DOI: 10.3842/umzh.v76i7.7518

UDC 517.54

Doan Cong Dinh[1] (School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Vietnam)

REPRESENTATION OF SOLUTIONS OF THE LAMÉ – NAVIER SYSTEM BY ENDOMORPHISMS ON QUATERNIONS

ЗОБРАЖЕННЯ РОЗВ'ЯЗКІВ СИСТЕМИ ЛАМЕ – НАВ'Є ЕНДОМОРФІЗМАМИ НА КВАТЕРНІОНАХ

Solutions of the Lamé – Navier system in $\mathbb{R}^3$ are given in real analysis and in quatenionic analysis in different forms. We introduce a new method of using endomorphisms on quaternions with an aim to represent these solutions.

Розв'язки системи Ламе – Нав'є в $\mathbb{R}^3$ наведено у різних формах в рамках дійсного та кватерніонного аналізу. Запропоновано новий метод використання ендоморфізмів на кватерніонах для зображення розв'язків цієї системи.

1. Introduction. The displacement field u in a homogeneous isotropic linear elastic material without volume forces is described by the Lamé – Navier system in $\mathbb{R}^3$

$$\mu\Delta u + (\mu+\lambda)\,\mathrm{grad}(\mathrm{div}\, u) = 0,$$

where $\mu > 0,\ \lambda > -\dfrac{2}{3}\mu$ are the Lamé coefficients [16]. Quaternionic analysis provides useful tools for 3D elasticity problems. Various representations of solutions of the Lamé – Navier system are considered. Some Kolosov – Muskhelishvili formulae are constructed by using two monogenic functions [2, 4, 9, 12, 18]. Complete systems of polynomial solutions are introduced in [3, 5]. Somigliana formula in quaternionic analysis is introduced in [8].

In [15], the Lamé – Navier system is rewritten in quaternionic analysis with the appearance of the (α,β)-Dirac operator. In [7], the (α,β)-monogenic functions are represented by a new method of using the endomorphisms on Clifford algebras. From these relations, in this paper, we introduce a method of endomorphisms on quaternions to represent solutions of the Lamé – Navier system.

2. Preliminaries. We recall the notation of the quaternion algebra

$$\mathbb{H} = \big\{q_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\big\},$$

where the quaternion units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ obey the multiplication rules

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1, \qquad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3.$$

An element $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with the reduced quaternion variable $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \mathbb{H}$. In other papers and books, variable $x = x_1 + x_2\mathbf{e}_1 + x_3\mathbf{e}_2$ is called the reduced quaternionic variable [5, 11]. The Dirac operator in quaternionic analysis is given by

$$D = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}.$$

[1] E-mail: doancongdinhdhbk@gmail.com.

We refer the readers to the books [6, 10, 13] for details on quaternionic and Clifford analysis.

With each displacement field u, using the equalities

$$DDu = -\mathrm{grad}(\mathrm{div}\, u) + \mathrm{rot}(\mathrm{rot}\, u),$$

$$DuD = -\mathrm{grad}(\mathrm{div}\, u) - \mathrm{rot}(\mathrm{rot}\, u),$$

the Lamé – Navier system can be reformulated to the form [15]

$$(3\mu + \lambda)\Delta u - (\mu + \lambda)DuD = 0.$$

Define the operator

$$\mathcal{M}u := (3\mu + \lambda)Du + (\mu + \lambda)uD.$$

Integral representations of solutions of the operator $\mathcal{M}$ are constructed in [1] and in [7] by using the endomorphisms on Clifford algebras. The Lamé – Navier system is rewritten in the form

$$D\mathcal{M}u = 0.$$

Denote by $\mathrm{End}(\mathbb{H})$ all the $\mathbb{R}$-linear mappings from $\mathbb{H}$ into itself. When an element $f \in \mathrm{End}(\mathbb{H})$ is applied to an element $a \in \mathbb{H}$, we will use the notation $f\langle a\rangle$. For each $j = 1, 2, 3$, define the elements $e_j, e'_j \in \mathrm{End}(\mathbb{H})$ by

$$e_j\langle a\rangle = \mathbf{e}_j a,$$

$$e'_j\langle a\rangle = a\mathbf{e}_j \quad \forall a \in \mathbb{H}.$$

We obtain the following relations:

$$e_j^2 = e_j'^2 = -1, \quad 1 \le j \le 3,$$

$$e_i e_j + e_j e_i = 0, \quad 1 \le i < j \le 3,$$

$$e'_i e'_j + e'_j e'_i = 0, \quad 1 \le i < j \le 3,$$

$$e_i e'_j = e'_j e_i, \quad 1 \le i \le j \le 3.$$

By [14, 17], $\mathrm{End}(\mathbb{H})$ is generated by 16 elements: $1,\ e_i,\ e'_i,\ e_i e'_j,\ i, j \in \{1; 2; 3\}$. Define

$$\mathcal{L} = \sum_{j=1}^{3} \left[(3\mu + \lambda)e_i + (\mu + \lambda)e'_i\right] \frac{\partial}{\partial x_i}.$$

The adjoin operator of $\mathcal{L}$ is defined by

$$\overline{\mathcal{L}} = \frac{-1}{4\mu(2\mu + \lambda)} \sum_{j=1}^{3} \left[(3\mu + \lambda)e_i - (\mu + \lambda)e'_i\right] \frac{\partial}{\partial x_i}.$$

We obtain the relation $\mathcal{L}\overline{\mathcal{L}} = \overline{\mathcal{L}}\mathcal{L} = \Delta$. We define the Dirac operator for functions taking values in $\mathrm{End}(\mathbb{H})$ (by the same notation)

$$D = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$

An element $q_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3 \in \mathbb{H},\ q_0, q_1, q_2, q_3 \in \mathbb{R}$, is identified with $q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \in \mathrm{End}(\mathbb{H})$.

3. Integral representation formulae. We consider the equation $D\mathcal{M}u = 0$ with u is a $\mathbb{H}$-valued function. In this section, we construct an integral representation for its solutions by using the endomorphism on quaternions.

Lemma 3.1 (Stokes formulae). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $f, g \in \mathcal{C}^1(\overline{\Omega}, \mathrm{End}(\mathbb{H}))$. Then*

$$\int\limits_{\partial\Omega} f\mathbf{n}g\, dS = \int\limits_{\Omega} (fD \cdot g + f \cdot Dg)\, dx, \tag{3.1}$$

$$\int\limits_{\partial\Omega} f\boldsymbol{\nu}g\, dS = \int\limits_{\Omega} (f\mathcal{L} \cdot g + f \cdot \mathcal{L}g)\, dx, \tag{3.2}$$

where $\boldsymbol{\nu} = \sum_{j=1}^{3} \big[(3\mu+\lambda)e_i + (\mu+\lambda)e_i'\big]n_i$ and $\mathbf{n} = n_1e_1 + n_2e_2 + n_3e_3$ is the outer unit normal vector of the boundary (*see* [6, p. 52]).

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let also $f \in \mathcal{C}^4(\overline{\Omega}, \mathrm{End}(\mathbb{H}))$, $g \in \mathcal{C}^2(\overline{\Omega}, \mathrm{End}(\mathbb{H}))$. Then*

$$\int\limits_{\partial\Omega} \big[f\overline{\mathcal{L}}D \cdot \mathbf{n} \cdot \mathcal{L}g + \Delta f\overline{\mathcal{L}} \cdot \boldsymbol{\nu}g\big] dS = \int\limits_{\Omega} \big[f\overline{\mathcal{L}}D \cdot D\mathcal{L}g + \Delta^2 fg\big] dx, \tag{3.3}$$

where $\boldsymbol{\nu} = \sum_{j=1}^{3} \big[(3\mu+\lambda)e_i + (\mu+\lambda)e_i'\big]n_i$ and $\mathbf{n} = n_1e_1 + n_2e_2 + n_3e_3$ is the outer unit normal vector of the boundary.

Proof. Applying the Stokes formulae (3.1), (3.2), we have

$$\begin{aligned} &\int\limits_{\partial\Omega} f\overline{\mathcal{L}}D \cdot \mathbf{n} \cdot \mathcal{L}g = \int\limits_{\Omega} \big[-\Delta f\overline{\mathcal{L}} \cdot \mathcal{L}g + f\overline{\mathcal{L}}D \cdot D\mathcal{L}g\big] dx, \\ &\int\limits_{\partial\Omega} \Delta f\overline{\mathcal{L}} \cdot \boldsymbol{\nu}g dS = \int\limits_{\Omega} \big[\Delta^2 fg + \Delta f\overline{\mathcal{L}} \cdot \mathcal{L}g\big] dx. \end{aligned} \tag{3.4}$$

Substituting (3.4) into the left-hand side of (3.3), we complete the proof of the lemma.

Recall the fundamental solutions of the operators Δ^2 and Δ in $\mathbb{R}^3$, respectively, by

$$\Phi_2(x,y) = -\frac{|x-y|}{8\pi}, \qquad \Phi_1(x,y) = -\frac{1}{4\pi|x-y|}.$$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $g \in \mathcal{C}^2(\overline{\Omega}, \mathbb{H})$. Then, for all $y \in \Omega$,*

$$\begin{aligned} g(y) = &\int\limits_{\partial\Omega} \big[\overline{\mathcal{L}_x}D_x\Phi_2(x,y)\mathbf{n}(x)\langle\mathcal{M}g(x)\rangle + \overline{\mathcal{L}_x}\Phi_1(x,y)\boldsymbol{\nu}(x)\langle g(x)\rangle\big] dS \\ &- \int\limits_{\Omega} \overline{\mathcal{L}_x}D_x\Phi_2(x,y)\langle D\mathcal{M}g(x)\rangle dx, \end{aligned}$$

where $\boldsymbol{\nu}(x) = \sum_{j=1}^{3} \big[(3\mu+\lambda)e_i + (\mu+\lambda)e_i'\big]n_i$ and $\mathbf{n}(x) = n_1e_1 + n_2e_2 + n_3e_3$ is the outer unit normal vector of the boundary. The notations $\overline{\mathcal{L}_x}$, D_x indicate the variable x of derivation.

Proof. Let y be a fixed point in Ω. Let $\varepsilon > 0$ such that $B_\varepsilon(y) = \{x \in \mathbb{R}^3 \mid |x-y| \le \varepsilon\} \subset \Omega$. Applying the Lemma 3.2 for the functions Φ_2 and g in the domain $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(y)$, we obtain

$$\int\limits_{\partial\Omega_\varepsilon} \big[\Phi_2(x,y)\overline{\mathcal{L}_x} D_x \cdot \mathbf{n}(x) \cdot \mathcal{L}g(x) + \Delta\Phi_2(x,y)\overline{\mathcal{L}_x} \cdot \boldsymbol{\nu}(x) g(x)\big] dS$$

$$= \int\limits_{\Omega_\varepsilon} \Phi_2(x,y)\overline{\mathcal{L}_x} D_x \cdot D\mathcal{L}g(x) dx. \tag{3.5}$$

Since $\Phi_2(x,y)$ is a real-valued function and $\Delta_x\Phi_2(x,y) = \Phi_1(x,y)$, the equality (3.5) becomes

$$\int\limits_{\partial\Omega_\varepsilon} \big[\overline{\mathcal{L}_x} D_x \Phi_2(x,y) \cdot \mathbf{n}(x) \cdot \mathcal{L}g(x) + \overline{\mathcal{L}_x}\Phi_1(x,y) \cdot \boldsymbol{\nu}(x) g(x)\big] dS$$

$$= \int\limits_{\Omega_\varepsilon} \overline{\mathcal{L}_x} D_x \Phi_2(x,y) \cdot D\mathcal{L}g(x) dx. \tag{3.6}$$

The left-hand side of (3.6) is equal to

$$\int\limits_{\partial\Omega_\varepsilon} \big[\overline{\mathcal{L}_x} D_x \Phi_2(x,y) \cdot \mathbf{n}(x) \cdot \mathcal{L}g(x) + \overline{\mathcal{L}_x}\Phi_1(x,y) \cdot \boldsymbol{\nu}(x) g(x)\big] dS$$

$$- \int\limits_{|x-y|=\varepsilon} \big[\overline{\mathcal{L}_x} D_x \Phi_2(x,y) \cdot \mathbf{n}(x) \cdot \mathcal{L}g(x) + \overline{\mathcal{L}_x}\Phi_1(x,y) \cdot \boldsymbol{\nu}(x) g(x)\big] dS.$$

We calculate the second order derivatives of the function $\Phi_2(x,y)$:

$$\frac{\partial \Phi_2(x,y)}{\partial x_i} = -\frac{x_i - y_i}{8\pi|x-y|}, \quad 1 \le i \le 3,$$

$$\frac{\partial^2 \Phi_2(x,y)}{\partial x_i^2} = \frac{(x_i - y_i)^2 - |x-y|^2}{8\pi|x-y|^3}, \quad 1 \le i \le 3,$$

$$\frac{\partial^2 \Phi_2(x,y)}{\partial x_i \partial x_j} = \frac{(x_i - y_i)(x_j - y_j)}{8\pi|x-y|^3}, \quad 1 \le i < j \le 3.$$

We have the estimate

$$\left|\frac{\partial^2 \Phi_2(x,y)}{\partial x_i \partial x_j}\right| \le \frac{1}{8\pi|x-y|} \quad \forall x, y \in \Omega, \quad x \ne y, \quad 1 \le i \le j \le 3.$$

Then we obtain

$$\lim_{\varepsilon \to 0^+} \int\limits_{|x-y|=\varepsilon} \overline{\mathcal{L}_x} D_x \Phi_2(x,y) \cdot \mathbf{n}(x) \cdot \mathcal{L}g(x) dS = 0. \tag{3.7}$$

Since on the sphere $|x-y| = \varepsilon$, $\overline{\mathcal{L}_x}\Phi_1(x,y) \cdot \boldsymbol{\nu}(x) = \dfrac{1}{4\pi\varepsilon^2}$, we have

$$\lim_{\varepsilon\to 0^+} \int\limits_{|x-y|=\varepsilon} \overline{\mathcal{L}_x}\Phi_1(x,y)\cdot \boldsymbol{\nu}(x)g(x)dS = \lim_{\varepsilon\to 0^+} \int\limits_{|x-y|=\varepsilon} \frac{g(x)}{4\pi\varepsilon^2}dS = g(y). \tag{3.8}$$

Let $\varepsilon \to 0^+$ from both sides of the equality (3.6). By using (3.7), (3.8), we get

$$g(y) = \int\limits_{\partial\Omega} \big[\overline{\mathcal{L}_x}D_x\Phi_2(x,y)\mathbf{n}(x)\cdot\mathcal{L}g(x) + \overline{\mathcal{L}_x}\Phi_1(x,y)\boldsymbol{\nu}(x)g(x)\big]dS$$

$$- \int\limits_{\Omega} \overline{\mathcal{L}_x}D_x\Phi_2(x,y)\cdot D\mathcal{L}g(x)dx. \tag{3.9}$$

Apply both sides of the equality (3.9) to the element 1. By using the equality

$$\mathcal{L}g(x)\langle 1\rangle = \mathcal{M}g(x),$$

we complete the proof of the theorem.

***Remark* 3.1.** Theorem 3.1 can be seen as an analogue of the classical Borel – Pompeiu formula. The values of a function $g \in \mathcal{C}^2(\overline{\Omega},\mathbb{H})$ are recovered from the values of $\mathcal{M}g$ and g on the boundary $\partial\Omega$ and the values of $D\mathcal{M}g$ in Ω. Solutions of the Lamé – Navier system $D\mathcal{M}g = 0$ in quaternionic analysis are represented by an analogue of the classical the Cauchy integral formula in the following corollary.

***Corollary* 3.1.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $u \in \mathcal{C}^2(\overline{\Omega},\mathbb{H})$ be a solution of the equation $D\mathcal{M}u = 0$. Then, for all $y \in \Omega$,*

$$u(y) = \int\limits_{\partial\Omega} \big[\overline{\mathcal{L}_x}D_x\Phi_2(x,y)\mathbf{n}(x)\langle\mathcal{M}u(x)\rangle + \overline{\mathcal{L}_x}\Phi_1(x,y)\boldsymbol{\nu}(x)\langle u(x)\rangle\big]dS.$$

4. Conclusions. The representation of the solutions of the Lamé – Navier system in classical analysis is known as the Betti – Somigliana representation formula, which plays an important role in boundary integral equation problems in elasticity. The development of quaternionic analysis provides useful tools for 3D elasticity problems. In this paper, we introduce a new application of the endomorphisms on quaternions to represent solutions of the Lamé – Navier system in a compact form.

Acknowledgement. The author would like to express his sincere gratitude to the editors and the anonymous referees for their very important comments to improve this paper.

Conflict of interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

Funding. This research was funded by the Vietnam Ministry of Education and Training.

References

1. R. Abreu-Blaya, J. Bory-Reyes, M. A. Herrera-Peláez, J. M. Sigarreta, *Integral representation formulas related to the Lamé – Navier system*, Acta Math. Sin. (Engl. Ser.), **36**, 1341 – 1356 (2020).
2. S. Bock, K. Gürlebeck, *On a spatial generalization of the Kolosov – Muskhelishvili formulae*, Math. Methods Appl. Sci., **32**, 223 – 240 (2009).
3. S. Bock, *On monogenic series expansions with applications to linear elasticity*, Adv. Appl. Clifford Algebras, **24**, 931 – 943 (2014).

4. S. Bock, K. Gürlebeck, D. Legatiuk, H. M. Nguyen, *Ψ-hyperholomorphic functions and a Kolosov – Muskhelishvili formula*, Math. Methods Appl. Sci., **38**, № 18, 5114 – 5123 (2015).
5. S. Bock, K. Gürlebeck, *On a polynomial basis generated from the generalized Kolosov – Muskhelishvili formulae*, Adv. Appl. Clifford Algebras, **19**, 191 – 209 (2009).
6. F. Brackx, R. Delanghe, F. Sommen, *Clifford analysis, vol. 76, Research Notes in Mathematics*, Pitman (Advanced Publishing Program), Boston, MA (1982).
7. D. C. Dinh, *Applications of endomorphisms on Clifford algebras to (α, β)-monogenic functions and isotonic functions*, Complex Anal. and Oper. Theory, **17**, 21 (2023).
8. D. C. Dinh, *Somigliana formula in quaternion analysis*, Mech. Research Commun., **126**, Article 104018 (2022).
9. Y. Grigor'ev, *Three-dimensional analogue of Kolosov – Muskhelishvili formulae*, in: Bernstein S., Kähler U., Sabadini I., Sommen F. (eds.) Modern Trends in Hypercomplex Analysis, Trends Math., Birkhäuser, Cham (2016), 203 – 215.
10. K. Gürlebeck, K. Habetha, W. Sprößig, *Holomorphic functions in the plane and n-dimensional space*, Birkhäuser, Basel (2008).
11. K. Gürlebeck, K. Habetha, W. Sprößig, *Application of holomorphic functions in two and higher dimensions*, Springer Internat. Publ., Switzerland (2016).
12. K. Gürlebeck, H. M. Nguyen, *Ψ-hyperholomorphic functions and an application to elasticity problems*, AIP Conf. Proc., **1648**, № 1, Article 440005 (2015).
13. K. Gürlebeck, W. Sprößig, *Quaternionic analysis and elliptic boundary value problems*, Internat. Ser. Numer. Math., vol. 89, Birkhäuser-Verlag, Basel (1990).
14. M. Laoues, *Linear operators in Clifford algebras*, Lett. Math. Phys., **23**, 325 – 331 (1991).
15. A. Moreno García, T. Moreno García, R. Abreu Blaya, J. Bory Reyes, *Inframonogenic functions and their applications in 3-dimensional elasticity theory*, Math. Methods Appl. Sci., **41**, № 10, 3622 – 3631 (2018).
16. N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity*, Springer (1977).
17. W. Waterhouse, *Linear operators in Clifford algebras*, Lett. Math. Phys., **30**, 187 – 188 (1994).
18. D. Weisz-Patrault, S. Bock, K. Gürlebeck, *Three-dimensional elasticity based on quaternion-valued potentials*, Int. J. Solids Struct., **51**, № 19, 3422 – 3430 (2014).

Received 07.03.23