DOI: 10.3842/umzh.v76i7.7546

UDC 517.5

Babar Sultan[1] (Department of Mathematics, Quaid-I-Azam University, Islamabad, Pakistan),
Mehvish Sultan (Department of Mathematics, Capital University of Science and Technology, Islamabad, Pakistan)

SOBOLEV-TYPE THEOREM FOR COMMUTATORS OF HARDY OPERATORS IN GRAND HERZ SPACES

ТЕОРЕМА ТИПУ СОБОЛЄВА ДЛЯ КОМУТАТОРІВ ОПЕРАТОРІВ ГАРДІ У ВЕЛИКИХ ПРОСТОРАХ ГЕРЦА

The higher-order commutators of fractional Hardy-type operators of variable order $\zeta(z)$ are shown to be bounded from the grand variable Herz spaces $\dot{K}_{p(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$ into the weighted space $\dot{K}_{\rho,q(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$, where $\rho=(1+|z_1|)^{-\lambda}$ and $\dfrac{1}{q(z)}=\dfrac{1}{p(z)}-\dfrac{\zeta(z)}{n}$ if $p(z)$ is not necessarily constant at infinity.

Показано, що комутатори вищих порядків дробових операторів типу Гарді змінного порядку $\zeta(z)$ обмежені з великих змінних просторів Герца $\dot{K}_{p(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$ у ваговий простір $\dot{K}_{\rho,q(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$, де $\rho=(1+|z_1|)^{-\lambda}$ і $\dfrac{1}{q(z)}=\dfrac{1}{p(z)}-\dfrac{\zeta(z)}{n}$, якщо $p(z)$ не обов'язково є сталою на нескінченності.

1. Introduction. Let g is a nonnegative integrable function in $\mathbb{R}^+$. The classical Hardy operators are defined as

$$Hg(z):=\frac{1}{z}\int\limits_0^z g(x)dx,\quad H^*g(z):=\int\limits_x^\infty \frac{g(x)}{x}dx,\quad z>0.$$

The Hardy operators H and H^* are mutually adjoint:

$$\int\limits_0^\infty g(z)Hf(z)dz=\int\limits_0^\infty f(z)H^*g(z)dz,$$

where $f\in L^p(\mathbb{R}^+)$, $g\in L^q(\mathbb{R}^+)$ with $1<p<\infty$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Nowadays there is a vast boom of research related to both the study of the Herz spaces themselves and the operator theory in these spaces. This is caused under the influence of some possible applications in modeling with nonstandard local growth (in differential equations, fluid mechanics, elasticity theory, see, for example, [1, 2]). Boundedness of various operators has been intensively studied in the last years, and one the remarkable results was on the boundedness of the Hardy – Littlewood maximal operator in these spaces.

In [3] authors proved a Sobolev-type theorem for the potential operator $I^{\zeta(x)}$ from the Lebesgue space $L^{p(\cdot)}$ into the weighted Lebesgue space $L^{q(\cdot)}_w$ in $\mathbb{R}^n$, under the conditions that $p(x)$ is satisfying the logarithmic condition locally and at infinity. It was not only supposed that $p(x)$ is constant at infinity but also assumed that $p(x)$ took its minimal value at infinity.

[1] Corresponding author, e-mail: babarsultan40@yahoo.com.

In [4], remarkable results were proved in which authors discussed the boundedness of sublinear operators on homogeneous Herz spaces $K^{\zeta(\cdot),u}_{v(\cdot)}(\mathbb{R}^n)$ and nonhomogeneous Herz spaces $\dot{K}^{\zeta(\cdot),u}_{v(\cdot)}(\mathbb{R}^n)$ with variable exponents $\zeta(\cdot)$ and $v(\cdot)$. Meanwhile, they also established Hardy – Littlewood Sobolev theorems for fractional integrals on Herz spaces with variable exponent. In [5], authors considered the Herz – Morrey spaces $M\dot{K}^{\zeta(\cdot),\lambda}_{u,v(\cdot)}(\mathbb{R}^n)$ with variable exponent and investigated mapping properties for the fractional Hardy-type operators in these spaces.

Grand variable Herz spaces was introduced in [6] and proved boundedness for sublinear operators in these spaces. Sultan el al. in [7] introduced the idea of grand variable Herz – Morrey spaces and obtained boundedness of Riesz potential operator in these spaces. Boundedness of other integral operators on grand function spaces can be seen in [8 – 12]. Inspired by the concept, in this paper we will demonstrate the boundedness of higher order commutators of the fractional Hardy-type operators of variable order from grand Herz spaces to weighted space under some proper assumptions on weights.

We divided this paper into different sections. Apart from introduction, a section is dedicated to basic lemmas and definitions. One section is for Sobolev-type theorem for the fractional Hardy-type operators of variable order in grand variable Herz spaces.

2. Preliminaries. For this section we refer to [13 – 17] unless and until stated otherwise.

2.1. Lebesgue space with variable exponent.

Definition 2.1. *If H is a measurable set in $\mathbb{R}^n$ and $p(\cdot): H \to [1,\infty)$ is a measurable function.*

(a) *Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ is defined as*

$$L^{p(\cdot)}(H) = \left\{ f \ \textit{measurable}: \int\limits_H \left(\frac{|f(y)|}{\gamma}\right)^{p(y)} dy < \infty \ \textit{ where } \ \gamma \ \textit{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(H)$ can be defined as

$$\|f\|_{L^{p(\cdot)}(H)} = \inf\left\{ \gamma > 0: \int\limits_H \left(\frac{|f(y)|}{\gamma}\right)^{p(y)} dy \le 1 \right\}.$$

(b) *The space $L^{p(\cdot)}_{\rm loc}(H)$ is defined as*

$$L^{p(\cdot)}_{\rm loc}(H) := \Big\{ f: f \in L^{p(\cdot)}(K) \ \textit{ for all compact subsets } \ K \subset H \Big\}.$$

We use these notations in this article:

(i) Let $f \in L^1_{\rm loc}(H)$ be a locally integrable function, then the Hardy – Littlewood maximal operator M is defined as

$$Mf(y) := \sup_{s>0} s^{-n} \int\limits_{B(y,s)} |f(y)| dy, \quad y \in H,$$

where $B(y,s) := \{x \in H: |y-x| < s\}$.

(ii) The set $\mathfrak{P}(H)$ is consists of all measuable functions $p(\cdot)$ satisfying

$$p_- := \operatorname{ess}\inf_{h\in H} p(h) > 1, \qquad p_+ := \operatorname{ess}\sup_{h\in H} p(h) < \infty. \tag{2.1}$$

(iii) $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(H)$ is consists of all functions $q \in \mathfrak{P}(H)$ satisfying (2.1) and $\log$-condition defined as

$$|q(x) - q(y)| \leq \frac{C(q)}{-\ln|x-y|}, \qquad |x-y| \leq \frac{1}{2}, \quad x, y \in H. \tag{2.2}$$

(iv) Let H is unbounded, $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ are the subsets of $\mathfrak{P}(H)$ and values of these subsets lies in $[1, \infty)$. $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ satisfy conditions

$$|q(h) - q_\infty| \leq \frac{C}{\ln(e + |h|)}, \tag{2.3}$$

where $q_\infty \in (1, \infty)$, and

$$|q(h) - q_0| \leq \frac{C}{\ln|h|}, \quad |h| \leq \frac{1}{2}, \tag{2.4}$$

respectively.

(v) $\chi_l = \chi_{F_l}$, $F_l = D_l \setminus D_{l-1}$, $D_l = D(0, 2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$ for all $l \in \mathbb{Z}$.

C is a constant, its value varies from line to line and independent of main parameters involved.

We are assuming that order of Riesz potential operator $\zeta(x)$ is not continuous rather we are assuming that it is a measurable function in $\mathbb{R}^n$ satisfying the following conditions:

(1) $\zeta_0 := \inf_{x \in \mathbb{R}^n} \zeta(x) > 0$,

(2) $\operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(\infty)\zeta(x) < n$,

(3) $\operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)\zeta(x) < n$,

where $p(\infty) = \lim_{|x| \to \infty} p(x)$.

The following proposition is the one of the main requirement to prove our main results. The given proposition was proved in [3] and commonly known as Sobolev theorem for Riesz potential operator in Lebesgue spaces under the some necessary assumptions on exponent.

Proposition 2.1. *Suppose that* $p(\cdot) \in \mathfrak{B}^{\log}(\mathbb{R}^n) \cap \mathfrak{B}_{0,\infty}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$ *and*

$$1 < p(\infty) \leq p(x) \leq p_+ < \infty.$$

Let $\zeta(x)$ *is satisfying the above conditions* (1), (2), and (3). *Then we have following weighted Sobolev-type estimate for the fractional operator* $I^{\zeta(z)}$:

$$\|(1+|z|)^{-\lambda(z)} I^{\zeta(z)}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(z)} = \frac{1}{p(z)} - \frac{\zeta(z)}{n}$$

is the Sobolev exponent

$$\lambda(z) = C\zeta(z)\left(1 - \frac{\zeta(z)}{n}\right) \leq C\frac{n}{4}.$$

Here, C *is denoting the Dini–Lipschitz constant from the inequality* (2.3) *in which* $q(\cdot)$ *replaced by* $p(\cdot)$.

***Remark* 2.1.** (i) If $\zeta(z)$ is satisfying the condition (2.3): $|\zeta(z)-\zeta_\infty| \le \dfrac{C_\infty}{\ln(e+|z|)}$ for $x \in \mathbb{R}^n$, then $(1+|z|)^{-\lambda(z)}$ is equivalent to the weight $(1+|z|)^{-\lambda_\infty}$. (ii) One can replace the variable order of Riesz potential operator $\zeta(x)$ by $\zeta(y)$ in the case of potentials over bounded domain, such potentials differ unessential if the function $\zeta(x)$ is satisfying the smoothness logarithmic condition as (2.2), since

$$C_1|z_1-z_2|^{n-\zeta(z_2)}| \le |z_1-z_2|^{n-\zeta(z_1)} \le C_2|z_1-z_2|^{n-\zeta(z_2)}.$$

2.2. Variable exponent Herz spaces and Herz–Morrey spaces. In this subsection we define variable exponent Herz spaces.

Definition 2.2. *Let $u, v \in [1,\infty)$, $\zeta \in \mathbb{R}$, the classical versions of Herz spaces commonly known as homogeneous and nonhomogeneous are defined by their norms such as*

$$\|g\|_{K^{\zeta}_{u,v}(\mathbb{R}^n)} := \|g\|_{L^u(D(0,1))} + \left\{\sum_{l\in\mathbb{N}} 2^{l\zeta v}\left(\int\limits_{F_{2^{l-1},2^l}} |g(y)|^u dy\right)^{\frac{v}{u}}\right\}^{\frac{1}{v}},$$

$$\|g\|_{\dot{K}^{\zeta}_{u,v}(\mathbb{R}^n)} := \left\{\sum_{l\in\mathbb{Z}} 2^{l\zeta v}\left(\int\limits_{F_{2^{l-1},2^l}} |g(y)|^u dy\right)^{\frac{v}{u}}\right\}^{\frac{1}{v}},$$

respectively, such that $F_{t,\tau} := D(0,\tau)\backslash D(0,t)$.

Definition 2.3. *Let $u \in [1,\infty)$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}$. $\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ is the homogeneous version of Herz space and its norm is given as*

$$\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\}) : \|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{l=-\infty}^{\infty} \|2^{l\zeta} g\chi_l\|^u_{L^{v(\cdot)}}\right)^{\frac{1}{u}}.$$

Definition 2.4. *Let $u \in [1,\infty)$, $\zeta \in \mathbb{R}$ and $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The nonhomogeneous Herz space $K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ is defined as*

$$K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\}) : \|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{v(\cdot)}(D(0,1))} + \left(\sum_{k=-\infty}^{\infty} \|2^{k\zeta} g\chi_k\|^u_{L^{v(\cdot)}}\right)^{\frac{1}{u}}.$$

Now we will define variable Herz–Morrey spaces.

Definition 2.5. *For $a(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$, $0 < u < \infty$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $0 \le \beta < \infty$. A variable Herz–Morrey space $M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$ is defined by*

$$M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n) = \Big\{ g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \colon \|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} < \infty \Big\},$$

where

$$\|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\beta} \left(\sum_{k=-\infty}^{k_0} \|2^{ka(\cdot)} g \chi_k\|^{u(1+\psi)}_{L^{v(\cdot)}} \right)^{\frac{1}{u(1+\psi)}}.$$

2.3. Important lemmas. Following are the important lemmas to prove our main results.

Lemma 2.1 [18]. *Let $D > 1$ and $q \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0} s^{\frac{n}{q(0)}} \le \|\chi_{F_{s,D_s}}\|_{q(\cdot)} \le t_0 s^{\frac{n}{q(0)}} \quad \textit{for} \quad 0 < s \le 1$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{q_\infty}} \le \|\chi_{F_{s,D_s}}\|_{q(\cdot)} \le t_\infty s^{\frac{n}{q_\infty}} \quad \textit{for} \quad s \ge 1,$$

respectively, where $t_0 \ge 1$ and $t_\infty \ge 1$ is depending on D but not depending on s.

Lemma 2.2 [13] (Generalized Hölder's inequality). *Consider a measurable subset H such that $H \subseteq \mathbb{R}^n$ and $1 \le p_-(H) \le p_+(H) \le \infty$. Then*

$$\|fg\|_{L^{r(\cdot)}(H)} \le \|f\|_{L^{p(\cdot)}(H)} \|g\|_{L^{q(\cdot)}(H)}$$

holds, where $f \in L^{p(\cdot)}(H)$, $g \in L^{q(\cdot)}(H)$ and $\dfrac{1}{r(z)} = \dfrac{1}{p(z)} + \dfrac{1}{q(z)}$ for every $z \in H$.

Lemma 2.3 [19]. *Let k is a positive integers. Then, for all $b \in BMO(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ for $j > i$,*

$$C^{-1}\|b\|^k_{BMO(\mathbb{R}^n)} \le \sup_{D:ball} \frac{1}{\|\chi_D\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \big\|(b - b_D)^k \chi_D\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\le C\|b\|^k_{BMO(\mathbb{R}^n)},$$

$$\big\|(b - b_{D_i})^k \chi_{D_j}\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)^k \|b\|^k_{BMO(\mathbb{R}^n)} \|\chi_{D_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

2.4. Grand variable Herz spaces. Next we define grand variable Herz spaces.

Definition 2.6 (BMO space). *A BMO function is a locally integrable function u whose mean oscillation given by $\dfrac{1}{|Q|}\displaystyle\int_Q |u(y) - u_Q|dy$ is bounded. Mathematically,*

$$\|u\|_{BMO} = \sup_{Q} \frac{1}{|Q|} \int_{Q} |u(y) - u_Q| dy < \infty.$$

Definition 2.7. *Let* $a(\cdot) \in L^{\infty}(\mathbb{R}^n)$, $u \in [1, \infty)$, $v \colon \mathbb{R}^n \to [1, \infty)$, $\theta > 0$. *A grand Herz spaces with variable exponent* $\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}$ *is defined by*

$$\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)} = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}} < \infty \right\},$$

where

$$\|g\|_{\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}} = \sup_{\psi > 0} \left(\psi^{\theta} \sum_{k \in \mathbb{Z}} \|2^{ka(\cdot)} g \chi_k\|^{u(1+\psi)}_{L^{v(\cdot)}} \right)^{\frac{1}{u(1+\psi)}}$$

$$= \sup_{\psi > 0} \psi^{\frac{\theta}{u(1+\psi)}} \|g\|_{\dot{K}^{a(\cdot),u(1+\psi)}_{v(\cdot)}}.$$

We consider the Riesz potential operator

$$I^{\zeta(z_1)} f(z_1) = \int_{\mathbb{R}^n} \frac{f(z_2)}{|z_1 - z_2|^{n - \zeta(z_1)}} dz_2, \quad 0 < \zeta(z_1) < n.$$

Note that the $\zeta(z_1)$ is the order of the Riesz potential operator which is variable.

Definition 2.8. *Let* g *is a locally integrable function on* $\mathbb{R}^n$, $b \in BMO(\mathbb{R}^n)$, $0 \le \zeta(z) < n$ *and* $m > 0$. *The* n*-dimensional fractional Hardy operators of variable order* $\zeta(z)$ *can be defined as*

$$\mathcal{H}^m_b g(z) := \frac{1}{|z|^{n - \zeta(z)}} \int_{|x| < |z|} [b(z) - b(x)]^m g(x) dx,$$

$$\mathcal{H}^{m*}_b g(z) := \int_{|x| \ge |z|} \frac{[b(z) - b(x)]^m g(x)}{|x|^{n - \zeta(z)}} dx, \quad z \in \mathbb{R}^n \setminus \{0\}.$$

3. Sobolev-type theorem for grand variable Herz spaces.

Theorem 3.1. *Let* $1 < u < \infty$, $a, q_2 \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$, *and* $1/q_1(z) - 1/q_2(z) = \zeta(\cdot)/n$. *If* $0 < u < \min\{1/(q_1)_+, 1(q_2')_+\}$, $0 < \zeta(\cdot) < n$, *and*

$$\frac{-n}{q_{1\infty}} < a_{\infty} < \frac{n}{q'_{1\infty}}, \qquad \frac{-n}{q_1(0)} < a(0) < \frac{n}{q'_1(0)},$$

then

$$\left\| (1 + |z|)^{-\lambda(z)} \mathcal{H}^m_b(f) \right\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C \|b\|^m_{BMO(\mathbb{R}^n)} \|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in \dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)$ and $f(z) = \sum_{j=-\infty}^{\infty} f(z)\chi_j(z) = \sum_{j=-\infty}^{\infty} f_j(z)$. Then have

$$|\mathcal{H}_b^m(f)(z).\chi_k(z)| \leq \frac{1}{|z|^{n-\zeta(\cdot)}} \int\limits_{D_k} |f(x)|[b(z)-b(x)]^m dx \chi_k(z)$$

$$\leq \frac{1}{|z|^{n-\zeta(\cdot)}} \left(|b(z)-b_{D_j}|^m \int\limits_{D_j} |f(x)|dx + \int\limits_{D_j} |b(x)-b_{D_j}|^m |f(x)|dx \right) \chi_k(z)$$

$$\leq \frac{1}{|z|^{n-\zeta(\cdot)}} \left(|b(z)-b_{D_j}|^m \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}} \|\chi_j\|_{L^{q_1'(\cdot)}} \right.$$

$$\left. + \sum_{j=-\infty}^{k} \|(b-b_{D_j})^m \chi_j\|_{L^{q_1'(\cdot)}} \|f_j\|_{L^{q_1(\cdot)}} \right) \chi_k(z)$$

$$\leq \frac{1}{|z|^{n-\zeta(\cdot)}} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}} \Big(|b(z)-b_{D_j}|^m \|\chi_j\|_{L^{q_1'(\cdot)}} + \|(b-b_{D_j})^m \chi_j\|_{L^{q_1'(\cdot)}} \Big) \chi_k(z)$$

$$\leq C z^{-n} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}} \Big(|b(z)-b_{D_j}|^m \|\chi_j\|_{L^{q_1'(\cdot)}} + \|(b-b_{D_j})^m \chi_j\|_{L^{q_1'(\cdot)}} \Big) |z|^{\zeta(z)} \chi_k(z).$$

It is known (see, e.g., [5]) that

$$I^{\zeta(\cdot)}((b(z)-b_{D_j})^m \chi_k)(z) \geq I^{\zeta(\cdot)}((b(z)-b_{D_j})^m \chi_k)(z)(\chi_k)(z)$$

$$= \int\limits_{D_k} \frac{|b(z)-b_{D_j}|^m}{|z-z_2|^{\zeta(z)-n}} dz_2 \chi_k(z)$$

$$\geq C|b(z)-b_{D_j}|^m |z|^{\zeta(z)} \chi_k(z)$$

$$\geq C|b(z)-b_{D_j}|^m |z|^{\zeta(z)} \chi_k(z).$$

Consequently, by Proposition 2.1 we obtain

$$\|\chi_k |z_1|^{\zeta(z_1)} (1+|z_1|)^{-\lambda(z_1)}\|_{L^{q_2(\cdot)}} \leq \|(1+|z_1|)^{-\lambda(z_1)} I^{\zeta(\cdot)}(\chi_k)\|_{L^{q_2(\cdot)}} \leq \|\chi_k\|_{L^{q_1(\cdot)}}.$$

As a result, we have

$$\Big\| \chi_k (1+|z|)^{-\lambda(z)} \mathcal{H}_b^m(f_j) \Big\|_{L^{q_2(\cdot)}}$$

$$\leq C \sum_{j=-\infty}^{k} z^{-n} \|f_j\|_{L^{q_1(\cdot)}} \Big(\|(1+|z|)^{-\lambda(z)} |z|^{\zeta(z)} (b-b_{D_j})^m \chi_k\|_{L^{q_2(\cdot)}} \|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$+ \|(b-b_{D_j})^m\chi_j\|_{L^{q_1'(\cdot)}}\|(1+|z|)^{-\lambda(z)}I^{\zeta(\cdot)}(\chi_k)\|_{L^{q_2(\cdot)}}\Big)$$

$$\leq C\sum_{j=-\infty}^{k} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\Big(\|(1+|z|)^{-\lambda(z)}I^{\zeta(\cdot)}((b-b_{D_j})^m\chi_k)\|_{L^{q_2(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$+ \|(b-b_{D_j})^m\chi_j\|_{L^{q_1'(\cdot)}}\|(1+|z|)^{-\lambda(z)}I^{\zeta(\cdot)}(\chi_k)\|_{L^{q_2(\cdot)}}\Big)$$

$$\leq C\sum_{j=-\infty}^{k} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\big((k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$+ \|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\big)$$

$$\leq C\sum_{j=-\infty}^{k} z^{-n}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}},$$

$$\|(1+|z|)^{-\lambda(z)}\mathcal{H}_b^m(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)}$$

$$= \sup_{\psi>0}\left(\psi^\theta\sum_{k\in\mathbb{Z}}\|2^{ka(\cdot)}\chi_k(1+|z|)^{-\lambda(z)}\mathcal{H}_b^m(f)\|^{u(1+\psi)}_{L^{q_2(\cdot)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq \sup_{\psi>0}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{k} z^{-n}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+ \sup_{\psi>0}\left(\psi^\theta\sum_{k=0}^{\infty}2^{ka_\infty u(1+\psi)}\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{k} z^{-n}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$=: E_1 + E_2.$$

Now we will find the estimate for E_1, by using the fact $z \in F_k$ and $j \leq k$.
By the Lemma 2.1, we have

$$z^{-n}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}} \leq C2^{-kn}2^{\frac{kn}{q_1(0)}}2^{\frac{jn}{q_1'(0)}} \leq C2^{\frac{(j-k)n}{q_1'(0)}}.$$

Applying above results to E_1, we get

$$E_1 \le \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{k}2^{-kn}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\le C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\left(\sum_{j=-\infty}^{k}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}2^{\frac{(j-k)n}{q_1'(0)}}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}.$$

Let $b := \dfrac{n}{q_1'(0)} - a(0)$. Applying the fact $2^{-u(1+\psi)} < 2^{-u}$, the Hölder's inequality and Fubini's theorem, we obtain

$$\le C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k}2^{a(0)j}\|f_j\|_{L^{q_1(\cdot)}}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}2^{b(j-k)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\le C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k}2^{a(0)j}\|f_j\|_{L^{q_1(\cdot)}}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}2^{b(j-k)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left[\psi^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k}2^{a(0)u(1+\psi)j}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}(k-j)^{mu(1+\psi)/2}2^{bu(1+\psi)(j-k)/2}\right)\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{k}(k-j)^{mu(1+\psi)/2}2^{b(u(1+\psi))'(j-k)/2}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right]^{\frac{1}{u(1+\psi)}}$$

$$\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left[\psi^{\theta}\sum_{k=-\infty}^{-1}\sum_{j=-\infty}^{k}2^{a(0)u(1+\psi)j}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}(k-j)^{mu(1+\psi)/2}2^{bu(1+\psi)(j-k)/2}\right]^{\frac{1}{u(1+\psi)}}$$

$$\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left[\psi^{\theta}\sum_{j=-\infty}^{-1}2^{a(\cdot)u(1+\psi)j}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\sum_{k=j}^{-1}(k-j)^{mu(1+\psi)/2}2^{bu(1+\psi)(j-k)/2}\right]^{\frac{1}{u(1+\psi)}}$$

$$\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left(\psi^{\theta}\sum_{j=-\infty}^{-1}2^{a(0)u(1+\psi)j}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\sum_{k=j}^{-1}(k-j)^{mu(1+\psi)/2}2^{bu(1+\psi)(j-k)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{l=-\infty}^{-1} 2^{a(0)u(1+\psi)j}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$= C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{j\in\mathbb{Z}} \|2^{a(\cdot)j} f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Now, for E_2, by using Minkowski's inequality, we have

$$E_2 \leq \sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty} 2^{ka_\infty u(1+\psi)}\left(\sum_{j=-\infty}^{k}\|\chi_k(1+|z|)^{-\lambda(z)}\mathcal{H}^m_b(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq \sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty} 2^{ka_\infty u(1+\psi)}\left(\sum_{j=-\infty}^{-1}\|\chi_k(1+|z|)^{-\lambda(z)}\mathcal{H}^m_b(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$+ \sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty} 2^{ka_\infty u(1+\psi)}\left(\sum_{j=0}^{k}\|\chi_k(1+|z|)^{-\lambda(z)}\mathcal{H}^m_b(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$:= A_1 + A_2.$$

The estimate for A_2 follows in a similar manner to E_1 by replacing $q_1'(0)$ with $q'_{1\infty}$ and by the use of the fact $b := \dfrac{n}{q_{1'_\infty}} - a_\infty > 0$. For A_1 we obtain

$$2^{-kn}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}} \leq C2^{-kn}2^{\frac{kn}{q_{1\infty}}}2^{\frac{jn}{q_1'(0)}} \leq C2^{\frac{-kn}{q_{1\infty}}}2^{\frac{jn}{q_1'(0)}},$$

as $a_\infty - \dfrac{n}{q'_{1_\infty}} < 0$ we get

$$A_1 \leq C\sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty} 2^{ka_\infty u(1+\psi)}\left(\sum_{j=-\infty}^{-1}\|\chi_k(1+|z|)^{-\lambda(z)}\mathcal{H}^m_b(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left(\psi^\theta\sum_{k=0}^{\infty}2^{ka_\infty u(1+\psi)}\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{-1}(k-j)^m 2^{-kn}2^{\frac{kn}{q_{1\infty}}}2^{\frac{jn}{q_1'(0)}}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}$$

$$\times \sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}(k-j)^m 2^{-kn}2^{\frac{kn}{q_{1\infty}}+ka_\infty}2^{ja_\infty}2^{\frac{jn}{q_1'(0)}-ja_\infty}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}2^{a_\infty ju(1+\psi)}((k-j)^m 2^{-kn}2^{\frac{kn}{q_{1\infty}}+ka_\infty}2^{\frac{jn}{q_1'(0)}-ja_\infty})^{u(1+\psi)/2}\right)\right.$$

$$\left.\times\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\left(\sum_{j=-\infty}^{-1}((k-j)^m 2^{-kn}2^{\frac{kn}{q_{1\infty}}+ka_\infty}2^{\frac{jn}{q_1'(0)}-ja_\infty})^{u(1+\psi)'/2}\right)^{\frac{u(1+\psi)}{u(1+\psi)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty}\sum_{j=-\infty}^{-1}2^{a_\infty ju(1+\psi)}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right.$$

$$\left.\times\left((k-j)^m 2^{-kn}2^{\frac{kn}{q_{1\infty}}+ka_\infty}2^{\frac{jn}{q_1'(0)}-ja_\infty}\right)^{u(1+\psi)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left(\psi^\theta \sum_{j=-\infty}^{-1}2^{a_\infty ju(1+\psi)}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right.$$

$$\left.\times\sum_{k=l}^{\infty}((k-j)^m 2^{-kn}2^{\frac{kn}{q_{1\infty}}+ka_\infty}2^{\frac{jn}{q_1'(0)}-ja_\infty})^{\frac{1}{2}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\psi>0}\left(\psi^\theta \sum_{j=-\infty}^{-1}2^{a_\infty ju(1+\psi)}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Combining these estimates, we get

$$\|(1+|z|)^{-\lambda(z)}\mathcal{H}^m_b(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

The theorem is proved.

Theorem 3.2. *Let* $1 < u < \infty$, $a, q_2 \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$, *and* $1/q_1(z) - 1/q_2(z) = \zeta(\cdot)/n$. *If* $0 < u < \min\{1/(q_1)_+, 1(q_2')_+\}$, $0 < \zeta(\cdot) < n$ *and*

$$\frac{-n}{q_{1\infty}} < a_\infty < \frac{n}{q_{1\infty}'}, \qquad \frac{-n}{q_1(0)} < a(0) < \frac{n}{q_1'(0)},$$

then

$$\|(1+|z|)^{-\lambda(z)}\mathcal{H}_b^{m*}(f)\|_{\dot K_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)} \le C\|b\|_{BMO(\mathbb{R}^n)}^m\|f\|_{\dot K_{q_1(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.$$

Proof. This proof is similar to our previous result, therefore we only give a simple proof. Let $f\in \dot K_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$ and $f(z)=\sum_{j=-\infty}^{\infty} f(z)\chi_j(z)=\sum_{j=-\infty}^{\infty} f_j(z)$. Then we have

$$\begin{aligned}
&\|\chi_k(1+|z|)^{-\lambda(z)}\mathcal{H}_b^{m^*}(f\chi_j)\|_{L^{q_2(\cdot)}}\\
&\quad\le C\sum_{j=k+1}^{\infty} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\Big(\|(1+|z|)^{-\lambda(z)}|z|^{\zeta(z)}(b-b_{D_j})^m\chi_k\|_{L^{q_2(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\\
&\qquad+\|(b-b_{D_j})^m\chi_j\|_{L^{q_1'(\cdot)}}\|(1+|z|)^{-\lambda(z)}I^{\zeta(\cdot)}(\chi_k)\|_{L^{q_2(\cdot)}}\Big)\\
&\quad\le C\sum_{j=k+1}^{\infty} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\Big(\|(1+|z|)^{-\lambda(z)}I^{\zeta(\cdot)}((b-b_{D_j})^m\chi_k)\|_{L^{q_2(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\\
&\qquad+\|(b-b_{D_j})^m\chi_j\|_{L^{q_1'(\cdot)}}\|(1+|z|)^{-\lambda(z)}I^{\zeta(\cdot)}(\chi_k)\|_{L^{q_2(\cdot)}}\Big)\\
&\quad\le C\sum_{j=k+1}^{\infty} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\big((k-j)^m\|b\|_{BMO(\mathbb{R}^n)}^m\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\\
&\qquad+\|b\|_{BMO(\mathbb{R}^n)}^m\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\big)\\
&\quad\le C\sum_{j=k+1}^{\infty} z^{-n}(k-j)^m\|b\|_{BMO(\mathbb{R}^n)}^m\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}},
\end{aligned}$$

$$\begin{aligned}
&\|(1+|z|)^{-\lambda(z)}\mathcal{H}_b^{m*}(f)\|_{\dot K_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}\\
&\quad\le \sup_{\psi>0}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\left(\sum_{j=k+1}^{\infty} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\qquad+\sup_{\psi>0}\left(\psi^\theta\sum_{k=0}^{\infty}2^{ka_\infty u(1+\psi)}\left(\sum_{j=k+1}^{\infty} z^{-n}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\quad=: E_1+E_2.
\end{aligned}$$

We will find estimate of E_2. Now by using the fact that for each $k\in\mathbb{Z}$ and $j\ge k+1$ with $z\in F_k$

$$2^{-jn}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1'(\cdot)}}\le C2^{-jn}2^{\frac{kn}{q_{1\infty}}}2^{\frac{jn}{q_{1\infty}'}}\le C2^{\frac{(k-j)n}{q_{1\infty}}},$$

we get

$$E_2 \leq C \sup_{\psi>0} \left(\psi^\theta \sum_{k=0}^{\infty} 2^{k a_\infty u(1+\psi)} \left(\sum_{j=k+1}^{\infty} \|\chi_k (1+|z|)^{-\lambda(z)} \mathcal{H}_b^{m^*}(f_j)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C \|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} 2^{a_\infty j} \|f_j\|_{L^{q_1(\cdot)}} (k-j)^m 2^{d(k-j)} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}},$$

where $d = \dfrac{n}{q_{1_\infty}} + a_\infty > 0$. Then we use Hölder's theorem for series and the inequality $2^{-u(1+\psi)} < 2^{-u}$ to obtain

$$E_2 \leq C \|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \left[\psi^\theta \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} 2^{a_\infty u(1+\psi) j} \right.\right.$$

$$\left. \times \|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}} (k-j)^{m u(1+\psi)/2} 2^{d u(1+\psi)(k-j)/2} \right)$$

$$\left. \times \left(\sum_{j=k+1}^{\infty} (k-j)^{m u(1+\psi)'/2} 2^{d (u(1+\psi))'(k-j)/2} \right)^{\frac{u(1+\psi)}{(u(1+\psi))'}} \right]^{\frac{1}{u(1+\psi)}}$$

$$\leq C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left[\psi^\theta \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} 2^{a_\infty u(1+\psi) j} \right.$$

$$\left. \times \|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}} (k-j)^{m u(1+\psi)/2} 2^{d u(1+\psi)(k-j)/2} \right]^{\frac{1}{u(1+\psi)}}$$

$$\leq C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{j=0}^{\infty} 2^{a_\infty u(1+\psi) j} \|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}} \right.$$

$$\left. \times \sum_{k=0}^{l-2} (k-j)^{m u(1+\psi)/2} 2^{d u(1+\psi)(k-j)/2} \right)^{\frac{1}{u(1+\psi)}}$$

$$< C \|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{j\in\mathbb{Z}} 2^{a_\infty u(1+\psi) j} \right.$$

$$\times \|f_j\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \sum_{k=-\infty}^{l-2} (k-j)^{mu(1+\psi)/2} 2^{du(1+\psi)(k-j)/2} \Bigg)^{\frac{1}{u(1+\psi)}}$$

$$= C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\psi>0} \left(\psi^\theta \sum_{j\in\mathbb{Z}} \|2^{a(\cdot)j} f_j\|_{L^{q_1(\cdot)}}^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^m \|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.$$

For E_1, by using Minkowski's inequality, we have

$$E_1 \leq \sup_{\psi>0} \left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\psi)} \left(\sum_{j=k+1}^{\infty} \|\chi_k (1+|z|)^{-\lambda(z)} \mathcal{H}_b^{m*}(f_j)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$\leq \sup_{\psi>0} \left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\psi)} \left(\sum_{j=k+1}^{-1} \|\chi_k (1+|z|)^{-\lambda(z)} \mathcal{H}_b^{m*}(f_j)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$+ \sup_{\psi>0} \left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\psi)} \left(\sum_{j=0}^{\infty} \|\chi_k (1+|z|)^{-\lambda(z)} \mathcal{H}_b^{m*}(f_j)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$:= I_1 + I_2.$$

The estimate of I_1 can be obtained in a similar way to E_2 by replacing $q_{1\infty}$ with $q_1(0)$ and using the inequality $\dfrac{n}{q_1(0)} + a(0) > 0$, $\dfrac{n}{q_{1_\infty}} + a_\infty > 0$. For I_2 we have

$$2^{-jn} \|\chi_k\|_{L^{q_1(\cdot)}} \|\chi_j\|_{L^{q_1'(\cdot)}} \leq C 2^{-jn} 2^{\frac{kn}{q_1(0)}} 2^{\frac{jn}{{q'}_{1\infty}}} \leq C 2^{\frac{kn}{q_1(0)}} 2^{\frac{-jn}{q_{1\infty}}},$$

$$I_2 \leq C \sup_{\psi>0} \left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\psi)} \left(\sum_{j=0}^{\infty} \|\chi_k (1+|z|)^{-\lambda(z)} \mathcal{H}_b^{m^*}(f_j)\|_{L^{q_2(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\psi>0} \Bigg(\psi^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\psi)}$$

$$\times \left(\sum_{j=0}^{\infty} (k-j)^m 2^{-jn} 2^{\frac{kn}{q_1(0)}} 2^{\frac{jn}{{q'}_{1\infty}}} \|f_j\|_{L^{q_1(\cdot)}} \right)^{u(1+\psi)} \Bigg)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}$$

$$\times \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=0}^{\infty} 2^{ja(0)}(k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(^\theta \sum_{k=-\infty}^{-1}\left(\sum_{j=0}^{\infty} 2^{ja(0)(u(1+\psi))}\left((k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)}\right)^{u(1+\psi)/2}\right)\right.$$

$$\left.\times \|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\left(\sum_{j=0}^{\infty}((k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)})^{u(1+\psi)'/2}\right)^{\frac{u(1+\psi)}{u(1+\psi)'}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{-1}\sum_{j=0}^{\infty} 2^{ja(0)(u(1+\psi))}\|f_j\|^{u(1+\psi)/2}_{L^{q_1(\cdot)}}\right.$$

$$\left.\times\left((k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{j=0}^{\infty} 2^{ja(0)(u(1+\psi))}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right.$$

$$\left.\times \sum_{k=-\infty}^{j}\left((k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)}\right)^{u(1+\psi)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{j=0}^{\infty} 2^{ja(0)(u(1+\psi))}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right.$$

$$\left.\times \sum_{k=-\infty}^{j}\left((k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)}\right)^{u(1+\psi)/2}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{j=0}^{\infty} 2^{ja(0)(u(1+\psi))}\|f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right.$$

$$\left.\times \sum_{k=-\infty}^{j}(k-j)^m 2^{\frac{kn}{q_1(0)}+ka(0)} 2^{\frac{-jn}{q_{1\infty}}-ja(0)}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta\sum_{j\in\mathbb{Z}}\|2^{a(\cdot)j}f_j\|^{u(1+\psi)}_{L^{q_1(\cdot)}}\right)^{\frac{1}{u(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Combining these estimates yields

$$\|(1+|z|)^{-\lambda(z)}\mathcal{H}^{m*}_b(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},$$

we finish the proof of the theorem.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

Funding. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Author contributions. All authors have contributed equally to the work.

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Received 28.03.23