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THE FEKETE – SZEGÖ FUNCTIONAL ASSOCIATED WITH M-TH ROOT TRANSFORMATION USING CONICAL DOMAINS

ФУНКЦІОНАЛ ФЕКЕТЕ – СЕГЬО, АСОЦІЙОВАНИЙ З M-М КОРЕНЕВИМ ПЕРЕТВОРЕННЯМ ІЗ ВИКОРИСТАННЯМ КОНІЧНИХ ОБЛАСТЕЙ

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{R}_\alpha^p$ be the operator defined on $\mathcal{A}$ by

$$\mathcal{R}_\alpha^p = f(z) * \frac{z}{(1-z)^{2(1-\alpha)}}.$$

A function f in $\mathcal{A}$ is said to be in the class k-$\mathcal{SP}_\alpha^p$ if $\mathcal{R}_\alpha^p(f)$ is a *k-parabolic starlike function*. We focus on the Fekete – Szegö inequality associated with m-th root transformation using conical domains for this class.

Нехай $\mathcal{A}$ — клас аналітичних функцій у відкритому одиничному диску $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, а $\mathcal{R}_\alpha^p$ — оператор, визначений на $\mathcal{A}$ за допомогою формули

$$\mathcal{R}_\alpha^p = f(z) * \frac{z}{(1-z)^{2(1-\alpha)}}.$$

Кажуть, що функція f з $\mathcal{A}$ належить до класу k-$\mathcal{SP}_\alpha^p$, якщо $\mathcal{R}_\alpha^p(f)$ є *k-параболічною зіркоподібною функцією*. Наша увага сфокусована на нерівності Фекете – Сегьо, що асоційована з перетворенням m-го кореня з використанням конічних областей для цього класу.

1. Introduction. Let $\mathcal{A}$ be the class of functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}_0$ be the family of functions f in $\mathcal{A}$ satisfying the normalization condition [12]. Thus, the functions in $\mathcal{A}_0$ are given by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of analytic univalent functions in $\mathbb{U}$. For fixed k, $0 \le k < \infty$, the function $f \in \mathcal{A}_0$ is said to be in k-$\mathcal{UCV}$, the class of *k-uniformly convex functons* in $\mathbb{U}$, if the image of every circular arc γ contained in $\mathbb{U}$ with center ξ where $|\xi| \le k$, convex arc. This interesting unification of the concepts of univalent convex functions [12] and uniformily convex functions [9] is due to Kanas and Wisniowska [10].

The class k-$\mathcal{SP}$, consisting of *k-parabolic starlike functons* is defined from k-$\mathcal{UCV}$ via the Alexandar transforms [11], that is,

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$$f \in k\text{-}\mathcal{UCV} \Leftrightarrow g \in k\text{-}\mathcal{SP},$$

where $g(z) = zf'(z),\ z \in \mathbb{U}$.

The one variable characterization theorem [10] of the class k-$\mathcal{UCV}$ gives that $f \in k$-$\mathcal{UCV}$ (respectively, $f \in k$-$\mathcal{SP}$) if and only if the values of $p(z) = 1 + \dfrac{zf''(z)}{f'(z)}$ $\left(\text{respectively, } p(z) = \dfrac{zf'(z)}{f(z)}\right)$, $z \in \mathbb{U}$, lie in the conic region Ω_k in the w-plane, where

$$\Omega_k = \left\{w = u + iv \in C : u^2 > k^2(u-1)^2 + k^2v^2,\ u > 0,\ 0 \le k < \infty\right\}.$$

For details of the geometric description of Ω_k see [10, 11].

A function is said to be univalent in a simply connected domain if the images of distinct points are distinct there. A function is said to be starlike of order α, denoted by $\mathcal{S}^*(\alpha)$, if and only if $\operatorname{Re}\left[\dfrac{zf'(z)}{f(z)}\right] \ge \alpha$ in $\mathbb{U}$. Also the set $\mathcal{S}^*(\alpha)$ for $0 \le \alpha < 1$ is a subset of $\mathcal{S}$ and the function

$$\frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n = z + \sum_{n=2}^{\infty} \left(\frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}\right) z^n$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$.

The convolution or Hadamard product of two power series $\phi(z) = \sum_{n=0}^{\infty} u_n z^n$ analytic in $|z| < r$ and $\varphi(z) = \sum_{n=0}^{\infty} v_n z^n$ analytic in $|z| < r_2$ is the power series $(\phi * \varphi)(z) = \phi(z) * \varphi(z) = \sum_{n=0}^{\infty} u_n v_n z^n$ which is analytic in $|z| < r_1 r_2$ [12].

For any natural number m, the m-th root transform is defined by

$$F(z) = [f(z^m)]^{1/m} = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}, \quad z \in \mathbb{U}. \tag{1.2}$$

Let m be a positive integer. A domain D is said to be *m-fold symmetric* if a rotation of D about the origin through an angle $\dfrac{2\pi}{m}$ carries D to itself. A function $f(z)$ is siad to be *m-fold symmetric* in $\mathbb{U}$ if for every z in $\mathbb{U}$

$$f\left(e^{\frac{2\pi i}{m}} z\right) = e^{\frac{2\pi i}{m}} f(z).$$

In 1916, Gronwall shows that if $f(z)$ is regular and *m-fold symmetric* in $\mathbb{U}$, then it has a power series expansion of the form

$$f(z) = b_1 z + b_{m+1} z^{m+1} + b_{2m+1} z^{2m+1} + \ldots = \sum_{n=0}^{\infty} b_{nm+1} z^{nm+1}. \tag{1.3}$$

Conversely, if $f(z)$ is given by the power series (1.3), then $f(z)$ is *m-fold symmetric* inside the circle of convergence of the series.

For different subclasses of $\mathcal{S}$, the Fekete – Szegö problem has been investigated by many authors including [4, 5, 8, 13 – 23].

In the present paper, the Fekete – Szegö inequality for the class k-$\mathcal{SP}_\alpha^p$, $0 \le k < \infty$, associated with m-th root transformation using conical domains is settled completely. Also, for particular values of k, we obtain the Fekete – Szegö inequality.

2. Preliminary results.

Lemma 2.1 [6]. *Let* $k \in [0, \infty)$ *be fixed and* $q_k(z)$ *be the Riemann map of* $\mathbb{U} = \{z \in \mathbb{C} \colon |z| < 1\}$ *on to* Ω_k *satisfying* $q_k(0) = 1,\ q_k'(0) > 0$. *If*

$$q_k(z) = 1 + Q_1(k)z + Q_2(k)z^2 + Q_3(k)z^3 + \ldots, \quad z \in \mathbb{U}, \tag{2.1}$$

then

$$Q_1 = Q_1(k) = \begin{cases} \dfrac{2A^2}{1-k^2}, & 0 \le k < 1, \\ \dfrac{8}{\pi^2}, & k = 1, \\ \dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}, & k > 1, \end{cases}$$

$$Q_2 = Q_2(k) = D(k)Q_1(k),$$

where

$$D = D(k) = \begin{cases} \dfrac{A^2+2}{3}, & 0 \le k < 1, \\ \dfrac{2}{3}, & k = 1, \\ \dfrac{4\kappa^2(t)\left(t^2+6t+1\right) - \pi^2}{24\kappa^2(t)(1+t)\sqrt{t}}, & k > 1, \end{cases}$$

$$A = \frac{2}{\pi}\arccos k,$$

and $\kappa(t) = \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$, $t \in (0,1)$, *is the complete elliptic integral of first kind (for details see* [1 – 3]).

Lemma 2.2 [7]. *Let the function* $\omega \in B_0$ *be given by*

$$\omega(z) = d_1 z + d_2 z^2 + d_3 z^3 + \ldots, \quad z \in \mathbb{U},$$

where

$$B_0 = \{\omega \in A \colon \omega(0) = 0,\ |\omega(z)| < 1,\ z \in \mathbb{U}\}.$$

Then, for every complex number μ,

$$\left|d_2 - \mu d_1^2\right| \le 1 + (|\mu| - 1)\left|d_1^2\right|.$$

3. Main results.

Definition 3.1. *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}_\alpha^p$, $0 \le k < \infty$, *if* $\mathcal{R}_\alpha^p(f) \in k$-$\mathcal{SP}$ *or*, *equivalently*,

$$\operatorname{Re}\left(\frac{z\left(\mathcal{R}_\alpha^p(f)'(z)\right)}{\mathcal{R}_\alpha^p(f)(z)}\right) > k\left|\frac{z\left(\mathcal{R}_\alpha^p(f)'(z)\right)}{\mathcal{R}_\alpha^p(f)(z)} - 1\right|, \quad z \in \mathbb{U}.$$

We first state and prove the following theorem.

Theorem 3.1. *The function $f \in \mathcal{A}_0$, given by* (1.1), *is said to be in the class k-$\mathcal{SP}^p_\alpha$, $0 \le k < \infty$, $0 \le \alpha < 1$, and if F is the m-th root transformation of f given by* (1.2). *Then*

$$\left|b_{2m+1} - \mu b^2_{m+1}\right|$$
$$\le \begin{cases} \dfrac{Q_1}{2m(1-\alpha)(3-2\alpha)}\left\{\dfrac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)} - Q_1 - D\right\}, & \text{if } \mu \ge \sigma_1, \\ \dfrac{Q_1}{2m(1-\alpha)(3-2\alpha)}, & \text{if } \sigma_2 \le \mu \le \sigma_1, \\ \dfrac{Q_1}{2m(1-\alpha)(3-2\alpha)}\left\{Q_1 + D - \dfrac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right\}, & \text{if } \mu \le \sigma_2, \end{cases} \tag{3.1}$$

where

$$\sigma_1 = \frac{2m(1-\alpha)}{(3-2\alpha)Q_1}\left(1 + Q_1 + D - \frac{(m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right),$$
$$\sigma_2 = \frac{2m(1-\alpha)}{(3-2\alpha)Q_1}\left(Q_1 + D - 1 - \frac{(m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right)$$

and Q_1 and D are given by Lemma 2.1.

Proof. The expansion of $\dfrac{z}{(1-z)^{2(1-\alpha)}}$ is given by

$$\frac{z}{(1-z)^{2(1-\alpha)}} = z + \frac{2-2\alpha}{1!}z^2 + \frac{(2-2\alpha)(3-2\alpha)}{2!}z^3 + \ldots .$$

So, by the definition of Hadamard product, we have

$$\mathcal{R}^p_\alpha(f)(z) = z + \frac{(2-2\alpha)a_2}{1!}z^2 + \frac{(2-2\alpha)(3-2\alpha)a_3}{2!}z^3 + \ldots .$$

By using simple calculations, we get

$$\frac{z\big(\mathcal{R}^p_\alpha(f)'(z)\big)}{\mathcal{R}^p_\alpha(f)(z)} = 1 + (2-2\alpha)a_2 z + \Big((2-2\alpha)(3-2\alpha)a_3 - (2-2\alpha)^2 a_2^2\Big)z^2 + \ldots . \tag{3.2}$$

By the definition, there are analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{z\big(\mathcal{R}^p_\alpha(f)'(z)\big)}{\mathcal{R}^p_\alpha(f)(z)} = q_k(\omega(z)),$$

where $\omega(z)$ is given by

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots . \tag{3.3}$$

From the equations (2.1) and (3.3), we obtain

$$q_k(\omega(z)) = 1 + Q_1(k)c_1 z + \big(Q_1(k)c_2 + Q_2(k)c_1^2\big)z^2 + \ldots . \tag{3.4}$$

Note that, throughout, we shall write $Q_1 = Q_1(k)$, $Q_2 = Q_2(k)$ and $D(k)$. Comparing the corresponding coefficients from (3.2) and (3.4), we have

$$a_2 = \frac{Q_1 c_1}{2(1-\alpha)} \tag{3.5}$$

and

$$a_3 = \frac{Q_1 c_2 + \big(Q_2 + Q_1^2\big) c_1^2}{2(1-\alpha)(3-2\alpha)}. \tag{3.6}$$

Since $Q_2 = DQ_1$, then (3.6) becomes

$$a_3 = \frac{Q_1 c_2 + \big(DQ_1 + Q_1^2\big) c_1^2}{2(1-\alpha)(3-2\alpha)}, \tag{3.7}$$

where $D(k)$ is as in Lemma 2.1.

For a function f given by (1.1), a computation shows that

$$[f(z^m)]^{1/m} = z + \frac{1}{m} a_2 z^{m+1} + \left[\frac{1}{m} a_3 - \frac{1}{2}\left(\frac{m-1}{m^2}\right) {a_2}^2\right] z^{2m+1} + \ldots . \tag{3.8}$$

From (1.2) and (3.8), equating the coefficients of z^{m+1} and z^{2m+1}, we get

$$b_{m+1} = \frac{1}{m} a_2 \tag{3.9}$$

and

$$b_{2m+1} = \frac{a_3}{m} - \frac{1}{2}\left(\frac{m-1}{m^2}\right) {a_2}^2. \tag{3.10}$$

From (3.5), (3.7), (3.9) and (3.10), we obtain

$$\begin{aligned} b_{2m+1} - \mu b_{m+1}^2 &= \frac{Q_1}{2m(1-\alpha)(3-2\alpha)} \\ &\quad \times \left\{ c_2 + \left(Q_1 + D - \frac{(2\mu + m - 1)(3-2\alpha)Q_1}{4m(1-\alpha)} \right) c_1^2 \right\}, \end{aligned} \tag{3.11}$$

which gives us

$$\begin{aligned} b_{2m+1} - \mu b_{m+1}^2 &= \frac{Q_1}{2m(1-\alpha)(3-2\alpha)} \\ &\quad \times \left\{ c_2 - c_1^2 + \left(1 + Q_1 + D - \frac{(2\mu + m - 1)(3-2\alpha)Q_1}{4m(1-\alpha)} \right) c_1^2 \right\}. \end{aligned} \tag{3.12}$$

Therefore,

$$\begin{aligned} b_{2m+1} - \mu b_{m+1}^2 &= \frac{Q_1}{2m(1-\alpha)(3-2\alpha)} \\ &\quad \times \left\{ \left|c_2 - c_1^2\right| + \left(1 + Q_1 + D - \frac{(2\mu + m - 1)(3-2\alpha)Q_1}{4m(1-\alpha)} \right) |c_1|^2 \right\}. \end{aligned} \tag{3.13}$$

Suppose that $\mu \geq \sigma_1$, the expression inside the second modulus symbol on the right-hand side of (3.13) is nonnegative. Then, using the estimate $\left|c_2 - c_1^2\right| \leq 1$ from Lemma 2.2 and the well-known esimate $|c_1| \leq 1$ of the Schwarz lemma, we have

$$\begin{aligned} \left|b_{2m+1} - \mu b_{m+1}^2\right| &\leq \frac{Q_1}{2m(1-\alpha)(3-2\alpha)} \\ &\quad \times \left\{ 1 + \left(\frac{(2\mu + m - 1)(3-2\alpha)Q_1}{4m(1-\alpha)} - Q_1 - D - 1 \right) \right\}, \end{aligned} \tag{3.14}$$

which implies that

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}\left\{\frac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}-Q_1-D\right\}.$$

This is precisely the first inequality in (3.1).

On the other hand, $\mu \leq \sigma_2$, then (3.11) gives

$$b_{2m+1}-\mu b_{m+1}^2 = \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}$$

$$\times\left\{|c_2|+\left(Q_1+D-\frac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right)|c_1|^2\right\}.$$

Applying estimate $|c_2| \leq 1-|c_1|^2$ of Lemma 2.2 and $|c_1| \leq 1$, we have

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}$$

$$\times\left\{1-|c_1|^2+\left(Q_1+D-\frac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right)|c_1|^2\right\},$$

which implies that

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}\left\{Q_1+D-\frac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right\}, \tag{3.15}$$

this is the last inequality in (3.1).

Lastly, if $\sigma_2 \leq \mu \leq \sigma_1$, then

$$\left|Q_1+D-\frac{(2\mu+m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right| \leq 1.$$

Therefore, (3.11) yields

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}.$$

This is the middle inequality in (3.1).

Now, we discuss the sharpness of the inequalities in (3.1). Suppose that $\mu > \sigma_1$. Then equality holds in (3.1) if and only if the equality holds in (3.14). This happens if and only if $|c_1| = 1$ and $\left|c_2 - c_1^2\right| = 1$. Therefore, $u(z) = z$ and the extremal function is $q_k(z)$.

Next, if $\mu < \sigma_2$, then equality holds (3.15) if and only if $c_1^2 = -1$ (and, hence, $c_2 = 0$ in (3.12)). Thus, $\omega(z) = e^{i\theta} z$, where $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$.

Lastly, if $\sigma_2 \le \mu \le \sigma_1$, then equality holds (3.1), if $c_1 = 0$ and $|c_2| = 1$. Thus, $\omega(z)$ is a rotation of z^2.

Theorem 3.1 is proved.

If $m = 1$ in Theorem 3.1, then we have the following corollary.

***Corollary* 3.1.** *The function $f \in \mathcal{A}_0$, given by* (1.1), *is said to be in the class k-$\mathcal{SP}^p_\alpha$, $0 \le k < \infty$, $0 \le \alpha < 1$. Then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} \dfrac{Q_1}{2(1-\alpha)(3-2\alpha)}\left\{\dfrac{\mu(3-2\alpha)Q_1}{2(1-\alpha)} - Q_1 - D\right\}, & \text{if } \mu \ge \alpha_1, \\ \dfrac{Q_1}{2(1-\alpha)(3-2\alpha)}, & \text{if } \alpha_2 \le \mu \le \alpha_1, \\ \dfrac{Q_1}{2(1-\alpha)(3-2\alpha)}\left\{Q_1 + D - \dfrac{\mu(3-2\alpha)Q_1}{2(1-\alpha)}\right\}, & \text{if } \mu \le \alpha_2, \end{cases}$$

where

$$\alpha_1 = \frac{2(1-\alpha)}{(3-2\alpha)Q_1}(1 + Q_1 + D),$$

$$\alpha_2 = \frac{2(1-\alpha)}{(3-2\alpha)Q_1}(Q_1 + D - 1)$$

and Q_1 and D are given by Lemma 2.1.

If $m = 1$ and $\alpha = 0$ in Theorem 3.1, then we have the following corollary.

***Corollary* 3.2.** *The function $f \in \mathcal{A}_0$, given by* (1.1), *is said to be in the class k-$\mathcal{SP}^p_\alpha$, $0 \le k < \infty$. Then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} \dfrac{Q_1}{6}\left(Q_1 + D - \dfrac{3\mu Q_1}{2}\right), & \text{if } \mu \le \dfrac{2}{3Q_1}(Q_1 + D - 1), \\ \dfrac{Q_1}{6}, & \text{if } \dfrac{2}{3Q_1}(Q_1 + D - 1) \le \mu \le \dfrac{2}{3Q_1}(1 + Q_1 + D), \\ \dfrac{Q_1}{6}\left(\dfrac{3\mu Q_1}{2} - Q_1 - D\right), & \text{if } \mu \ge \dfrac{2}{3Q_1}(1 + Q_1 + D), \end{cases}$$

where Q_1 and D are given by Lemma 2.1.

Putting the values of $Q_1 = Q_1(k)$ and $D = D(k)$ from Lemma 2.1 in Theorem 3.1 for $0 \le k < 1$, $k = 1$ and $k > 1$, respectively, we ge the following theorems.

Theorem 3.2. *The function $f \in \mathcal{A}_0$, given by* (1.1), *is said to be in the class k-$\mathcal{SP}^p_\alpha$, $0 \le k < 1$, $0 \le \alpha < 1$. Then*

$$\left|b_{2m+1}-\mu b_{m+1}^2\right|$$
$$\leq\begin{cases}\dfrac{1}{m(1-\alpha)(3-2\alpha)}\left(\dfrac{A^2}{1-k^2}\right)\\ \qquad\times\left(\dfrac{2A^2}{1-k^2}+\left(\dfrac{A^2+2}{3}\right)-\dfrac{(3-2\alpha)(2\mu+m-1)}{2m(1-\alpha)}\left(\dfrac{A^2}{1-k^2}\right)\right), & \text{if } \mu\leq\rho_2,\\ \dfrac{1}{m(1-\alpha)(3-2\alpha)}\left(\dfrac{A^2}{1-k^2}\right), & \text{if } \rho_2\leq\mu\leq\rho_1,\\ \dfrac{1}{m(1-\alpha)(3-2\alpha)}\left(\dfrac{A^2}{1-k^2}\right)\\ \qquad\times\left(\dfrac{(3-2\alpha)(2\mu+m-1)}{2m(1-\alpha)}\left(\dfrac{A^2}{1-k^2}\right)-\left(\dfrac{2A^2}{1-k^2}\right)-\left(\dfrac{A^2+2}{3}\right)\right), & \text{if } \mu\geq\rho_1,\end{cases}\tag{3.16}$$

where

$$\rho_1=\frac{2m(1-\alpha)}{3-2\alpha}\left(\frac{1-k^2}{A^2}\right)\left(\left(\frac{2A^2}{1-k^2}\right)+\left(\frac{A^2+2}{3}\right)-\frac{(m-1)(3-2\alpha)}{2m(1-\alpha)}\left(\frac{A^2}{1-k^2}\right)+1\right),$$
$$\rho_2=\frac{2m(1-\alpha)}{3-2\alpha}\left(\frac{1-k^2}{A^2}\right)\left(\left(\frac{2A^2}{1-k^2}\right)+\left(\frac{A^2+2}{3}\right)-\frac{(m-1)(3-2\alpha)}{2m(1-\alpha)}\left(\frac{A^2}{1-k^2}\right)-1\right)$$

and A is given by Lemma 2.1.

Proof. We omit the proofs since it is similar to Theorem 3.1.

If $m=1$ in Theorem 3.2, then we have the following corollary.

***Corollary* 3.3.** *The function $f\in\mathcal{A}_0$, given by* (1.1), *is said to be in the class k-$\mathcal{SP}_\alpha^p$, $0\leq k<1,\ 0\leq\alpha<1$. Then*

$$\left|b_3-\mu b_2^2\right|\leq\begin{cases}\dfrac{1}{(1-\alpha)(3-2\alpha)}\left(\dfrac{A^2}{1-k^2}\right)\\ \qquad\times\left(\dfrac{2A^2}{1-k^2}+\left(\dfrac{A^2+2}{3}\right)-\dfrac{(3-2\alpha)\mu}{(1-\alpha)}\left(\dfrac{A^2}{1-k^2}\right)\right), & \text{if } \mu\leq\rho_4,\\ \dfrac{1}{(1-\alpha)(3-2\alpha)}\left(\dfrac{A^2}{1-k^2}\right), & \text{if } \rho_4\leq\mu\leq\rho_3,\\ \dfrac{1}{(1-\alpha)(3-2\alpha)}\left(\dfrac{A^2}{1-k^2}\right)\\ \qquad\times\left(\dfrac{(3-2\alpha)\mu}{(1-\alpha)}\left(\dfrac{A^2}{1-k^2}\right)-\left(\dfrac{2A^2}{1-k^2}\right)-\left(\dfrac{A^2+2}{3}\right)\right), & \text{if } \mu\geq\rho_3,\end{cases}$$

where

$$\rho_3=\frac{1-\alpha}{3-2\alpha}\left(\frac{1-k^2}{A^2}\right)\left(\left(\frac{2A^2}{1-k^2}\right)+\left(\frac{A^2+2}{3}\right)+1\right),$$
$$\rho_4=\frac{1-\alpha}{3-2\alpha}\left(\frac{1-k^2}{A^2}\right)\left(\left(\frac{2A^2}{1-k^2}\right)+\left(\frac{A^2+2}{3}\right)-1\right)$$

and A is given by Lemma 2.1.

If $m=1$ and $\alpha=0$ in Theorem 3.2, then we have the following corollary.

***Corollary* 3.4.** *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}_\alpha^p$ *and* $0 \le k < 1$. *Then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} \dfrac{1}{3}\left(\dfrac{A^2}{1-k^2}\right)\left(\dfrac{2A^2}{1-k^2} + \left(\dfrac{A^2+2}{3}\right) - \left(\dfrac{3\mu A^2}{1-k^2}\right)\right), & \text{if } \mu \le \rho_6, \\ \dfrac{1}{3}\left(\dfrac{A^2}{1-k^2}\right), & \text{if } \rho_6 \le \mu \le \rho_5, \\ \dfrac{1}{3}\left(\dfrac{A^2}{1-k^2}\right)\left(\left(\dfrac{3\mu A^2}{1-k^2}\right) - \left(\dfrac{2A^2}{1-k^2}\right) - \left(\dfrac{A^2+2}{3}\right)\right), & \text{if } \mu \ge \rho_5, \end{cases}$$

where

$$\rho_5 = \frac{1}{3}\left(\frac{1-k^2}{A^2}\right)\left(1 + \left(\frac{2A^2}{1-k^2}\right) + \left(\frac{A^2+2}{3}\right)\right),$$

$$\rho_6 = \frac{1}{3}\left(\frac{1-k^2}{A^2}\right)\left(\left(\frac{2A^2}{1-k^2}\right) + \left(\frac{A^2+2}{3}\right) - 1\right)$$

and A *is given by Lemma* 2.1.

Theorem 3.3. *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}_\alpha^p$, $k = 0$, $0 \le \alpha < 1$. *Then*

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \le \begin{cases} \dfrac{4}{m(1-\alpha)(3-2\alpha)\pi^2}\left(\dfrac{8}{\pi^2} + \dfrac{2}{3} - \dfrac{2(3-2\alpha)(2\mu+m-1)}{m(1-\alpha)\pi^2}\right), & \text{if } \mu \le \beta_2, \\ \dfrac{4}{m(1-\alpha)(3-2\alpha)\pi^2}, & \text{if } \beta_2 \le \mu \le \beta_1, \\ \dfrac{4}{m(1-\alpha)(3-2\alpha)\pi^2}\left(\dfrac{2(3-2\alpha)(2\mu+m-1)}{m(1-\alpha)\pi^2} - \dfrac{8}{\pi^2} - \dfrac{2}{3}\right), & \text{if } \mu \ge \beta_1, \end{cases} \tag{3.17}$$

where

$$\beta_1 = \frac{m(1-\alpha)\pi^2}{4(3-2\alpha)}\left(\frac{8}{\pi^2} + \frac{5}{3} - \frac{2(3-2\alpha)(m-1)}{m(1-\alpha)\pi^2}\right)$$

and

$$\beta_2 = \frac{m(1-\alpha)\pi^2}{4(3-2\alpha)}\left(\frac{8}{\pi^2} - \frac{1}{3} - \frac{2(3-2\alpha)(m-1)}{m(1-\alpha)\pi^2}\right).$$

Proof. We omit the proofs since it is similar to Theorem 3.1.

If $m = 1$ in Theorem 3.3, then we have the following corollary.

***Corollary* 3.5.** *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}_\alpha^p$, $k = 0$, $0 \le \alpha < 1$. *Then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} \dfrac{4}{(1-\alpha)(3-2\alpha)\pi^2}\left(\dfrac{8}{\pi^2} + \dfrac{2}{3} - \dfrac{4(3-2\alpha)\mu}{(1-\alpha)\pi^2}\right), & \text{if } \mu \le \beta_4, \\ \dfrac{4}{(1-\alpha)(3-2\alpha)\pi^2}, & \text{if } \beta_4 \le \mu \le \beta_3, \\ \dfrac{4}{(1-\alpha)(3-2\alpha)\pi^2}\left(\dfrac{4(3-2\alpha)\mu}{(1-\alpha)\pi^2} - \dfrac{8}{\pi^2} - \dfrac{2}{3}\right), & \text{if } \mu \ge \beta_3, \end{cases}$$

where

$$\beta_3 = \frac{(1-\alpha)\pi^2}{4(3-2\alpha)}\left(\frac{8}{\pi^2}+\frac{5}{3}\right),$$
$$\beta_4 = \frac{(1-\alpha)\pi^2}{4(3-2\alpha)}\left(\frac{8}{\pi^2}-\frac{1}{3}\right).$$

If $m=1$ and $\alpha=0$ in Theorem 3.3, then we have the following corollary.

***Corollary* 3.6.** *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}^p_\alpha$ *and* $k=0$. *Then*

$$\left|b_3-\mu b_2^2\right| \le \begin{cases} \dfrac{4}{3\pi^2}\left(\dfrac{8}{\pi^2}+\dfrac{2}{3}-\dfrac{12\mu}{\pi^2}\right), & \text{if } \mu \le \beta_6, \\ \dfrac{4}{3\pi^2}, & \text{if } \beta_6 \le \mu \le \beta_5, \\ \dfrac{4}{3\pi^2}\left(\dfrac{12\mu}{\pi^2}-\dfrac{8}{\pi^2}-\dfrac{2}{3}\right), & \text{if } \mu \ge \beta_5, \end{cases}$$

where

$$\beta_5 = \frac{\pi^2}{12}\left(\frac{8}{\pi^2}+\frac{5}{3}\right),$$
$$\beta_6 = \frac{\pi^2}{12}\left(\frac{8}{\pi^2}-\frac{1}{3}\right).$$

Theorem 3.4. *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}^p_\alpha$, $k>0$, $0 \le \alpha < 1$ *and* $t \in (0,1)$. *Then*

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \le \begin{cases} \dfrac{1}{2m(1-\alpha)(3-2\alpha)}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right)(A(t)), & \text{if } \mu \le \delta_2, \\ \dfrac{1}{2m(1-\alpha)(3-2\alpha)}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right), & \text{if } \delta_2 \le \mu \le \delta_1, \\ \dfrac{1}{2m(1-\alpha)(3-2\alpha)}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right)(-A(t)), & \text{if } \mu \ge \delta_1, \end{cases} \tag{3.18}$$

where

$$A(t) = \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\left(t^2+6t+1\right)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - \frac{(3-2\alpha)(2\mu+m-1)\pi^2}{16m(1-\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}},$$
$$\delta_1 = \frac{8m(1-\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{(3-2\alpha)\pi^2}A_1(t),$$

$$A_1(t) = 1 + \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - \frac{(3-2\alpha)(m-1)}{4m(1-\alpha)}\left(\frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right),$$

$$\delta_2 = \frac{8m(1-\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{(3-2\alpha)\pi^2}A_2(t),$$

and

$$A_2(t) = \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - \frac{(3-2\alpha)(m-1)}{4m(1-\alpha)}\left(\frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right) - 1.$$

Proof. We omit the proofs since it is similar to Theorem 3.1.

If $m = 1$ in Theorem 3.4, then we have the following corollary.

Corollary **3.7.** *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}^p_\alpha$, $k > 0$, $0 \le \alpha < 1$ *and* $t \in (0,1)$. *Then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} \dfrac{1}{2(1-\alpha)(3-2\alpha)}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right)(A_3(t)), & \textit{if } \mu \le \delta_4, \\ \dfrac{1}{2(1-\alpha)(3-2\alpha)}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right), & \textit{if } \delta_4 \le \mu \le \delta_3, \\ \dfrac{1}{2(1-\alpha)(3-2\alpha)}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right)(-A_3(t)), & \textit{if } \mu \ge \delta_3, \end{cases}$$

where

$$A_3(t) = \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - \frac{(3-2\alpha)\pi^2\mu}{8(1-\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}},$$

$$\delta_3 = \frac{8(1-\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{(3-2\alpha)\pi^2} \times \left(1 + \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}}\right),$$

and

$$\delta_4 = \frac{8(1-\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{(3-2\alpha)\pi^2} \times \left(\frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - 1\right).$$

If $m = 1$ and $\alpha = 0$ in Theorem 3.4, then we have the following corollary.

***Corollary* 3.8.** *The function* $f \in \mathcal{A}_0$, *given by* (1.1), *is said to be in the class* k-$\mathcal{SP}^p_\alpha$, $t \in (0,1)$ *and* $k > 0$. *Then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} \dfrac{1}{6}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right)(A_4(t)), & \text{if } \mu \le \delta_6, \\ \dfrac{1}{6}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right), & \text{if } \delta_6 \le \mu \le \delta_5, \\ \dfrac{1}{6}\left(\dfrac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}\right)(-A_4(t)), & \text{if } \mu \ge \delta_5, \end{cases}$$

where

$$A_4(t) = \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - \frac{3\pi^2\mu}{8\kappa^2(t)(k^2-1)(1-t)\sqrt{t}},$$

$$\delta_5 = \frac{8\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{3\pi^2}\left(1 + \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}}\right),$$

and

$$\delta_6 = \frac{8\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{3\pi^2}\left(\frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}} + \frac{4\kappa^2(t)\big(t^2+6t+1\big)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}} - 1\right).$$

4. Improvements of the main results. In this section we discuss some improvements of the second part of assertion in (3.1).

***Remark* 4.1.** The second inequality in (3.1) can be improved as follows for $0 \le \alpha < 1$:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\mu - \sigma_2)\left|b_{m+1}\right|^2 \le \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}, \quad \sigma_2 \le \mu \le \sigma_3, \tag{4.1}$$

and

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\sigma_1 - \mu)\left|b_{m+1}\right|^2 \le \frac{Q_1}{2m(1-\alpha)(3-2\alpha)}, \quad \sigma_3 \le \mu \le \sigma_1, \tag{4.2}$$

where

$$\sigma_3 = \frac{2m(1-\alpha)}{(3-2\alpha)Q_1}\left\{Q_1 + D - \frac{(m-1)(3-2\alpha)Q_1}{4m(1-\alpha)}\right\}$$

and Q_1 and D are given by Lemma 2.1.

***Remark* 4.2.** If $m = 1$ then the second inequality in (3.1) can be improved as follows for $0 \le \alpha < 1$:

$$\left|b_3 - \mu b_2^2\right| + (\mu - \alpha_2)\left|b_2\right|^2 \le \frac{Q_1}{2(1-\alpha)(3-2\alpha)}, \quad \alpha_2 \le \mu \le \alpha_3,$$

and

$$\left|b_3 - \mu b_2^2\right| + (\alpha_1 - \mu)\left|b_2\right|^2 \le \frac{Q_1}{2(1-\alpha)(3-2\alpha)}, \quad \alpha_3 \le \mu \le \alpha_1,$$

where

$$\alpha_3 = \frac{2(1-\alpha)(Q_1+D)}{(3-2\alpha)Q_1}$$

and Q_1 and D are given by Lemma 2.1.

***Remark* 4.3.** By putting the values of $Q_1(k)$ and $D(k)$ for $0 \le k < 1$ in (4.1) and (4.2), the second part of the estimate in (3.16) can be improved as follows for $0 \le \alpha < 1$:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\mu - \rho_2)|b_{m+1}|^2 \le \frac{A^2}{m(1-\alpha)(3-2\alpha)(1-k^2)}, \quad \rho_2 \le \mu \le \rho_7,$$

and

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\rho_1 - \mu)|b_{m+1}|^2 \le \frac{A^2}{m(1-\alpha)(3-2\alpha)(1-k^2)}, \quad \rho_7 \le \mu \le \rho_1,$$

where

$$\rho_7 = \frac{2m(1-\alpha)}{3-2\alpha}\left(1 + \frac{(A^2+2)(1-k^2)}{6A^2} - \frac{(m-1)(3-2\alpha)}{4m(1-\alpha)}\right).$$

***Remark* 4.4.** By putting the values of $Q_1(k)$ and $D(k)$ for $k = 1$ and $0 \le \alpha < 1$ in (4.1) and (4.2), the second part of the estimate in (3.17) can be improved as follows:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\mu - \beta_2)|b_{m+1}|^2 \le \frac{4}{m(1-\alpha)(3-2\alpha)\pi^2}, \quad \beta_2 \le \mu \le \beta_7,$$

and

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\beta_1 - \mu)|b_{m+1}|^2 \le \frac{4}{m(1-\alpha)(3-2\alpha)\pi^2}, \quad \beta_7 \le \mu \le \beta_1,$$

where

$$\beta_7 = \frac{2m(1-\alpha)}{(3-2\alpha)}\left(1 + \frac{\pi^2}{12} - \frac{(m-1)(3-2\alpha)}{4m(1-\alpha)}\right).$$

***Remark* 4.5.** By putting the values of $Q_1(k)$ and $D(k)$ for $0 \le \alpha < 1$, $k > 1$ and $t \in (0,1)$ in (4.1) and (4.2), the second part of the estimate in (3.18) can be improved as follows:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\mu - \delta_2)|b_{m+1}|^2$$
$$\le \frac{\pi^2}{8m(1-\alpha)(3-2\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}, \quad \delta_2 \le \mu \le \delta_7,$$

and

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + (\delta_1 - \mu)|b_{m+1}|^2$$
$$\le \frac{\pi^2}{8m(1-\alpha)(3-2\alpha)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}, \quad \delta_7 \le \mu \le \delta_1,$$

where

$$\delta_7 = \frac{2m(1-\alpha)}{3-2\alpha}\left(1 + \frac{\big(4\kappa^2(t)\big(t^2+6t+1\big) - \pi^2\big)4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}}{24\pi^2\kappa^2(t)(1+t)\sqrt{t}} - \frac{(m-1)(3-2\alpha)}{4m(1-\alpha)}\right).$$

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References

1. N. I. Ahiezer, *Elements of theory of elliptic functions* (in Russian), Moscow (1970).
2. G. D. Anderson, M. K. Vamanamurthy, M. K. Vourinen, *Conformal invariants, inequalities and quasiconformal maps*, Wiley-Interscience (1997).
3. H. M. Srivastava, S. Owa, *Current topics in analytic function theory*, World Sci. Publ. Co., Singapore (1992).
4. A. K. Mishra, P. Gochhayat, *A coefficient inequality for a subclass of the Caratheodory functions defined using conical domains*, Comput. Math. Appl., 2816 – 2820 (2011).
5. W. Ma, D. Minda, *Uniformly convex functions. II*, Ann. Polon. Math., **58**, № 3, 275 – 285 (1993).
6. S. Kanas, *Coefficient estimates in subclasses of the Caratheodory class related to conic domains*, Acta Math. Comenian., **74**, 149 – 161 (2005).
7. F. R. Keogh, E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20**, 8 – 12 (1969).
8. A. K. Mishra, P. Gochhayat, *Fekete – Szego problem for k-uniformly convex functions and for a class defined by Owa – Srivastava operator*, J. Math. Anal. and Appl., **347**, № 2, 563 – 572 (2008).
9. A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., **56**, 87 – 92 (1991).
10. S. Kanas, A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. and Appl. Math., **105**, 327 – 336 (1999).
11. S. Kanas, A. Wisniowska, *Conic regions and starlike functions*, Rev. Roumaine Math. Pures Appl., **45**, 647 – 657 (2000).
12. P. L. Duren, *Univalent functions*, Springer-Verlag, New York etc. (1983).
13. J. H. Choi, Y. C. Kim, T. Sugawa, *A general approach to the Fekete – Szegö problem*, J. Math. Soc. Japan, **59**, № 3, 707 – 727 (2007).
14. M. Darus, T. N. Shanmugam, S. Sivasubramanian, *Fekete – Szegö inequality for a certain class of analytic functions*, Mathematica, **49(72)**, № 1, 29 – 34 (2007).
15. O. P. Ahuja, M. Jahangiri, *Fekete – Szegö problem for a unified class of analytic functions*, PanAmer. Math. J., **7**, № 2, 67 – 78 (1997).
16. M. Darus, N. Tuneski, *On the Fekete – Szegö problem for generalized close-to-convex functions*, Int. Math. J., **4**, № 6, 561 – 568 (2003).
17. S. Kanas, *An unified approach to the Fekete – Szegö problem*, Appl. Math. and Comput., **218**, 8453 – 8461 (2012).
18. S. Kanas, H. E. Darwish, *Fekete – Szegö problem for starlike and convex functions of complex order*, Appl. Math. Lett., **23**, № 7, 777 – 782 (2010).
19. A. K. Bakhtin, G. P. Bakhtina, Yu. B. Zelinskii, *Topological-algebraic structures and geometric methods in complex analysis*, Zb. Prats Inst. Math. NASU (2008).
20. A. K. Bakhtin, I. V. Denega, *Extremal decomposition of the complex plane with free poles*, J. Math. Sci., **246**, № 1, 1 – 17 (2020).
21. I. Denega, *Extremal decomposition of a multidimensional complex space with poles on the boundary of a polydisk*, in: P. Cerejeiras, M. Reissig, I. Sabadini, J. Toft (eds), Current Trends in Analysis, Its Applications and Computation, Trends in Mathematics, Birkhäuser, Cham (2022), p. 143 – 151.
22. A. K. Bakhtin, I. V. Denega, *Generalized M. A. Lavrentiev's inequality*, J. Math. Sci., **262**, 138 – 153 (2022).
23. Ya. V. Zabolotnii, I. V. Denega, *An extremal problem on non-overlapping domains containing ellipse points*, Eurasian Math. J., **12**, № 4, 82 – 91 (2021).

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