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Fatih Hezenci[1], **Hüseyin Budak** (Department of Mathematics, Faculty of Science and Arts, Duzce University, Turkey)

ON ERROR BOUNDS FOR MILNE'S FORMULA IN CONFORMABLE FRACTIONAL OPERATORS

ПРО МЕЖІ ПОХИБКИ ФОРМУЛИ МІЛНА ДЛЯ КОНФОРМНИХ ДРОБОВИХ ОПЕРАТОРІВ

Milne's formula is a mathematical expression used to approximate the value of a definite integral. The formula is particularly useful for problems encountered in physics, engineering, and various other scientific disciplines. We establish an equality for conformable fractional integrals. With the help of this equality, we obtain error bounds for one of the open Newton – Cotes formulas, namely, Milne's formula for the case of differentiable convex functions within the framework of fractional and classical calculus. Furthermore, we provide our results by using special cases of the obtained theorems.

Формула Мілна — це математичний вираз, який використовується для наближеного обчислення значення означеного інтеграла. Ця формула особливо корисна для задач, що зустрічаються у фізиці, техніці та інших наукових дисциплінах. У цій статті доведено рівність для конформних дробових інтегралів. За допомогою цієї рівності отримано оцінку похибки однієї з відкритих формул Ньютона – Котеса, а саме формули Мілна, для випадку диференційовних опуклих функцій в рамках дробового та класичного числення. Навіть більше, ми наводимо наші результати, використовуючи окремі випадки отриманих теорем.

1. Introduction. Fractional calculus is a branch of mathematics that investigates the properties of derivatives and integrals of fractional order. It generalizes the classical calculus of integers, which is useful in modelling certain phenomena in physics, engineering, and other fields. The most common definitions of fractional integrals are the Riemann – Liouville fractional, conformable fractional, and tempered fractional integrals and so on. They can be established the bounds of new formulas by using not only Hermite – Hadamard and Simpson inequalities but also Newton's and Milne's formula.

New inequalities of Simpson-type and their application to quadrature inequalities in numerical analysis were given in [10]. Some fractional Simpson-type inequalities investigated for the case of function whose second derivatives in absolute value are convex given in [13]. Budak et al. [6] proved some variants of Simpson-type for differentiable convex functions by generalized fractional integrals (see [7] and the references therein for further information).

Simpson's second rule has the rule of three-point Newton – Cotes quadrature, thus evaluations for three steps quadratic kernel are mostly called Newton type results. These results are known as Newton-type inequalities in the literature. Large number of mathematicians have been considered to Newton-type inequalities extensively. For example, in paper [11], some Newton-type inequalities have been considered for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex. In addition, some Newton-type inequalities for the case of differentiable convex functions associated with the Riemann – Liouville fractional integrals were proved and some inequalities of Riemann – Liouville fractional Newton-type for functions of bounded variation were given in [14]. Furthermore, new Newton-type inequalities based on convexity

[1] Corresponding author, e-mail: fatihezenci@gmail.com.

were given and several applications for special cases of real functions were also established in paper [12]. It can be referred to [15 – 17, 21, 25] and the references therein for the case of further informations associated with Newton-type inequalities connected with convex differentiable functions.

In paper [9], Djenaoui and Meftah established some new estimates of Milne's quadrature rule for functions whose first derivative is s-convex. In paper [5], Budak et al. obtained fractional versions of Milne's formula by using the differentiable convex functions. Moreover, Alomari and Liu established error estimations for Milne's rule for mappings of bounded variation and for absolutely continuous mappings in [2]. Furthermore, in paper [4], Ali et al. computed the error bounds via one of the open Newton – Cotes formulas, namely Milne's formula for differentiable convex functions in fractional and classical calculus.

This study is mainly devoted to propose Milne's formula for the case of differentiable convex functions involving conformable fractional integrals. This paper is organized as follows. In Section 2, the basic definition of fractional calculus and other relevant research in this discipline will be presented. In Section 3, an integral equality will be established that is critical in establishing the primary results of the presented paper. Furthermore, it will be proved several Milne's formula for the case of differentiable convex functions connected with conformable fractional integrals. By using the special cases of the obtained results, it will be presented several important inequalities. Finally, in Section 4, several ideas about Milne's formula for the further directions of research.

2. Preliminaries. Let us put forth some preliminaries which will be utilized in the sequel.

(i) Simpson's quadrature formula (Simpson's $1/3$ rule) is formulated as follows:

$$\int_{\sigma}^{\delta} \mathcal{F}(x)dx \approx \frac{\delta-\sigma}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]. \tag{1}$$

(ii) Simpson's second formula or Newton – Cotes quadrature formula (Simpson's $3/8$ rule (cf. [8])) is formulated as follows:

$$\int_{\sigma}^{\delta} \mathcal{F}(x)dx \approx \frac{\delta-\sigma}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]. \tag{2}$$

Formulae (1) and (2) are provided for any function $\mathcal{F}$ with continuous $4^{\mu h}$ derivative on $[\sigma,\delta]$.

The most popular Newton – Cotes quadrature involving three-point is Simpson-type inequality is as follows.

Theorem 1. *If* $\mathcal{F}:[\sigma,\delta]\rightarrow\mathbb{R}$ *is a four times continuously differentiable function on* (σ,δ) *and* $\left\|\mathcal{F}^{(4)}\right\|_{\infty}=\sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$, *then one has the following inequality*:

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right|\leq\frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\delta-\sigma)^{4}.$$

One of the classical closed type quadrature rules is the Simpson $3/8$ rule based on the Simpson $3/8$ inequality as follows.

Theorem 2 (see [8]). *If $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a four times continuously differentiable function on (σ,δ) and $\left\|\mathcal{F}^{(4)}\right\|_\infty=\sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$, then one has the inequality*

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int\limits_\sigma^\delta\mathcal{F}(x)dx\right|\le\frac{1}{6480}\left\|\mathcal{F}^{(4)}\right\|_\infty(\delta-\sigma)^4.$$

By using the Newton – Cotes formulas, Milne's formula which is of open type is parallel to the Simpson's formula which is of closed type, since they are held under the same conditions.

Theorem 3 (see [3]). *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ denote a four times continuously differentiable mapping on (σ,δ) and $\left\|\mathcal{F}^{(4)}\right\|_\infty=\sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$. Then one has the inequality*

$$\left|\frac{1}{3}\left[2\mathcal{F}(\sigma)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int\limits_\sigma^\delta\mathcal{F}(x)dx\right|\le\frac{7(\delta-\sigma)^4}{23040}\left\|\mathcal{F}^{(4)}\right\|_\infty.$$

For building our main results, the main definitions of Riemann – Liouville integrals and conformable integrals, which are well-known in the literature, are presented as follows.

Definition 1. *The gamma function, beta function, and incomplete beta function are defined as*

$$\Gamma(x):=\int\limits_0^\infty\mu^{x-1}e^{-\mu}d\mu,$$

$$\delta(x,y):=\int\limits_0^1\mu^{x-1}(1-\mu)^{y-1}d\mu,$$

and

$$\mathscr{B}(x,y,r):=\int\limits_0^r\mu^{x-1}(1-\mu)^{y-1}d\mu,$$

respectively, for $x,y\in\mathbb{R}^+$.

Definition 2 (see [20]). *The Riemann – Liouville integrals $J^\beta_{\sigma+}\mathcal{F}(x)$ and $J^\beta_{\delta-}\mathcal{F}(x)$ of order $\beta>0$ are given by*

$$J^\beta_{\sigma+}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int\limits_\sigma^x(x-\mu)^{\beta-1}\mathcal{F}(\mu)d\mu,\quad x>\sigma, \tag{3}$$

and

$$J^\beta_{\delta-}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int\limits_x^\delta(\mu-x)^{\beta-1}\mathcal{F}(\mu)d\mu,\quad x<\delta, \tag{4}$$

respectively, for $\mathcal{F}\in L_1[\sigma,\delta]$. The Riemann – Liouville integrals equals to the classical integrals for the case of $\beta=1$.

Jarad et al. investigated the fractional conformable integral operators in paper [19]. They also provided certain characteristics and relationships between these operators and several other fractional operators. The fractional conformable integral operators are defined as follows.

Definition 3 (see [19]). *The fractional conformable integral operator* ${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(x)$ *and* ${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(x)$ *of order* $\beta \in C,\ \operatorname{Re}(\beta) > 0$ *and* $\alpha \in (0,1]$ *are described by*

$$ {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int\limits_{\sigma}^{x} \left(\frac{(x-\sigma)^{\alpha} - (\mu-\sigma)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}} d\mu, \quad \mu > \sigma, \tag{5} $$

and

$$ {}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int\limits_{x}^{\delta} \left(\frac{(\delta-x)^{\alpha} - (\delta-\mu)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}} d\mu, \quad \mu < \delta, \tag{6} $$

respectively, for $\mathcal{F} \in L_1[\sigma,\delta]$.

If we choose $\alpha = 1$, then the fractional integral in (5) and (6) coincides with the Riemann – Liouville fractional integral in (3) and (4), respectively. Some recent results according to fractional integral inequalities, see [1, 18, 22 – 24] and the references cited therein.

3. Principal outcomes.

Lemma 1. *Let us note that* $\mathcal{F}\colon [\sigma,\delta] \to \mathbb{R}$ *is an absolutely continuous function* (σ,δ) *so that* $\mathcal{F}' \in L_1[\sigma,\delta]$. *Then the following equality holds*:

$$ \frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right) - \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right) \right] - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta) \right] $$

$$ = \frac{\alpha^{\beta}(\delta-\sigma)}{2} \sum_{i=1}^{4} I_i, $$

where

$$ I_1 = \int\limits_{0}^{\frac{1}{4}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha} \right)^{\beta} \left[\mathcal{F}'(\mu\delta + (1-\mu)\sigma) - \mathcal{F}'(\mu\sigma + (1-\mu)\delta) \right] d\mu, $$

$$ I_2 = \int\limits_{\frac{1}{4}}^{\frac{1}{2}} \left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha} \right)^{\beta} - \frac{2}{3\alpha^{\beta}} \right] \left[\mathcal{F}'(\mu\delta + (1-\mu)\sigma) - \mathcal{F}'(\mu\sigma + (1-\mu)\delta) \right] d\mu, $$

$$ I_3 = \int\limits_{\frac{1}{2}}^{\frac{3}{4}} \left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha} \right)^{\beta} - \frac{1}{3\alpha^{\beta}} \right] \left[\mathcal{F}'(\mu\delta + (1-\mu)\sigma) - \mathcal{F}'(\mu\sigma + (1-\mu)\delta) \right] d\mu, $$

$$ I_4 = \int\limits_{\frac{3}{4}}^{1} \left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha} \right)^{\beta} - \frac{1}{\alpha^{\beta}} \right] \left[\mathcal{F}'(\mu\delta + (1-\mu)\sigma) - \mathcal{F}'(\mu\sigma + (1-\mu)\delta) \right] d\mu. $$

Proof. From the facts of integration by parts, we readily obtain

$$I_1=\int\limits_0^{\frac{1}{4}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\big[\mathcal{F}'(\mu\delta+(1-\mu)\sigma)-\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\big]d\mu$$

$$=\frac{1}{\delta-\sigma}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]\Bigg|_0^{\frac{1}{4}}$$

$$-\frac{\beta}{\delta-\sigma}\int\limits_0^{\frac{1}{4}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]d\mu$$

$$=\frac{1}{\delta-\sigma}\left(\frac{1-\left(\frac{3}{4}\right)^{\alpha}}{\alpha}\right)^{\beta}\left[\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)+\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$

$$-\frac{\beta}{\delta-\sigma}\int\limits_0^{\frac{1}{4}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]d\mu. \quad (7)$$

Similar to foregoing process, we readily get

$$I_2=\frac{2}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right]\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)$$

$$-\frac{1}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{3}{4}\right)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right]\left[\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)+\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$

$$-\frac{\beta}{\delta-\sigma}\int\limits_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]d\mu,$$

$$I_3=\frac{1}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{1}{4}\right)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]\left[\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)+\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$

$$-\frac{2}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)$$

$$-\frac{\beta}{\delta-\sigma}\int\limits_{\frac{1}{2}}^{\frac{3}{4}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]d\mu,$$

and

$$I_4=-\frac{1}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{1}{4}\right)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right]\left[\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)+\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$
$$-\frac{\beta}{\delta-\sigma}\int\limits_{\frac{3}{4}}^{1}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]d\mu. \tag{8}$$

If we combine from (7) to (8), then we readily get

$$\sum_{i=1}^{4}I_i=\frac{2}{3\alpha^{\beta}(\delta-\sigma)}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$
$$-\frac{\beta}{\delta-\sigma}\int\limits_{0}^{1}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}[\mathcal{F}(\mu\delta+(1-\mu)\sigma)+\mathcal{F}(\mu\sigma+(1-\mu)\delta)]d\mu. \tag{9}$$

If we use the change of the variable $x=\mu\delta+(1-\mu)\sigma$ and $x=\mu\sigma+(1-\mu)\delta$ for $\mu\in[0,1]$, then the equality (9) can be rewritten as follows:

$$\sum_{i=1}^{4}I_i=\frac{2}{3\alpha^{\beta}(\delta-\sigma)}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$
$$-\left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int\limits_{\sigma}^{\delta}\left(\frac{(\delta-\sigma)^{\alpha}-(\delta-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(\delta-x)^{1-\alpha}}dx$$
$$-\left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int\limits_{\sigma}^{\delta}\left(\frac{(\delta-\sigma)^{\alpha}-(x-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(x-\sigma)^{1-\alpha}}dx$$
$$=\frac{2}{3\alpha^{\beta}(\delta-\sigma)}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]$$
$$-\frac{\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta+1}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]. \tag{10}$$

Finally, the multiplication both sides of (10) by $\dfrac{\alpha^{\beta}(\delta-\sigma)}{2}$ completes the proof of the Lemma 1.

Theorem 4. *Under the assumptions that Lemma* 1 *holds and the function* $|\mathcal{F}'|$ *is convex on* $[\sigma,\delta]$. *Then the following Milne's formula holds*:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+\,{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}(\Omega_1(\alpha,\beta)+\Omega_2(\alpha,\beta)+\Omega_3(\alpha,\beta)+\Omega_4(\alpha,\beta))\big[|\mathcal{F}'(\sigma)|+|\mathcal{F}'(\delta)|\big].$$

Here,

$$\Omega_1(\alpha,\beta)=\int\limits_0^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu=\frac{1}{\alpha^{\beta+1}}\left[\delta\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{3}{4}\right)^{\alpha}\right)\right],$$

$$\Omega_2(\alpha,\beta)=\int\limits_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|d\mu,$$

$$\Omega_3(\alpha,\beta)=\int\limits_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|d\mu,$$

$$\Omega_4(\alpha,\beta)=\int\limits_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|d\mu=\frac{1}{4\alpha^{\beta}}-\frac{1}{\alpha^{\beta+1}}\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{4}\right)^{\alpha}\right).$$

Proof. Let us take modulus in Lemma 1. Then we readly have

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+\,{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\int\limits_0^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\big|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)-\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\big|d\mu\right.$$
$$+\int\limits_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\big|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)-\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\big|d\mu$$
$$+\int\limits_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\big|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)-\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\big|d\mu$$
$$\left.+\int\limits_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\big|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)-\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\big|d\mu\right]. \qquad (11)$$

It is known that $|\mathcal{F}'|$ is convex. Then it follows

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\int\limits_0^{\frac14}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|+(1-\mu)\left|\mathcal{F}'(\sigma)\right|+\mu\left|\mathcal{F}'(\sigma)\right|+(1-\mu)\left|\mathcal{F}'(\delta)\right|\right]d\mu\right.$$

$$+\int\limits_{\frac14}^{\frac12}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|+(1-\mu)\left|\mathcal{F}'(\sigma)\right|+\mu\left|\mathcal{F}'(\sigma)\right|+(1-\mu)\left|\mathcal{F}'(\delta)\right|\right]d\mu$$

$$+\int\limits_{\frac12}^{\frac34}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|+(1-\mu)\left|\mathcal{F}'(\sigma)\right|+\mu\left|\mathcal{F}'(\sigma)\right|+(1-\mu)\left|\mathcal{F}'(\delta)\right|\right]d\mu$$

$$\left.+\int\limits_{\frac34}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|+(1-\mu)\left|\mathcal{F}'(\sigma)\right|+\mu\left|\mathcal{F}'(\sigma)\right|+(1-\mu)\left|\mathcal{F}'(\delta)\right|\right]d\mu\right]$$

$$=\frac{(\delta-\sigma)\alpha^{\beta}}{2}(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha))\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

***Remark* 1.** If we assign $\alpha=1$ in Theorem 4, then the following Milne's formula holds:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J^{\beta}_{\delta-}\mathcal{F}(\sigma)+J^{\beta}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\delta-\sigma}{2}(\Omega_1(1,\beta)+\Omega_2(1,\beta)+\Omega_3(1,\beta)+\Omega_4(1,\beta))\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

which is given in paper [4, Theorem 1].

***Remark* 2.** Let us consider $\beta=1$ and $\alpha=1$ in Theorem 4. Then we get Milne's formula

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int\limits_{\sigma}^{\delta}\mathcal{F}(\mu)d\mu\right|$$

$$\leq\frac{5(\delta-\sigma)}{48}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

which is given in paper [4, Corollary 1]. This inequality helps us find the error bound of Milne's rule.

Theorem 5. *Assume that the assumptions of Lemma* 1 *valid and the function* $|\mathcal{F}'|^{q}$, $q>1$, *is convex on* $[\sigma,\delta]$. *Then the following Milne's formula holds*:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\varphi_1^p(\alpha,\beta)+\varphi_4^p(\alpha,\beta)\right)\left[\left(\frac{7|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^q+7|\mathcal{F}'(\delta)|^q}{32}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\varphi_2^p(\alpha,\beta)+\varphi_3^p(\alpha,\beta)\right)\left[\left(\frac{3|\mathcal{F}'(\sigma)|^q+5|\mathcal{F}'(\delta)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{32}\right)^{\frac{1}{q}}\right]\right\}.$$

Here, $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and*

$$\varphi_1^p(\alpha,\beta)=\left(\int_0^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}},$$

$$\varphi_2^p(\alpha,\beta)=\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|^p d\mu\right)^{\frac{1}{p}},$$

$$\varphi_3^p(\alpha,\beta)=\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|^p d\mu\right)^{\frac{1}{p}},$$

$$\varphi_4^p(\alpha,\beta)=\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^p d\mu\right)^{\frac{1}{p}}.$$

Proof. Applying Hölder inequality in (11), we obtain

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\int_0^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{4}}\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^q d\mu\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_0^{\frac{1}{4}}\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^q d\mu\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|^p d\mu\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^q d\mu\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^q d\mu\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\int_{\frac{3}{4}}^{1}\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}+\left(\int_{\frac{3}{4}}^{1}\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]\right\}.$$

By using convexity of $|\mathcal{F}'|^q$, we readily have

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^{p}d\mu\right)^{\frac{1}{p}}\left[\left(\int_{0}^{\frac{1}{4}}\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_{0}^{\frac{1}{4}}\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\int_{\frac{3}{4}}^{1}\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}+\left(\int_{\frac{3}{4}}^{1}\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]\right\}$$

$$= \frac{\alpha^{\beta}(\delta-\sigma)}{2}\Bigg\{\Bigg(\Bigg(\int_0^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^{p}d\mu\Bigg)^{\frac{1}{p}}+\Bigg(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\mu\Bigg)^{\frac{1}{p}}\Bigg)$$

$$\times\left[\left(\frac{7|\mathcal{F}'(\sigma)|^{q}+|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^{q}+7|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}\right]$$

$$+\Bigg(\Bigg(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|^{p}d\mu\Bigg)^{\frac{1}{p}}+\Bigg(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|^{p}d\mu\Bigg)^{\frac{1}{p}}\Bigg)$$

$$\times\left[\left(\frac{3|\mathcal{F}'(\sigma)|^{q}+5|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathcal{F}'(\sigma)|^{q}+3|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}\right]\Bigg\},$$

***Remark* 3.** Consider $\alpha=1$ in Theorem 4. Then the following Milne's formula holds:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J^{\beta}_{\delta-}\mathcal{F}(\sigma)+J^{\beta}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\delta-\sigma}{2}\Bigg\{(\varphi_1^p(1,\beta)+\varphi_4^p(1,\beta))\left[\left(\frac{7|\mathcal{F}'(\sigma)|^{q}+|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^{q}+7|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}\right]$$

$$+(\varphi_2^p(1,\beta)+\varphi_3^p(1,\beta))\left[\left(\frac{3|\mathcal{F}'(\sigma)|^{q}+5|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathcal{F}'(\sigma)|^{q}+3|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}\right]\Bigg\},$$

which is given in paper [4, Theorem 2].

***Remark* 4.** Let us note that $\beta=1$ and $\alpha=1$ in Theorem 5. Then we obtain the following Milne's formula:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\mu)d\mu\right|$$

$$\leq(\delta-\sigma)\Bigg\{\left(\frac{1}{(p+1)4^{p+1}}\right)^{\frac{1}{p}}\left[\left(\frac{7|\mathcal{F}'(\sigma)|^{q}+|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^{q}+7|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}\right]$$

$$+\left(\frac{5^{p+1}}{12^{p+1}(p+1)}-\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\sigma)|^{q}+5|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathcal{F}'(\sigma)|^{q}+3|\mathcal{F}'(\delta)|^{q}}{32}\right)^{\frac{1}{q}}\right]\Bigg\},$$

which is presented in paper [4, Corollary 2]. This inequality helps us find the error bound of Milne's rule.

Theorem 6. *Under the assumptions that Lemma* 1 *holds and the function* $|\mathcal{F}'|^{q}$, $q\geq 1$, *is convex on* $[\sigma,\delta]$. *Then we have the following Milne's formula*:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\Big\{(\Omega_1(\alpha,\beta))^{1-\frac{1}{q}}\Big[\big(\Omega_5(\alpha,\beta)\big|\mathcal{F}'(\delta)\big|^q+(\Omega_1(\alpha,\beta)-\Omega_5(\alpha,\beta))\big|\mathcal{F}'(\sigma)\big|^q\big)^{\frac{1}{q}}$$

$$+\ \big(\Omega_5(\alpha,\beta)\big|\mathcal{F}'(\sigma)\big|^q+(\Omega_1(\alpha,\beta)-\Omega_5(\alpha,\beta))\big|\mathcal{F}'(\delta)\big|^q\big)^{\frac{1}{q}}\Big]$$

$$+(\Omega_2(\alpha,\beta))^{1-\frac{1}{q}}\Big[\big(\Omega_6(\alpha,\beta)\big|\mathcal{F}'(\delta)\big|^q+(\Omega_2(\alpha,\beta)-\Omega_6(\alpha,\beta))\big|\mathcal{F}'(\sigma)\big|^q\big)^{\frac{1}{q}}$$

$$+\ \big(\Omega_6(\alpha,\beta)\big|\mathcal{F}'(\sigma)\big|^q+(\Omega_2(\alpha,\beta)-\Omega_6(\alpha,\beta))\big|\mathcal{F}'(\delta)\big|^q\big)^{\frac{1}{q}}\Big]$$

$$+(\Omega_3(\alpha,\beta))^{1-\frac{1}{q}}\Big[\big(\Omega_7(\alpha,\beta)\big|\mathcal{F}'(\delta)\big|^q+(\Omega_3(\alpha,\beta)-\Omega_7(\alpha,\beta))\big|\mathcal{F}'(\sigma)\big|^q\big)^{\frac{1}{q}}$$

$$+\ \big(\Omega_7(\alpha,\beta)\big|\mathcal{F}'(\sigma)\big|^q+(\Omega_3(\alpha,\beta)-\Omega_7(\alpha,\beta))\big|\mathcal{F}'(\delta)\big|^q\big)^{\frac{1}{q}}\Big]$$

$$+(\Omega_4(\alpha,\beta))^{1-\frac{1}{q}}\Big[\big(\Omega_8(\alpha,\beta)\big|\mathcal{F}'(\delta)\big|^q+(\Omega_4(\alpha,\beta)-\Omega_8(\alpha,\beta))\big|\mathcal{F}'(\sigma)\big|^q\big)^{\frac{1}{q}}$$

$$+\ \big(\Omega_8(\alpha,\beta)\big|\mathcal{F}'(\sigma)\big|^q+(\Omega_4(\alpha,\beta)-\Omega_8(\alpha,\beta))\big|\mathcal{F}'(\delta)\big|^q\big)^{\frac{1}{q}}\Big]\Big\}.$$

Here, $\Omega_1(\alpha,\beta),\ \Omega_2(\alpha,\beta),\ \Omega_3(\alpha,\beta)$ *and* $\Omega_4(\alpha,\beta)$ *are defined in Theorem* 4 *and*

$$\Omega_5(\alpha,\beta)=\int_0^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\mu d\mu$$

$$=\frac{1}{\alpha^{\beta+1}}\left[\delta\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{3}{4}\right)^{\alpha}\right)-\delta\left(\frac{2}{\alpha},\beta+1\right)+\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{3}{4}\right)^{\alpha}\right)\right],$$

$$\Omega_6(\alpha,\beta)=\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\mu d\mu,$$

$$\Omega_7(\alpha,\beta)=\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\mu d\mu,$$

$$\Omega_8(\alpha,\beta)=\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\mu d\mu$$

$$=\frac{7}{32\alpha^{\beta}}-\frac{1}{\alpha^{\beta+1}}\left[\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{4}\right)^{\alpha}\right)-\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{4}\right)^{\alpha}\right)\right].$$

Proof. If we first apply power-mean inequality in (11), then we have

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\Bigg\{\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$
\times\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}
$$

$$
+\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}
$$

$$
\times\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}
$$

$$
+\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}
$$

$$
\left.\times\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right\}.
$$

Using the fact that $|\mathcal{F}'|^{q}$ is convex, we get

$$
\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(\sigma)+\,{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(\delta)\right]\right|
$$

$$
\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu\right)^{1-\frac{1}{q}}\right.
$$

$$
\times\left[\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right.
$$

$$
\left.+\left(\int_{0}^{\frac{1}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right]
$$

$$
+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}
$$

$$\times\left[\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{2}{3\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{3}{4}}^{1}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left[\mu\left|\mathcal{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathcal{F}'(\delta)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right]\right\}.$$

***Corollary* 1.** *For $\alpha=1$ in Theorem 6, the following Milne's formula holds:*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\delta-}^{\beta}\mathcal{F}(\sigma)+J_{\sigma+}^{\beta}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\delta-\sigma}{2}\Big\{(\Omega_1(1,\beta))^{1-\frac{1}{q}}\Big[\big(\Omega_5(1,\beta)\left|\mathcal{F}'(\delta)\right|^{q}+(\Omega_1(1,\beta)-\Omega_5(1,\beta))\left|\mathcal{F}'(\sigma)\right|^{q}\big)^{\frac{1}{q}}$$

$$+\big(\Omega_5(1,\beta)\left|\mathcal{F}'(\sigma)\right|^{q}+(\Omega_1(1,\beta)-\Omega_5(1,\beta))\left|\mathcal{F}'(\delta)\right|^{q}\big)^{\frac{1}{q}}\Big]$$

$$+(\Omega_2(1,\beta))^{1-\frac{1}{q}}\Big[\big(\Omega_6(1,\beta)\left|\mathcal{F}'(\delta)\right|^{q}+(\Omega_2(1,\beta)-\Omega_6(1,\beta))\left|\mathcal{F}'(\sigma)\right|^{q}\big)^{\frac{1}{q}}$$

$$+ \left(\Omega_6(1,\beta)\left|\mathcal{F}'(\sigma)\right|^q + (\Omega_2(1,\beta) - \Omega_6(1,\beta))\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\Big]$$

$$+ (\Omega_3(1,\beta))^{1-\frac{1}{q}}\Big[\left(\Omega_7(1,\beta)\left|\mathcal{F}'(\delta)\right|^q + (\Omega_3(1,\beta) - \Omega_7(1,\beta))\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}$$

$$+ \left(\Omega_7(1,\beta)\left|\mathcal{F}'(\sigma)\right|^q + (\Omega_3(1,\beta) - \Omega_7(1,\beta))\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\Big]$$

$$+ (\Omega_4(1,\beta))^{1-\frac{1}{q}}\Big[\left(\Omega_8(\alpha,1)\left|\mathcal{F}'(\delta)\right|^q + (\Omega_4(1,\beta) - \Omega_8(1,\beta))\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}$$

$$+ \left(\Omega_8(1,\beta)\left|\mathcal{F}'(\sigma)\right|^q + (\Omega_4(1,\beta) - \Omega_8(1,\beta))\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\Big]\Big\}.$$

***Corollary* 2.** *Let us consider $\beta = 1$ and $\alpha = 1$ in Theorem* 6. *Then*, *the following Milne*'*s formula holds*:

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right) - \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right] - \frac{1}{\delta-\sigma}\int\limits_{\sigma}^{\delta}\mathcal{F}(\mu)d\mu\right|$$

$$\leq (\delta-\sigma)\left\{\frac{1}{32}\left[\left(\frac{|\mathcal{F}'(\delta)|^q + 5|\mathcal{F}'(\sigma)|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{F}'(\sigma)|^q + 5|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$

$$\left. + \frac{7}{96}\left[\left(\frac{5|\mathcal{F}'(\delta)|^q + 9|\mathcal{F}'(\sigma)|^q}{14}\right)^{\frac{1}{q}} + \left(\frac{5|\mathcal{F}'(\sigma)|^q + 9|\mathcal{F}'(\delta)|^q}{14}\right)^{\frac{1}{q}}\right]\right\}.$$

This inequality helps us find the error bound of Milne'*s rule*.

4. Summary and concluding remarks. Milne's formula has applications in various fields, including physics, engineering, and computational mathematics, where accurate numerical integration is essential for solving differential equations, simulating physical systems, and analysing data. Moreover, in the field of financial systems, Milne's formula may refer to its application in the field of mathematical finance and quantitative analysis. Financial derivatives, such as options and other complex instruments, often require the evaluation of integrals to determine their prices or sensitivities. Furthermore, Several new versions of Milne's formula are presented for the case of differentiable convex functions by using conformable fractional integrals. More precisely, Milne's formula are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, previous and new results are presented by using special cases of the obtained theorems.

In future works, the ideas and strategies for our results concerned with Milne's formula by conformable fractional integrals may open new ways for mathematicians. Moreover, one can try to generalize our results by utilizing a different version of convex function classes or another type fractional integral operators. Furthermore, one can obtain likewise corrected Milne's formula by conformable fractional integrals for convex functions by using quantum calculus.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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