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THE UNIT GROUP OF THE GROUP ALGEBRA $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$

ГРУПА ОДИНИЦЬ ГРУПОВОЇ АЛГЕБРИ $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$

Let p be a prime, $\mathbb{F}_q$ be a finite field with $q = p^n$ elements, and $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ be the semidirect product of the groups $\mathbb{Z}_9$ and $\mathbb{Z}_3$. The unit group $\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ of the group algebra $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ is completely characterized.

Нехай p — просте число, $\mathbb{F}_q$ — скінченне поле з $q = p^n$ елементами, а $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ — напівпрямий добуток груп $\mathbb{Z}_9$ і $\mathbb{Z}_3$. Наведено повну характеризацію групи одиниць $\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))$ групової алгебри $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$.

1. Introduction. Determination of the unit group of group algebras has been always very fascinating and challenging. A lot of work has been done for finding the algebraic structure of the unit group $\mathcal{U}(FG)$ of a group algebra FG, when G is a finite non-Abelian group. For example, Gildea [5, 6, 11], and Makhijani [20, 22, 23] have determined the unit group of group algebras of certain dihedral groups. Unit group of group algebra of the generalized dihedral group are in [22] by Makhijani. For algebras of circulant matrices and its unit group refer to Makhijani et al. [21], and Sharma [26]. Jordan regular units in group rings are given in [15]. Unit group of group algebras of certain permutation groups have determined in Gildea [8], Makhijani [24], and Sharma et al. [27, 28]. Recently, Gildea [7, 9, 10] has determined unit group of certain group algebras. Maheshwari and Sharma [25] characterized the unit group of the finite group algebra of special linear group $SL_2(\mathbb{Z}_3)$. In [14] we have characterized unit group of group algebra $\mathbb{F}S_5$. In this paper we have characterized completely unit group of the group algebra $\mathbb{F}(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ for any finite field $\mathbb{F}$, where $\mathbb{Z}_n$ denotes multiplicative group of integral modulo n, and $H \rtimes K$ denotes the semidirect product with H normal.

Units of group rings are very important from application point of view. Hurley [12] gives a relationship between ring of matrices and group rings. Again Hurley [13] and Dholakia [3] gives a method to construct convolutional codes by using units of group rings.

If $\mathbb{F}$ is a perfect field, then by using Wedderburn – Malcev theorem [2], we have

$$\mathcal{U}(\mathbb{F}G) \cong (1 + J(\mathbb{F}G)) \rtimes \mathcal{U}\left(\frac{\mathbb{F}G}{J(\mathbb{F}G)}\right),$$

where $J(\mathbb{F}G)$ denotes the Jacobson radical of the group algebra $\mathbb{F}G$. Thus, a nice description of Wedderburn decomposition of $\dfrac{\mathbb{F}G}{J(\mathbb{F}G)}$ is always very helpful to determine unit group of $\mathbb{F}G$.

We compute the disjoint conjugacy classes of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. These are 11 in numbers. For this, we have used the the presentation of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ given by

$$\mathbb{Z}_9 \rtimes \mathbb{Z}_3 = \big\{a, b : a^9 = b^3 = e,\ bab^{-1} = a^4\big\}.$$

The 11 conjugacy classes of group $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ are given as follows:

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$$[e] = \{e\}, \qquad [a] = \{a, a^4, a^7\}, \qquad [b] = \{b, ba^3, ba^6\}, \qquad [a^3] = \{a^3\},$$
$$[b^2] = \{b^2, b^2a^3, b^2a^6\}, \qquad [ba] = \{ba, ba^4, ba^7\}, \qquad [a^2] = \{a^2, a^5, a^8\}, \qquad [a^6] = \{a^6\},$$
$$[b^2a] = \{b^2a, b^2a^4, b^2a^7\}, \qquad [ba^2] = \{ba^2, ba^5, ba^8\}, \qquad [b^2a^2] = \{b^2a^2, b^2a^5, b^2a^8\}.$$

Through out the paper, $\mathbb{F}$, $\mathbb{F}_q$, and $\mathbb{F}_{p^n}$ are used interchangeably and represent a finite field with p^n elements, G represents a finite group.

For p-regular elements g, denoted by γ_g, the sum of all conjugates of g in G. The cyclotomic $\mathbb{F}$-class of γ_g will denoted by the set

$$S_{\mathbb{F}}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G,\mathbb{F}}\},$$

where $T_{G,\mathbb{F}_q} = \{q^i \pmod{s} \mid 0 \le i \le d-1\}$, d is the order of $q \pmod{s}$. Further details are given in Ferraz [4]. Several books have been specifically dedicated to exploring the algebraic structure within noncommutative group ring: A. A. Bovdi [1], G. Karpilovsky [16–19].

2. The structure of $\boldsymbol{\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))}$, $\boldsymbol{p \neq 3}$. Since $|(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)| = 3^3$, by Maschke's theorem, the Jocobson radical $J(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)) = 0$. Also observe that $\mathbb{F}(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ being a finite ring, is Artinian. Further, by Wedderburn decomposition theorem

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{M}_{n_1}(D_1) \oplus \mathbb{M}_{n_2}(D_2) \oplus \ldots \oplus \mathbb{M}_{n_r}(D_r),$$

where $D_1, D_2, \ldots, D_r$ are finite dimensional division algebras over the finite field $\mathbb{F}$. Hence, $D_k = \mathbb{F}_{p^{n_k}}$ and

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{M}_{n_1}(\mathbb{F}_{p^{m_1}}) \oplus \mathbb{M}_{n_2}(\mathbb{F}_{p^{m_2}}) \oplus \ldots \oplus \mathbb{M}_{n_r}(\mathbb{F}_{p^{m_r}}).$$

This immediately gives

$$\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)) \cong GL_{n_1}(\mathbb{F}_{p^{m_1}}) \times GL_{n_2}(\mathbb{F}_{p^{m_2}}) \times \ldots \times GL_{n_r}(\mathbb{F}_{p^{m_r}}).$$

Thus, the structure of the unit group $\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))$ is determined completely by computing the integers $r, n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r$.

Lemma 2.1. *Let $q = p^n$, $p \neq 3$, be a prime. Then Wedderburn decomposition of $\mathbb{F}_q(C_3 \times C_3)$ (C_3 being cyclic group order 3) is given as follows*:

Table 1

Condition on n	Wedderburn decomposition of $\mathbb{F}_q(C_3 \times C_3)$
n is even	$\mathbb{F}_q \oplus 8\mathbb{F}_q$
n is odd with $p \equiv 1 \pmod{3}$	$\mathbb{F}_q \oplus 8\mathbb{F}_q$
n is odd with $p \equiv -1 \pmod{3}$	$\mathbb{F}_q \oplus 4\mathbb{F}_q(\omega)$

Proof. When n is even, $p^n \equiv 1 \pmod{3}$, i.e., $3/(p^n - 1)$, it implies that the field $\mathbb{F}_q$ has primitive third root of unity. Hence, we have

$$\mathbb{F}_q(C_3 \times C_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q.$$

Next, let n be odd. We divide this case into two subcases: $p \equiv 1 \pmod{3}$ and $p \equiv -1 \pmod{3}$.

Subcase 1. If $p \equiv 1 \pmod 3$ and n is odd, then $p^n \equiv 1 \pmod 3$, which indicates that the field $\mathbb{F}_q$ has a primitive third root of unity. Thus, we have

$$\mathbb{F}_q(C_3 \times C_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q.$$

Subcase 2. If $p \equiv -1 \pmod 3$ and n is odd, then $p^n \not\equiv 1 \pmod 3$. In this case, the field $\mathbb{F}_q$ doesn't have a primitive third root of unity. Therefore, we get

$$\mathbb{F}_q(C_3 \times C_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_q(\omega),$$

where $\omega \notin \mathbb{F}_q$ is a third root of unity.

Theorem 2.1. *Let* $q = p^n$, $p \neq 3$, *be a prime*, *then Wedderburn decomposition of* $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ *is given as follows*:

Table 2

Condition on n	Wedderburn decomposition of $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$
n is even	$\mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q)$
n is odd with $p \equiv 1 \pmod 3$	$\mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q)$
n is odd with $p \equiv -1 \pmod 3$	$\mathbb{F}_q \oplus 4\mathbb{F}_q(\omega) \oplus \mathbb{M}_3(\mathbb{F}_{q^2})$

Proof. Observe that $\dfrac{\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)}{\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)'} \cong \mathbb{F}_q(C_3 \times C_3)$, as $(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)' \cong C_3 \times C_3$. Also, $|\mathbb{Z}_9 \rtimes \mathbb{Z}_3| = 27 = 3^3$. Hence, the group algebra $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ is semisimple, as $p \nmid |G|$. Here, $q = p^n$, $p \neq 3$. Now

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) = \mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)e_{(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)'} \oplus \mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)((\mathbb{Z}_9 \rtimes \mathbb{Z}_3)' - 1)),$$

where

$$e_{(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)'} = e_{(C_3 \times C_3)} = \frac{(\widehat{C_3 \times C_3})'}{|C_3 \times C_3|} = \frac{\sum_{\sigma \in (C_3 \times C_3)} \sigma}{9},$$

and

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)e_{(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)'} = \text{sum of all commutative simple components of } \mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3).$$

However,

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)e_{(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)'} \cong \mathbb{F}_q\left(\frac{(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)}{(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)'}\right) \cong \mathbb{F}_q(C_3 \times C_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q \qquad \text{or} \qquad \mathbb{F}_q \oplus 4\mathbb{F}_q(\omega).$$

This gives the Wedderburn decomposition

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \sum_{i=1}^{2} \mathbb{M}_{n_i}(\mathbb{F}_{q^{k_i}})$$

or

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_q(\omega) \oplus \sum_{i=1}^{2} \mathbb{M}_{n_i}(\mathbb{F}_{q^{k_i}})$$

for $n_i \geq 2$.

We divide the proof in two cases.

Case 1. When n is even, $p^n \equiv 1 \pmod 3$, i.e., the field $\mathbb{F}_q$ contains primitive third root of unity. In this case, we get

$$S_{\mathbb{F}_q(\gamma_g)} = \{\gamma_g\} \quad \text{for each group element } g,$$

that is,

$$|S_{\mathbb{F}_q(\gamma_g)}| = 1$$

for each $g \in \mathbb{Z}_9 \rtimes \mathbb{Z}_3$. This gives

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \sum_{i=1}^{2} \mathbb{M}_{n_i}(\mathbb{F}_q)$$

by Lemma 2.1.

By dimension constraints, we have

$$\dim_{\mathbb{F}_q}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)) = 1 + 8 + n_1^2 + n_2^2,$$
$$27 = 1 + 8 + n_1^2 + n_2^2 \qquad \text{or} \qquad 18 = n_1^2 + n_2^2.$$

This equation has only one solution, namely,

$$n_1 = 3, \quad n_2 = 3.$$

Hence, in this case, we obtain

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q).$$

Case 2. Let n be odd. We further divide this case into two subcases:

(1) $p \equiv 1 \pmod 3$,

(2) $p \equiv -1 \pmod 3$.

Subcase 1. If $p \equiv 1 \pmod 3$ and n is odd, observe that

$$S_{\mathbb{F}_q(\gamma_g)} = \{\gamma_g\} \quad \text{for each group element } g,$$

that is,

$$|S_{\mathbb{F}_q(\gamma_g)}| = 1$$

for each $g \in \mathbb{Z}_9 \rtimes \mathbb{Z}_3$.

Again, as $p \equiv 1 \pmod 3$ and n is odd, $p^n \equiv 1 \pmod 3$, which says that the field $\mathbb{F}_q$ has primitive third root of unity. Hence, as in case 1 when n is even, by the above lemma, we get

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q).$$

Subcase 2. Let $p \equiv -1 \pmod 3$, and n is odd. Then

$$S_{\mathbb{F}_q(\gamma_e)} = \{\gamma_e\}, \quad \text{i.e., } |S_{\mathbb{F}_q(\gamma_e)}| = 1,$$
$$S_{\mathbb{F}_q(\gamma_a)} = \{\gamma_a, \gamma_{a^2}\}, \quad \text{i.e., } |S_{\mathbb{F}_q(\gamma_a)}| = 2,$$
$$S_{\mathbb{F}_q(\gamma_{a^3})} = \{\gamma_{a^3}, \gamma_{a^6}\}, \quad \text{i.e., } |S_{\mathbb{F}_q(\gamma_{a^3})}| = 2,$$
$$S_{\mathbb{F}_q(\gamma_b)} = \{\gamma_b, \gamma_{b^2}\}, \quad \text{i.e., } |S_{\mathbb{F}_q(\gamma_b)}| = 2,$$
$$S_{\mathbb{F}_q(\gamma_{b^2a})} = \{\gamma_{b^2a}, \gamma_{ab^2}\}, \quad \text{i.e., } |S_{\mathbb{F}_q(\gamma_{b^2a})}| = 2,$$
$$S_{\mathbb{F}_q(\gamma_{ba})} = \{\gamma_{ba}, \gamma_{b^2a^2}\}, \quad \text{i.e., } |S_{\mathbb{F}_q(\gamma_{ba})}| = 2,$$

that is, in this subcase, we have $2 = |S_i| = |K_i : \mathbb{F}|$ [4, Proposition 1.3] for $i \neq 1$. Again, since $p \equiv -1 \pmod 3$ and n is odd, $p^n \not\equiv 1 \pmod 3$, by using above lemma, we get

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_q(\omega) \oplus \mathbb{M}_{n_1}(\mathbb{F}_{q^2}).$$

Equating dimensions on both side, we obtain

$$\dim_{\mathbb{F}_q}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)) = 1 + 4 \cdot 2 + 2n_1^2,$$
$$27 = 9 + 2n_1^2 \qquad \text{or} \qquad 18 = 2n_1^2.$$

Only possible solution of the equation is $n_1 = 3$.

Hence, in this subcase, we have

$$\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_q(\omega) \oplus \mathbb{M}_3(\mathbb{F}_{q^2}).$$

***Corollary* 2.1.** *Let $q = p^n$, where $p \neq 3$ be a prime. Then $\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))$, i.e., unit group of the group ring $\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$ is given as follows*:

Table 3

Condition on n	$\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))$
n is even	$\mathbb{F}_q^* \times 8\mathbb{F}_q^* \times GL_3(\mathbb{F}_q) \times GL_3(\mathbb{F}_q)$
n is odd with $p \equiv 1 \pmod 3$	$\mathbb{F}_q^* \times 8\mathbb{F}_q^* \times GL_3(\mathbb{F}_q) \times GL_3(\mathbb{F}_q)$
n is odd with $p \equiv -1 \pmod 3$	$\mathbb{F}_q^* \times 4\mathbb{F}_{q^2}^* \times GL_3(\mathbb{F}_{q^2})$

Proof. Proof is simple application of the fact that for any two ring R_1 and R_2, $\mathcal{U}(R_1 \oplus R_2) = \mathcal{U}(R_1) \times \mathcal{U}(R_2)$ and $\mathbb{F}_q(\omega) \cong \mathbb{F}_{q^2}$.

3. The structure of $\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))$, $q = 3^n$.

Proposition 3.1. *Let G be a finite group and $\mathbb{F}$ be a finite field of prime characteristic p. If G has a normal p-Sylow subgroup N, then the Jacobson radical $J(FG)$ of the group algebra $\mathbb{F}G$ is the kernel of the homomorphism $\phi\colon \mathbb{F}G \to \mathbb{F}[G/N]$ induced by the projection $G \to G/N$.*

Proof. If N is A normal p-Sylow subgroup, then by Maschke's theorem $J(\mathbb{F}[G/N] = 0$, i.e., $\mathbb{F}[G/N]$ is semisimple and, hence, $\ker(\phi) \supseteq J(\mathbb{F}G)$. This proves the proposition in one direction. Conversely, let H be any simple $\mathbb{F}G$-module. Clifford's theorem [16], says that the restriction of H to FN is semisimple. The only simple $\mathbb{F}N$-module is the trivial module as N is a p-group, and hence N acts on H trivially. Now $\ker(\phi)$ annihilates H as the action of G on H factors through G/N. Now the Jacobson radical $J(\mathbb{F}G)$ is the intersection of annihilators of all simple $\mathbb{F}G$-modules, we get $\ker(\phi) \subseteq J(\mathbb{F}G)$. This proves that $\ker(\phi) = J(\mathbb{F}G)$.

Proposition 3.2. *Let G be a finite p-group and $\mathbb{F}$ be a finite field of prime characteristic p. Then the Jacobson radical $J(\mathbb{F}G)$ of the group algebra $\mathbb{F}G$ is equal to $\Delta(G)$.*

Proof. Proof follows from the fact that G/N is trivial group in this case.

Theorem 3.1. *Let $q = 3^n$, then $\dfrac{\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)}{J(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))} \cong \mathbb{F}_q$.*

Proof. An element $x \in \mathbb{Z}_9 \rtimes \mathbb{Z}_3$ is 3-regular if $3 \nmid |x|$. As each non identity element of G is of order 3 or 9, only one element is 3-regular, i.e., the identity element e. Hence, by [4, Proposition 1.2], we have that $\dfrac{\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)}{J(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))}$ has only one simple component of dimension 1 over $\mathbb{F}_q$, as dimension of $J(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))$ is equal to 26 over $\mathbb{F}_q$. This proves that $\dfrac{\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)}{J(\mathbb{F}_q(\mathbb{Z}_9 \rtimes \mathbb{Z}_3))} \cong \mathbb{F}_q$.

***Corollary* 3.1.** *Let $q = 3^n$ and $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$. Then $\mathcal{U}(\mathbb{F}_q(G) \cong (1 + J(\mathbb{F}_qG)) \rtimes \mathbb{F}_q^*$, where $1 + J(\mathbb{F}_qG)$ is a non-Abelian group of exponent 9.*

Proof. By Theorem 3.1, we have $\mathcal{U}\left(\dfrac{\mathbb{F}_qG}{J(\mathbb{F}_qG)}\right) \cong \mathbb{F}_q^*$. As G is non-Abelian, there exist elements a and b in G such that $ab \neq ba$. Now, since $a - 1,\ b - 1$ are elements of augmentation ideal $\Delta(G) \subseteq J(\mathbb{F}_qG)$, it follows that $a = a - 1 + 1$, and $b = b - 1 + 1$ are two non commutating elements of $1 + J(\mathbb{F}_qG)$. This proves that $1 + J(\mathbb{F}_qG)$ is non-Abelian. Since group G is 3-solvable, by [16, p. 110, Proposition 1.9] we have $(1 + J(\mathbb{F}_qG))^{27} = 1$. Observe that G has elements of order 9. Therefore, the exponent of $1 + J(\mathbb{F}_qG)$ is either 9 or 27. Since G is a non-Abelian 3-group, 3-Sylow subgroups of G are non cyclic. Hence, by [16, p. 419, Proposition 2.4], index of nilpotency of $J(\mathbb{F}_qG)$ must be less than 3^3. Thus, the exponent of $1 + J(\mathbb{F}_qG)$ must be equal to 9.

4. The structure of $\boldsymbol{\mathcal{U}(\mathbb{F}_q(\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p)),\ q = p^n}$. In this section, we generalize results of Section 3 to the group algebra of $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$. The group $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ is the semi direct product of groups $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p$, and has the following presentation:

$$\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p = \left\{a, b\colon a^{p^2} = b^p = 1,\ bab^{-1} = a^{p+1}\right\}.$$

Theorem 4.1. *Let $q = p^n$ and $G = \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ for an odd prime p. Then $\dfrac{\mathbb{F}_qG}{J(\mathbb{F}_q(G))} \cong \mathbb{F}_q$.*

Proof. An element $x \in G$ is p-regular if $p \nmid |x|$. As each non identity element of G is of order p or p^2, only identity element e is p-regular. By Proposition 3.2, we have $\dim J(\mathbb{F}_qG) = p^3 - 1$ over $\mathbb{F}_q$. Hence, $\dfrac{\mathbb{F}_qG}{J(\mathbb{F}_q(G))}$ has only one simple component of dimension 1. Thus, we have $\dfrac{\mathbb{F}_qG}{J(\mathbb{F}_q(G))} \cong \mathbb{F}_q$.

***Corollary* 4.1.** *Let $q = p^n$ and $G = \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ for an odd prime p. Then $\mathcal{U}(\mathbb{F}_q(G) \cong (1 + J(\mathbb{F}_qG)) \rtimes \mathbb{F}_q^*$, where $1 + J(\mathbb{F}_qG)$ is a non-Abelian group of exponent p^2.*

Proof. By Theorem 4.1, we have $\mathcal{U}\left(\dfrac{\mathbb{F}_qG}{J(\mathbb{F}_qG)}\right) \cong \mathbb{F}_q^*$. Now, since G is non-Abelian, there exist elements a and b in G such that $ab \neq ba$. Since $a - 1,\ b - 1$ are elements of augmentation ideal $\Delta(G) \subseteq J(\mathbb{F}_qG)$, $a = a - 1 + 1$ and $b = b - 1 + 1$ are two non commutating elements of $1 + J(\mathbb{F}_qG)$. This proves that $1 + J(\mathbb{F}_qG)$ is non-Abelian. Since G is p-solvable, using [16, p. 110, Proposition 1.9], we have $(1 + J(\mathbb{F}_qG))^{p^3} = 1$. Now, G has elements of order p^2, it implies that exponent of $1 + J(\mathbb{F}_qG)$ is either p^2 or p^3. Again G is a non-Abelian p-group, p-Sylow subgroups of G are non cyclic. Hence, by [16, p. 419, Proposition 2.4], index of nilpotency of $J(\mathbb{F}_qG)$ must be less than p^3. Therefore, exponent of $1 + J(\mathbb{F}_qG)$ must be less than p^3, i.e., equal to p^2.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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