = K(\epsilon) > 0$ such that, for all $k \geq K$,

$$|u_{\beta_k} - \ell| \leq \frac{\epsilon}{2}. \tag{35}$$

It follows from (19) that, for any $\epsilon > 0$, there exist $k_0 = k_0(\epsilon) > 0$ and $\gamma = \gamma(\epsilon) > 1$ such that, for all $k \geq k_0$ and $\beta_k < j \leq \beta_{\gamma_k}$,

$$|u_j - u_{\beta_k}| \leq \frac{\epsilon}{2}. \tag{36}$$

Combining (35) and (36) yields

$$|u_j - \ell| \leq |u_j - u_{\beta_k}| + |u_{\beta_k} - \ell| \leq \epsilon$$

for all $j \geq k \geq K_1 = \max\{k_0, K\}$.

A similar proof can be given if (20) is provided.

***Proof of Theorem* 5.** Assume that $\gamma > 1$ and k is necessarily large in the sense that $T_k^{\alpha\beta_\gamma} > T_k^{\alpha\beta}$. Hence, by Lemma 1 (i),

$$\frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j \in (\beta_k, \beta_{\gamma_k}]} t_j u_j = \sigma_k^{\alpha\beta}(u) + \frac{T_k^{\alpha\beta_\gamma}}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \big(\sigma_k^{\alpha\beta_\gamma}(u) - \sigma_k^{\alpha\beta}(u)\big).$$

Then, using (24) and the deferred weighted summability of (u_k), we deduce (21).

Assume that $0 < \gamma < 1$ and k is necessarily large in the sense that $T_k^{\alpha\beta} > T_k^{\alpha\beta_\gamma}$. Thus, from Lemma 1 (ii),

$$\frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j \in (\beta_{\gamma_k}, \beta_k]} t_j u_j = \sigma_k^{\alpha_\gamma\beta_\gamma}(u) + \frac{T_k^{\alpha_\gamma\beta}}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \big(\sigma_k^{\alpha_\gamma\beta}(u) - \sigma_k^{\alpha_\gamma\beta_\gamma}(u)\big).$$

Since (9) holds, we also have

$$\begin{aligned} \limsup_{k\to\infty} \frac{T_k^{\alpha_\gamma\beta}}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} &\leq \limsup_{k\to\infty} \frac{P_{\beta_{\gamma_k}}}{P_{\beta_k} - P_{\beta_{\gamma_k}}} \\ &= \left\{1 - \left(\liminf_{k\to\infty} \frac{P_{\beta_k}}{P_{\beta_{\gamma_k}}}\right)^{-1}\right\}^{-1} < \infty. \end{aligned} \tag{37}$$

Therefore, taking (37) and the deferred weighted summability of (u_k) into account, we establish (22).

5. Extension to ordered linear spaces. The theory developed in the Section 3 can be extended to sequences in ordered linear spaces.

Let $(X, \leq)$ be an ordered linear space over $\mathbb{R}$, in which with 0 we represent the zero of X and by τ, a given nonnegative element in X. We remind the definition below from [20].

A sequence (u_k) in X is called convergent to $\ell \in X$, relative to τ, if, for every $\epsilon > 0$, there exists $k_0 > 0$ such that

$$-\epsilon\tau \leq u_k - \ell \leq \epsilon\tau, \quad \text{whenever} \quad k \geq k_0.$$

Following Maddox [20], we say that (u_k) in X is deferred slowly decreasing, relative to τ, if, for every $\epsilon > 0$, there exists $k_0 > 0$ and $\gamma > 1$ such that

$$u_j - u_{\beta_k} \geq -\epsilon\tau, \quad \text{whenever} \quad k \geq k_0 \quad \text{and} \quad \beta_k < j \leq \beta_{\gamma_k}.$$

The deferred slow decrease property for $0 < \gamma < 1$ relative to τ is defined in a similar way. Based on the above concepts, we can derive the following generalizations of [16].

Theorem 6. *Assume that* (8) *holds. If* $\left(\sigma_k^{\alpha\beta}(u)\right)$ *converges to* $\ell \in X$, *relative to* τ, *then* (u_{β_k}) *converges to* ℓ, *relative to* τ, *if and only if the following two conditions are satisfied*:

for every $\epsilon > 0$, *there exists* $k_0 > 0$ *and* $\gamma > 1$ *such that, for all* $k \geq k_0$,

$$\frac{1}{T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}} \sum_{j \in (\beta_k, \beta_{\gamma_k}]} t_j (u_j - u_{\beta_k}) \geq -\epsilon\tau,$$

for every $\epsilon > 0$, *there exists* $k_0 > 0$ *and* $0 < \gamma < 1$ *such that, for all* $k \geq k_0$,

$$\frac{1}{T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}} \sum_{j \in (\beta_{\gamma_k}, \beta_k]} t_j (u_{\beta_k} - u_j) \geq -\epsilon\tau.$$

In that case the sequences $\left(\left(T_k^{\alpha\beta_\gamma} - T_k^{\alpha\beta}\right)^{-1} \sum_{j \in (\beta_k, \beta_{\gamma_k}]} t_j (u_j - u_{\beta_k})\right)$ *for every* $\gamma > 1$ *and* $\left(\left(T_k^{\alpha\beta} - T_k^{\alpha\beta_\gamma}\right)^{-1} \sum_{j \in (\beta_{\gamma_k}, \beta_k]} t_j (u_{\beta_k} - u_j)\right)$ *for every* $0 < \gamma < 1$, *necessarily converge to* $0 \in X$, *relative to* τ.

Theorem 7. *Assume that* (8) *holds and* (u_k) *is deferred slowly decreasing relative to* τ. *If* $\left(\sigma_k^{\alpha\beta}(u)\right)$ *converges to* $\ell \in X$, *relative to* τ, *then* (u_k) *converges to* ℓ *relative to* τ.

6. Conclusion. We established necessary and sufficient Tauberian conditions for sequences of real or complex numbers that are summable by deferred weighted mean methods. The main results of this paper are applicable to all such methods and provide a unifying framework for the results in the literature for specific methods, such as weighted mean summability [17], $(C, 1)$ summability [16], $(L, 1)$ summability [18], and deferred Cesàro summability [25]. We observed that the conditions in our theorems are consequences of the deferred slowly decreasing condition for sequences of real numbers, or the deferred slowly oscillating condition for sequences of complex numbers. Therefore, we obtained classical one-sided Landau-type as well as two-sided Hardy-type Tauberian conditions as corollaries of our main theorems. Finally, we introduced a general Tauberian theorem for sequences in ordered linear spaces.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

Funding. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Author contributions. All authors have contributed equally to the work.

References

1. A. Aasma, H. Dutta, P. N. Natarajan, *An introductory course in summability theory*, John Wiley & Sons, Hoboken, NJ (2017).
2. R. P. Agnew, *On deferred Cesàro means*, Ann. Math., **33**, 413 – 421 (1932).

3. M. Çınar, E. Yılmaz, M. Et, *Deferred statistical convergence on time scales*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci., **22**, 301 – 307 (2021).
4. S. Ercan, *On deferred Cesàro mean in paranormed spaces*, Korean J. Math., **29**, 169 – 177 (2021).
5. G. H. Hardy, *Theorems relating to the summability and convergence of slowly oscillating series*, Proc. London Math. Soc., **8**, 301 – 320 (1910).
6. B. B. Jena, S. K. Paikray, U. Misra, *Statistical deferred Cesàro summability and its applications to approximation theorems*, Filomat, **32**, 2307 – 2319 (2018).
7. B. B. Jena, S. K. Paikray, H. Dutta, *Statistically Riemann integrable and summable sequence of functions via deferred Cesàro mean*, Bull. Iran. Math. Soc., **48**, 1293 – 1309 (2022).
8. B. B. Jena, S. K. Paikray, *Statistical convergence of martingale difference sequence via deferred weighted mean and Korovkin-type theorems*, Miskolc Math. Notes, **22**, 273 – 286 (2021).
9. B. B. Jena, S. K. Paikray, M. Mursaleen, *On the degree of approximation of Fourier series based on a certain class of product deferred summability means*, J. Inequal. and Appl., **2023**, Article 18 (2023).
10. J. Karamata, *Sur un mode de croissance régulière, Théorèmes fondamentaux*, Bull. Soc. Math. France, **61**, 55 – 62 (1933).
11. V. A. Khan, B. Hazarika, I. A. Khan, U. Tuba, *I-deferred strongly Cesàro summable and μ-deferred I-statistically convergent sequence spaces*, Ric. Mat. (2021); https://doi.org/10.1007/s11587-021-00619-8.
12. V. A. Khan, B. Hazarika, I. A. Khan, Z. Rahman, *A study on I-deferred strongly Cesàro summable and μ-deferred I-statistical convergence for complex uncertain sequences*, Filomat, **36**, 7001 – 7020 (2022).
13. J. Korevaar, *Tauberian theory: a century of developments*, Springer-Verlag, Berlin (2004).
14. M. Küçükaslan, M. Yılmaztürk, *On deferred statistical convergence of sequences*, Kyungpook Math. J., **56**, 357 – 366 (2016).
15. E. Landau, *Über die Bedeutung einer neuerer Grenzwertss der Herren Hardy und Axel*, Prac. Mat.-Fiz., **21**, 97 – 177 (1910).
16. F. Móricz, *Necessary and sufficient Tauberian conditions, under which convergence follows from summability $(C, 1)$*, Bull. London Math. Soc., **26**, 288 – 294 (1994).
17. F. Móricz, B. E. Rhoades, *Necessary and sufficient Tauberian conditions for certain weighted mean methods of summability II*, Acta Math. Hungar., **102**, 279 – 285 (2004).
18. F. Móricz, *Necessary and sufficient Tauberian conditions for the logarithmic summability of functions and sequences*, Stud. Math., **219**, 109 – 121 (2013).
19. M. Mursaleen, F. Başar, *Sequence spaces: topics in modern summability theory*, CRC Press, Boca Raton, FL (2020).
20. I. J. Maddox, *A Tauberian theorem for ordered spaces*, Analysis, **9**, 297 – 302 (1989).
21. K. Raj, S. A. Mohiuddine, S. Jasrotia, *Characterization of summing operators in multiplier spaces of deferred Nörlund summability*, Positivity, **27**, Article 9 (2023).
22. R. Schmidt, *Über divergente Folgen und lineare Mittelbildungen*, Math. Z., **22**, 89 – 152 (1925).
23. K. Saini, K. Raj, M. Mursaleen, *Deferred Cesàro and deferred Euler equistatistical convergence and its applications to Korovkin-type approximation theorem*, Int. J. Gen. Syst., **50**, 567 – 579 (2021).
24. R. Savaş, *Multidimensional strongly deferred invariant convergence*, Numer. Funct. Anal. and Optim., **42**, 1323 – 1333 (2021).
25. S. A. Sezer, İ. Çanak, H. Dutta, *Necessary and sufficient Tauberian conditions under which convergence follows from deferred Cesàro summability*, Filomat, **36**, 921 – 931 (2022).
26. S. A. Sezer, Z. Önder, İ. Çanak, *Statistical deferred Cesàro summability and its applications to Tauberian theory*, Bull. Iran. Math. Soc., **49**, № 2, Article 19 (2023).
27. H. M. Srivastava, B. B. Jena, S. K. Paikray, U. Misra, *A certain class of weighted statistical convergence and associated Korovkin-type approximation theorems involving trigonometric functions*, Math. Methods Appl. Sci., **41**, 671 – 683 (2018).
28. H. M. Srivastava, B. B. Jena, S. K. Paikray, *Statistical probability convergence via the deferred Nörlund mean and its applications to approximation theorems*, Rev. R. Acad. Cienc. Exactas Fıs. Nat. Ser. A Mat. RACSAM, **114**, Article 144 (2020).

Received 26.02.23