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# THE SHARP BOUND OF THE THIRD HANKEL DETERMINANT FOR CERTAIN SUBFAMILIES OF ANALYTIC FUNCTIONS

# ТОЧНА ГРАНИЦЯ ДЛЯ ТРЕТЬОГО ВИЗНАЧНИКА ГАНКЕЛЯ ДЛЯ ДЕЯКИХ ПІДСІМЕЙ АНАЛІТИЧНИХ ФУНКЦІЙ

We study an upper bound and the sharp bound of the third-order Hankel determinant for some subfamilies of analytic functions for the parameters $\alpha \in (1, 4/3]$ and $\alpha = 3/2$, respectively, also proving the recent conjecture for the sharp bounds made in [Virendra Kumar, Sushil Kumar, V. Ravichandran, *Third Hankel determinant for certain classes of analytic functions*, Mathematical Analysis I: Approximation Theory, February (2020); DOI: 10.1007/978-981-15-1153-019].

Досліджено верхню та точну границю визначника Ганкеля третього порядку для певних підсімей аналітичних функцій та параметрів $\alpha \in (1, 4/3]$ та $\alpha = 3/2$, відповідно, а також доведено нещодавню гіпотезу для точних границь, наведену в [Virendra Kumar, Sushil Kumar, V. Ravichandran, *Third Hankel determinant for certain classes of analytic functions*, Mathematical Analysis I: Approximation Theory, February (2020); DOI: 10.1007/978-981-15-1153-019].

**1. Introduction.** Denote $\mathcal{H}$ the family of all analytic functions in unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and $\mathcal{A}$ be the subfamily of functions $f$ normilized by the conditions $f(0) = f'(0) - 1 = 0$, i.e., of the type

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \tag{1.1}$$

and $\mathcal{S}$ be the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. A typical problem in geometric function theory is to study a functional made up of combination of the coefficients of the original functions. For the positive integers $r,\ n$, Pommerenke [16] characterized the $r$th-Hankel determinant of $n$th-order for $f$ given in (1.1), defined as follows:

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \tag{1.2}$$

The problem of finding sharp estimates of the third Hankel determinant obtained for $r = 3$ and $n = 1$ in (1.2), given by

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$$H_{3,1}(f) := \begin{vmatrix} a_1 = 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5, \tag{1.3}$$

is technically much tough than $r = n = 2$, i.e., $|H_{2,2}(f)| := |a_2a_4 - a_3^2|$.

In recent years, many authors are working on obtaining upper bounds (see [2, 4, 9, 12, 14]) and a few papers were devoted to the estimation of sharp upper bound to $H_{3,1}(f)$, for certain subclasses of analytic functions (see [3, 7, 8, 10, 18–22]). Recently, Prajapat et al. [17] obtained an upper bound to $H_{3,1}(f)$, for the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, $\alpha > 1$, studied by Nishiwaki et al. [13] defined as $\operatorname{Re}\{zf'(z)/f(z)\} < \alpha$ and $\operatorname{Re}\{1 + zf''(z)/f'(z)\} < \alpha$, respectively, when $\alpha = 3/2$. Further, these bounds were refined by Kumar et al. [24], obtained as $(579 + 8\sqrt{3})/1728$ and $(144431 + 96\sqrt{141})/6497280$, respectively, which are not sharp, they also conjectured the sharp bounds for $\mathcal{M}(3/2)$ and $\mathcal{N}(3/2)$ as $119/576$ and $19/2160$, respectively. For $1 < \alpha \leq 4/3$, the classes $M(\alpha)$ and $N(\alpha)$, were investegated by Uralegaddi et al. [23]. Motivated by these results, in this paper we obtain sharp estimates for $H_{3,1}(f)$ for the classes $\mathcal{M}(3/2)$ and $\mathcal{N}(3/2)$, upper bounds to $H_{3,1}(f)$ for the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, $1 < \alpha \leq 4/3$.

The collection $\mathcal{P}$, of all functions $p$, each one called as Carathéodory function [5] of the form,

$$p(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.4}$$

having a positive real part in $\mathbb{D}$.

The foundation in proving our main results are the following lemmas and we adopt the procedure framed through Libera and Zlotkiewicz [11].

**Lemma 1.1** [6]. *If $p \in \mathcal{P}$, then $|c_i - \mu c_j c_{i-j}| \leq 2$ satisfies for the values $i, j \in \mathbb{N}$ with $i > j$ and $\mu \in [0, 1]$, which is same as $|c_{n+k} - \mu c_n c_k| \leq 2$ for $n, k \in \mathbb{N}$ with $\mu \in [0, 1]$.*

**Lemma 1.2** [15]. *For $p \in \mathcal{P}$, then $|c_t| \leq 2$ for $t \in \mathbb{N}$, equality occurs for the function $p(z) = (1 + z)/(1 - z)$, $z \in \mathbb{D}$.*

**Lemma 1.3.** *For $p \in \mathcal{P}$, then $|c_m c_n - \lambda c_k c_l| \leq 4$ for $m + n = k + l$ with $m, n, k, l \in \mathbb{N}$, where $\lambda \in [0, 1]$.*

***Proof.*** Consider

$$c_m c_n - \lambda c_k c_l = c_m c_n - c_{m+n} + c_{m+n} - \lambda c_k c_l.$$

By using the condition $m + n = k + l$ in the above expression, we have

$$c_m c_n - \lambda c_k c_l = c_m c_n - c_{m+n} + c_{k+l} - \lambda c_k c_l.$$

Taking modulus on both sides of the above expression, applying the triangle inequality and then Lemma 1.1, we obtain

$$|c_m c_n - \lambda c_k c_l| \leq |c_m c_n - c_{m+n}| + |c_{k+l} - \lambda c_k c_l| \leq 2 + 2 = 4.$$

**Lemma 1.4.** *If $p \in \mathcal{P}$, is of the form* (1.4) *with $c_1 \geq 0$, such that $c_1 \in [0,2]$, then*

$$2c_2 = c_1^2 + t\mu,$$

$$4c_3 = c_1^3 + 2c_1 t\mu - c_1 t\mu^2 + 2t\big(1 - |\mu|^2\big)\rho,$$

*and*

$$8c_4 = c_1^4 + 3c_1^2 t\mu + \big(4 - 3c_1^2\big)t\mu^2 + c_1^2 t\mu^3 + 4t\big(1 - |\mu|^2\big)\big(1 - |\rho|^2\big)\psi + 4t\big(1 - |\mu|^2\big)\big(c_1\rho - c_1\mu\rho - \bar{\mu}\rho^2\big),$$

*where $t := 4 - c_1^2$, for some $\mu$, $\rho$ and $\psi$ such that $|\mu| \leq 1$, $|\rho| \leq 1$ and $|\psi| \leq 1$.*

For $c_2$ (see [15, p. 166]), $c_3$ (see [11]) and $c_4$ can found in [22].

**Lemma 1.5** [1]. *If $p \in \mathcal{P}$, then*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2\big(|J| + |K - 2J| + |J - K + L|\big).$$

**2. Bounds for $\mathcal{M}(\alpha)$.**

**Theorem 2.1.** *If $f \in \mathcal{M}(\alpha)$, $1 < \alpha \leq \dfrac{4}{3}$, then*

$$\big|H_{3,1,\alpha}(f)\big| \leq \frac{1}{18}\big[4\alpha^6 - 24\alpha^5 + 76\alpha^4 - 117\alpha^3 + 84\alpha^2 - 25\alpha + 2\big].$$

***Proof.*** For $f \in \mathcal{M}(\alpha)$, there exists a analytic function $p \in \mathcal{P}$ such that

$$\alpha f(z) - zf'(z) = (\alpha - 1)p(z)f(z). \tag{2.1}$$

Using the series representation for $f$ and $p$ in (2.1), a simple calculation gives

$$a_2 = -(\alpha - 1)c_1, \qquad a_3 = \frac{\alpha - 1}{2}\big[(\alpha - 1)c_1^2 - c_2\big],$$

$$a_4 = -\frac{\alpha - 1}{3}\left[\frac{(\alpha - 1)^2c_1^3}{2} - \frac{3(\alpha - 1)c_1c_2}{2} + c_3\right] \tag{2.2}$$

and

$$a_5 = \frac{\alpha - 1}{4}\left[\frac{(\alpha - 1)^3c_1^4}{6} - (\alpha - 1)^2c_1^2c_2 + \frac{4(\alpha - 1)c_1c_3}{3} + \frac{(\alpha - 1)c_2^2}{2} - c_4\right].$$

Substituting the values of $a_2$, $a_3$, $a_4$ and $a_5$ from (2.2) in (1.3), after simplifying, we get

$$H_{3,1,\alpha}(f) = \left[-\frac{(\alpha - 1)^6c_1^6}{144} - \frac{(\alpha - 1)^5c_1^4c_2}{48} - \frac{(\alpha - 1)^4c_1^2c_2^2}{16} + \frac{(\alpha - 1)^4c_1^3c_3}{18} + \frac{(\alpha - 1)^3c_2^3}{16} - \frac{(\alpha - 1)^3c_1c_2c_3}{6} + \frac{(\alpha - 1)^3c_1^2c_4}{8} - \frac{(\alpha - 1)^2c_3^2}{9} + \frac{(\alpha - 1)^2c_2c_4}{8}\right]. \tag{2.3}$$

Rearranging the terms in (2.3) in order to apply the lemmas specified in this paper, we obtain

$$\begin{aligned} H_{3,1,\alpha}(f) = \Bigg[ c_1^3 \Bigg[ -\frac{(\alpha-1)^6 c_1^3}{144} - \frac{(\alpha-1)^5 c_1 c_2}{48} + \frac{(\alpha-1)^4 c_3}{18} \Bigg] \\ + \frac{(\alpha-1)^3 c_2^2}{16}[c_2 - (\alpha-1)c_1^2] + \frac{(\alpha-1)^3 c_1}{8}[c_1 c_4 - c_2 c_3] \\ + \frac{(\alpha-1)^2}{9}[c_2 c_4 - c_3^2] + \frac{(\alpha-1)^2 c_2}{72}[c_4 - 3(\alpha-1)c_1 c_3] \Bigg]. \end{aligned} \tag{2.4}$$

Taking modulus on both sides of the expression (2.4), we have

$$\begin{aligned} \big|H_{3,1,\alpha}(f)\big| \le \Bigg[ |c_1|^3 \Bigg| -\frac{(\alpha-1)^6 c_1^3}{144} - \frac{(\alpha-1)^5 c_1 c_2}{48} + \frac{(\alpha-1)^4 c_3}{18} \Bigg| \\ + \frac{(\alpha-1)^3 |c_2|^2}{16}|c_2 - (\alpha-1)c_1^2| + \frac{(\alpha-1)^3 |c_1|}{8}|c_1 c_4 - c_2 c_3| \\ + \frac{(\alpha-1)^2}{9}|c_2 c_4 - c_3^2| + \frac{(\alpha-1)^2 |c_2|}{72}|c_4 - 3(\alpha-1)c_1 c_3| \Bigg]. \end{aligned} \tag{2.5}$$

Applying Lemmas 1.1, 1.2, 1.3, and 1.5 in (2.5), it simplifies to

$$\big|H_{3,1,\alpha}(f)\big| \le \frac{1}{18}\big[4\alpha^6 - 24\alpha^5 + 76\alpha^4 - 117\alpha^3 + 84\alpha^2 - 25\alpha + 2\big]. \tag{2.6}$$

Theorem 2.1 is proved.

**Theorem 2.2.** *If* $f \in \mathcal{M}(3/2)$, *then*

$$\big|H_{3,1,\frac{3}{2}}(f)\big| \le \frac{3}{16}$$

*and the inequality is sharp for* $p_0(z) = (1+z^2)/(1-z^2)$.

***Proof.*** For $f \in \mathcal{M}(3/2)$, choosing $\alpha = 3/2$ in (2.3), we get

$$\begin{aligned} H_{3,1,\frac{3}{2}}(f) = \frac{1}{9216}\big[ -c_1^6 - 6c_1^4 c_2 + 32 c_1^3 c_3 - 36 c_1^2 c_2^2 + 72 c_2^3 \\ - 192 c_1 c_2 c_3 + 144 c_1^2 c_4 - 256 c_3^2 + 288 c_2 c_4 \big]. \end{aligned} \tag{2.7}$$

In view of (2.7), using the values of $c_2$, $c_3$ and $c_4$ from Lemma 1.4, gives

$$
\begin{aligned}
6c_1^4c_2 &= 3\big[c_1^6 + c_1^4t\mu\big],\\
32c_1^3c_3 &= 8\big[c_1^6 + 2c_1^4t\mu - c_1^4t\mu^2 + 2c_1^3t(1-|\mu|^2)\rho\big],\\
72c_2^3 &= 9\big[c_1^6 + 3c_1^4t\mu + 3c_1^2t^2\mu^2 + t^3\mu^3\big],\\
36c_1^2c_2^2 &= 9\big[c_1^6 + 2c_1^4t\mu + c_1^2t^2\mu^2\big],\\
192c_1c_2c_3 &= 24\big[c_1^6 + 3c_1^4t\mu + 2c_1^2t^2\mu^2 - c_1^4t\mu^2 - c_1^2t^2\mu^3\\
&\quad + 2t\big(c_1^3 + c_1t\mu\big)\big(1-|\mu|^2\big)\rho\big],\\
256c_3^2 &= 16\big[c^6 + 4c^4t\mu + 4c^2t^2\mu^2 - 2c^4t\mu^2 - 4c^2t^2\mu^3 + c^2t^2\mu^4\\
&\quad + 4t(c^3 + 2ct\mu - ct\mu^2)(1-|\mu|^2)\rho + 4t^2(1-|\mu|^2)^2\rho^2\big],\\
288c_2c_4 + 144c_1^2c_4 &= 18\big[2c_1^6 + 6c_1^4t\mu + 2c_1^2\big(4-3c_1^2\big)t\mu^2 + 2c_1^4t\mu^3\\
&\quad + 8c_1^3t\mu(1-\mu)\big(1-|\mu|^2\big)\rho - 8c_1^2t\big(1-|\mu|^2\big)\bar{\mu}\rho^2\\
&\quad + 8c_1^2t\big(1-|\mu|^2\big)\big(1-|\rho|^2\big)\xi + c_1^4t\mu + 3c_1^2t^2\mu^2\\
&\quad + \big(4-3c_1^2\big)t^2\mu^3 + c_1^2t^2\mu^4 + 4t^2c_1\mu(1-\mu)\big(1-|\mu|^2\big)\rho\\
&\quad - 4t^2\big(1-|\mu|^2\big)|\mu|^2\rho^2 + 4t^2\big(1-|\mu|^2\big)\big(1-|\rho|^2\big)\mu\psi\big].
\end{aligned}
\tag{2.8}
$$

Imputting the values from (2.8) in the expression (2.7), putting $u := c_1$ and taking $t = (4-u^2)$, after simplifying, we have

$$
\begin{aligned}
H_{3,1,\frac{3}{2}}(f) = \frac{4-u^2}{9216}&\big[u^2\mu\big(u^2(12-20\mu) - 8\mu(2+\mu-\mu^2) + u^2\mu^2(11-2\mu)\big)\\
&+ 8u\big(6u^2 - 4\mu(13+\mu) - u^2\mu(5-\mu)\big)(1-|\mu|^2)\rho\\
&+ 432\mu^3 - 8\big(18u^2\bar{\mu} + (4-u^2)(8+|\mu|^2)\big)(1-|\mu|^2)\rho^2\\
&+ 72(2u^2 + (4-u^2)\mu)\big(1-|\mu|^2\big)\big(1-|\rho|^2\big)\psi\big].
\end{aligned}
\tag{2.9}
$$

Taking modulus on both sides of (2.9), using $|\mu| = v \in [0,1]$, $|\rho| = w \in [0,1]$, $c_1 = u \in [0,2]$ and $|\psi| \le 1$, we obtain

$$
\big|H_{3,1,\frac{3}{2}}(f)\big| \le \frac{\vartheta_1(u,v,w)}{9216}, \tag{2.10}
$$

where $\vartheta_1 : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$
\begin{aligned}
\vartheta_1(u,v,w) = (4-u^2)&\big[12u^4v + (16u^2 + 20u^4)v^2\\
&+ (432 - 8u^2 + 11u^4)v^3 + (8u^2 - 2u^4)v^4\\
&+ \big(48u^3 + 144u^3v + 104u(4-u^2)v + 8u(4-u^2)v^2\big)(1-v^2)w\\
&+ \big(64(4-u^2) + 144u^2v + 8(4-u^2)v^2\big)(1-v^2)w^2
\end{aligned}
$$

$$+\left(144u^2+72(4-u^2)v\right)\left(1-v^2\right)\left(1-w^2\right)\Big]. \tag{2.11}$$

It remains to maximize $\vartheta_1$ on $\Omega_1 : [0,2]\times[0,1]\times[0,1]$.

**A.** On the vertices of $\Omega_1$, from (2.11), we get

$$\vartheta_1(0,0,0)=\vartheta_1(2,0,0)=\vartheta_1(2,1,0)=\vartheta_1(2,1,1)=\vartheta_1(2,0,1)=0,$$

$$\vartheta_1(0,0,1)=1024, \qquad \vartheta_1(0,1,0)=\vartheta_1(0,1,1)=1728.$$

**B.** On the edges of $\Omega_1$, from (2.11), we obtain:

(i) $u=0,\ v=0,\ 0<w<1$, then

$$\vartheta_1(0,0,w)=1024w^2<1024;$$

(ii) $u=0,\ v=1,\ 0<w<1$, then

$$\vartheta_1(0,1,w)=1728;$$

(iii) $u=0,\ w=0,\ 0<v<1$, then

$$\vartheta_1(0,v,0)=1152v+576v^3<1728;$$

(iv) $u=0,\ w=1,\ 0<v<1$, then

$$\begin{aligned}\vartheta_1(0,v,1)&=1024-896v^2+1728v^3-128v^4\\&=1024-896v^2+1024v^3+704v^3-128v^4\\&<1728-896v^2+1024v^3-128v^4\\&=1728-128(7-v)(1-v)v^2<1728;\end{aligned}$$

(v) $v=0,\ w=0,\ 0<u<2$, then

$$\vartheta_1(u,0,0)=576u^2-144u^4\le 576 \ \text{ for } u=\sqrt{2};$$

(vi) $v=0,\ w=1,\ 0<u<2$, then

$$\begin{aligned}\vartheta_1(u,0,1)&=1024-512u^2+192u^3+64u^4-48u^5\\&=1024-256(2-u)u^2-32(2-u)u^3+32u^4-48u^5\\&<1024+32u^4<1536;\end{aligned}$$

(vii) $v=1,\ w=0,\ 0<u<2$ or $v=1,\ w=1,\ 0<u<2$, then

$$\begin{aligned}\vartheta_1(u,1,0)=\vartheta_1(u,1,1)&=1728-368u^2+148u^4-41u^6\\&=1728-2u^2(46-7u^2)(4-u^2)-27u^6<1728;\end{aligned}$$

(viii) $u=2,\ v=0,\ 0<w<1$, or $u=2,\ v=1,\ 0<w<1$, or $u=2,\ w=0,\ 0<v<1$, or $u=2,\ w=1,\ 0<v<1$, then

$$\vartheta_1(2,v,w)=0.$$

**C.** On the faces of $\Omega_1,$ from (2.11), we have:
(i) $u = 2,\ v \in (0,1)$ and $w \in (0,1),$ then

$$\vartheta_1(2, v, w) = 0;$$

(ii) $u = 0,\ v \in (0,1)$ and $w \in (0,1),$ then

$$\begin{aligned}\vartheta_1(0, v, w) &= 1152v + 576v^3 + (1024 - 1152v - 896v^2 + 1152v^3 - 128v^4)w^2 \\ &= 1152v + 576v^3 + 128(8 - v)(-1 + v)^2(1 + v)w^2 \\ &< 1152v + 576v^3 + 128(8 - v)(-1 + v)^2(1 + v) \\ &= 1024 - 896v^2 + 1728v^3 - 128v^4 < 1728 \ \ (\because\ \text{B(iv)});\end{aligned}$$

(iii) $v = 0,\ u \in (0,2)$ and $w \in (0,1),$ then

$$\begin{aligned}\vartheta_1(u, 0, w) &= (4 - u^2)(48u^3w + 64(4 - u^2)w^2 + 144u^2(1 - w^2)) \\ &= (4 - u^2)(144u^2 + 48u^3w + 16(16 - 13u^2)w^2) \\ &< (4 - u^2)(144u^2 + 48u^3w + 16(16 - 4u^2)w^2) \\ &< (4 - u^2)(144u^2 + 48u^3 + 16(16 - 4u^2))) \\ &= 1024 + 64u^2 + 192u^3 - 80u^4 - 48u^5 \\ &< 1024 + 64u^2 + 192u^3 - 80u^4 < 1536;\end{aligned}$$

(iv) $v = 1,\ u \in (0,2),\ w \in (0,1),$ then, we observe that the function $\vartheta_1(u, 1, w)$ is independent of $w,$ from B(vii), we get

$$\vartheta_1(u, 1, w) < 1728;$$

(v) $w = 0, (u, v) \in (0,2) \times (0,1),$ then

$$\begin{aligned}\vartheta_1(u, v, 0) &= (4 - u^2)\big[(12u^4v + (16u^2 + 20u^4)v^2 + (432 - 8u^2 + 11u^4)v^3 \\ &\qquad + (8u^2 - 2u^4)v^4 + (144u^2 + 72(4 - u^2)v)(1 - v^2))\big] \\ &= (4 - u^2)\big[144u^2 + (288 - 72u^2 + 12u^4)v + (-128u^2 + 20u^4)v^2 \\ &\qquad + (144 + 64u^2 + 11u^4)v^3 + (8u^2 - 2u^4)v^4\big] \\ &< (4 - u^2)\big[144u^2 + (288 - 72u^2 + 12u^4)v + (-128u^2 + 20u^4)v^2 \\ &\qquad + (144 + 64u^2 + 11u^4)v^2 + (8u^2 - 2u^4)v^2\big] \\ &= (4 - u^2)\big[144u^2 + (288 - 72u^2 + 12u^4)v + (144 - 56u^2 + 29u^4)v^2\big] \\ &< (4 - u^2)\big[144u^2 + (288 - 72u^2 + 12u^4) + (144 - 56u^2 + 29u^4)\big] \\ &= (4 - u^2)(432 + 16u^2 + 41u^4) < 1728 \ \ (\because \text{B(vii)});\end{aligned}$$

(vi) $w = 1,\ u \in (0,2),\ v \in (0,1)$, then

$$\begin{aligned}\vartheta_1(u,v,1) = (4-u^2)\Big(&12u^4x + (16u^2+20u^4)v^2 \\ &+ (432-8u^2+11u^4)v^3 + (8u^2-2u^4)v^4 \\ &+ (1-v^2)(144u^2v + 8(4-u^2)(8+v^2)) \\ &+ (1-v^2)\big(48u^3 + 144u^3v + 8uv(4-u^2)(13+v)\big)\Big) \\ := g_1(u,v) &\text{ with } u \in (0,2) \text{ and } v \in (0,1).\end{aligned}$$

Note that all the real solution $(u,v) \in (0,2)\times(0,1)$ of the system of equation

$$\begin{aligned}\frac{\partial g_1}{\partial u} = &-1024u + 576u^2 + 256u^3 - 240u^4 + 1664v + 1152uv - 768u^2v \\ &- 384u^3v - 200u^4v - 72u^5v + 128v^2 + 1024uv^2 - 768u^2v^2 + 32u^3v^2 \\ &+ 280u^4v^2 - 120u^5v^2 - 1664v^3 - 2080uv^3 + 768u^2v^3 \\ &+ 784u^3v^3 + 200u^4v^3 - 66u^5v^3 - 128v^4 + 192uv^4 \\ &+ 192u^2v^4 - 96u^3v^4 - 40u^4v^4 + 12u^5v^4 = 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g_1}{\partial v} = &\ 1664u + 576u^2 - 256u^3 - 96u^4 - 40u^5 - 12u^6 - 1792v \\ &+ 256uv + 1024u^2v - 512u^3v + 16u^4v + 112u^5v - 40u^6v + 5184v^2 \\ &- 4992uv^2 - 3120u^2v^2 + 768u^3v^2 + 588u^4v^2 + 120u^5v^2 - 33u^6v^2 \\ &- 512v^3 - 512uv^3 + 384u^2v^3 + 256u^3v^3 - 96u^4v^3 - 32u^5v^3 + 8u^6v^3 = 0\end{aligned}$$

by a numerical computation are the following:

$$(u_0,v_0) \approx (0.782015, 0.889028) \qquad \text{and} \qquad (u_1,v_1) \approx (1.09323, 0.750188).$$

But $\vartheta_1(u_0,v_0,1) \approx 1565.83$ and $\vartheta_1(u_1,v_1,1) \approx 1590.42$.

**D.** Now we consider the interior of $\Omega_1$.

We can rewrite $\vartheta_1(u,v,w)$ as

$$\begin{aligned}\vartheta_1(u,v,w) = (4-u^2)\big[&12u^4v + (16u^2+20u^4)v^2 \\ &+ (432-8u^2+11u^4)v^3 + (8u^2-2u^4)v^4 \\ &+ \big(48u^3 + 144u^3v + 104u(4-u^2)v + 8u(4-u^2)v^2\big)(1-v^2)w \\ &+ \big[64(4-u^2) + 144u^2v + 8(4-u^2)v^2 \\ &- (144u^2 + 72(4-u^2)v)\big](1-v^2)w^2 \\ &+ (144u^2 + 72(4-u^2)v)\big(1-v^2)\big].\end{aligned}$$

*Case* 1. If $64(4-u^2)+144u^2v+8(4-u^2)v^2-(144u^2+72(4-u^2)v)>0$, then $\vartheta_1(u,v,w)$ is an increase function of $w$. Hence, $\vartheta_1(u,v,w)<\vartheta_1(u,v,1)$.

*Case* 2. If $64(4-u^2)+144u^2v+8(4-u^2)v^2-(144u^2+72(4-u^2)v)\le 0$, then

$$\begin{aligned}
\vartheta_1(u,v,w) \le{} & (4-u^2)\big[12u^4v+(16u^2+20u^4)v^2 \\
&\quad +(432-8u^2+11u^4)v^3+(8u^2-2u^4)v^4 \\
&\quad +\big(48u^3+144u^3v+104u(4-u^2)v+8u(4-u^2)v^2\big)(1-v^2)w \\
&\quad +(144u^2+72(4-u^2)v)\big(1-v^2\big)\big] \\
\le{} & (4-u^2)\big[12u^4v+(16u^2+20u^4)v^2 \\
&\quad +(432-8u^2+11u^4)v^3+(8u^2-2u^4)v^4 \\
&\quad +\big(48u^3+144u^3v+104u(4-u^2)v+8u(4-u^2)v^2\big)(1-v^2) \\
&\quad +(144u^2+72(4-u^2)v)\big(1-v^2\big)\big] \\
:={} & h_1(u,v) \text{ with } u\in(0,2) \text{ and } v\in(0,1).
\end{aligned}$$

Note that all the real solution $(u,v)\in(0,2)\times(0,1)$ of the system of equation

$$\begin{aligned}
\frac{\partial h_1}{\partial u} ={} & 1152u+576u^2-576u^3-240u^4+1664v-1152uv \\
& -768u^2v+480u^3v-200u^4v-72u^5v+128v^2 \\
& -1024uv^2-768u^2v^2+832u^3v^2+280u^4v^2-120u^5v^2 \\
& -1664v^3+224uv^3+768u^2v^3-80u^3v^3+200u^4v^3 \\
& -66u^5v^3-128v^4+64uv^4+192u^2v^4-64u^3v^4 \\
& -40u^4v^4+12u^5v^4=0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial h_1}{\partial v} ={} & 1152+1664u-576u^2-256u^3+120u^4-40u^5-12u^6 \\
& +256uv-1024u^2v-512u^3v+416u^4v+112u^5v \\
& -40u^6v+1728v^2-4992uv^2+336u^2v^2+768u^3v^2 \\
& -60u^4v^2+120u^5v^2-33u^6v^2-512uv^3+128u^2v^3 \\
& +256u^3v^3-64u^4v^3-32u^5v^3+8u^6v^3=0
\end{aligned}$$

by a numerical computation are the following:

$$(u_2,v_2)\approx(1.20468,0.70818) \text{ and } (u_3,v_3)\approx(0.822847,0.893599).$$

But $h_1(u_2, v_2) \approx 1602.12$ and $h_1(u_3, v_3) \approx 1558.5$.

Therefore, from cases 1 and 2, we have $\vartheta_1(u, v, w) < 1728$.

In review of cases **A**, **B**, **C** and **D**, we obtained

$$\max\left\{\vartheta_1(u, v, w) : u \in [0, 2], v \in [0, 1] \text{ and } w \in [0, 1]\right\} = 1728. \tag{2.12}$$

From expression (2.10) and (2.12), we get

$$\left|H_{3,1,,\frac{3}{2}}(f)\right| \leq \frac{3}{16}.$$

For $p_0 \in \mathcal{P}$, we obtain $a_2 = a_4 = 0,\ a_3 = -1/2$ and $a_5 = -1/8$, which follows the result.

Theorem 2.2 is proved.

**3. Bounds for $\mathcal{N}(\alpha)$.**

**Theorem 3.1.** *If $f \in \mathcal{N}(\alpha),\ 1 < \alpha \leq \dfrac{4}{3}$, then*

$$\left|H_{3,1,\alpha}(f)\right| \leq \frac{(\alpha-1)^2}{540}\Big(2(\alpha-1)^4 + 12(\alpha-1)^2 + 37(\alpha-1) + 18\Big).$$

***Proof.*** For $f \in \mathcal{N}(\alpha)$, there exists a analytic function $p \in \mathcal{P}$ such that

$$(\alpha-1)f'(z) - zf''(z) = (\alpha-1)p(z)f'(z). \tag{3.1}$$

Using the series representation for $f$ and $p$ in (3.1), a simple calculation gives

$$\begin{gathered} a_2 = -\frac{(\alpha-1)c_1}{2}, \qquad a_3 = \frac{\alpha-1}{6}\big((\alpha-1)c_1^2 - c_2\big), \\ a_4 = -\frac{\alpha-1}{12}\left[\frac{(\alpha-1)^2c_1^3}{2} - \frac{3(\alpha-1)c_1c_2}{2} + c_3\right] \end{gathered} \tag{3.2}$$

and

$$a_5 = \frac{\alpha-1}{20}\left[\frac{(\alpha-1)^3c_1^4}{6} - (\alpha-1)^2c_1^2c_2 + \frac{4(\alpha-1)c_1c_3}{3} + \frac{(\alpha-1)c_2^2}{2} - c_4\right].$$

Substituting the values of $a_2,\ a_3,\ a_4$ and $a_5$ from (3.2) in (1.3), we have

$$\begin{aligned} H_{3,1,\alpha}(f) = \Bigg(&-\frac{(\alpha-1)^6c_1^6}{8640} - \frac{(\alpha-1)^5c_1^4c_2}{1440} - \frac{7(\alpha-1)^4c_1^2c_2^2}{2880} \\ &+ \frac{(\alpha-1)^4c_1^3c_3}{720} + \frac{(\alpha-1)^3c_2^3}{2160} - \frac{(\alpha-1)^3c_1c_2c_3}{240} \\ &+ \frac{(\alpha-1)^3c_1^2c_4}{240} - \frac{(\alpha-1)^2c_3^2}{144} + \frac{(\alpha-1)^2c_2c_4}{120}\Bigg). \end{aligned} \tag{3.3}$$

Grouping the terms in (3.3) in order to apply the lemmas specified in this paper, we obtain

$$H_{3,1,\alpha}(f) = \Bigg(c_1^3\bigg(-\frac{(\alpha-1)^6c_1^3}{8640} - \frac{(\alpha-1)^5c_1c_2}{1400} + \frac{(\alpha-1)^4c_3}{720}\bigg)$$

$$+\frac{(\alpha-1)^3c_2^2}{2160}[c_2-(\alpha-1)c_1^2]-\frac{17(\alpha-1)^3c_1^2}{4320}\left(c_4-\frac{\alpha-1}{2}c_2^2\right)$$
$$-\frac{c_1(\alpha-1)^3}{240}\left(c_2c_3-\frac{1}{18}c_1c_4\right)+\frac{(\alpha-1)^2}{120}\left(c_2c_4-\frac{120}{144}c_3^2\right)\Bigg). \tag{3.4}$$

Taking modulus on both sides of the expression (3.4), we get

$$\big|H_{3,1,\alpha}(f)\big| \le \Bigg(|c_1|^3\left|-\frac{(\alpha-1)^6c_1^3}{8640}-\frac{(\alpha-1)^5c_1c_2}{1400}+\frac{(\alpha-1)^4c_3}{720}\right|$$
$$+\frac{(\alpha-1)^3|c_2|^2}{2160}|c_2-(\alpha-1)c_1^2|+\frac{(\alpha-1)^3|c_1|}{240}\left|c_2c_3-\frac{1}{18}c_1c_4\right|$$
$$+\frac{(\alpha-1)^2}{120}\left|c_2c_4-\frac{120}{144}c_3^2\right|+\frac{17(\alpha-1)^3|c_1|^2}{4320}\left|c_4-\frac{\alpha-1}{2}c_2^2\right|\Bigg). \tag{3.5}$$

Applying Lemmas 1.1, 1.2, 1.3, and 1.5 in (3.5), upon simplification we have

$$\big|H_{3,1,\alpha}(f)\big| \le \frac{(\alpha-1)^2}{540}\Big(2(\alpha-1)^4+12(\alpha-1)^2+37(\alpha-1)+18\Big).$$

Theorem 3.1 is proved.

**Theorem 3.2.** *If $f\in\mathcal{N}(3/2)$, then*

$$\big|H_{3,1,\frac{3}{2}}(f)\big| \le \frac{19}{2160}$$

*and the inequality is sharp for $p_1(z)=(1+z^2)/(1-z^2)$.*

***Proof.*** For $f\in\mathcal{N}(3/2)$, choosing $\alpha=\dfrac{3}{2}$ in (3.3), we obtain

$$H_{3,1,\frac{3}{2}}(f)=\frac{1}{552960}\big(-c_1^6-12c_1^4c_2+48c_1^3c_3-84c_1^2c_2^2+32c_2^3$$
$$-288c_1c_2c_3+288c_1^2c_4-960c_3^2+1152c_2c_4\big). \tag{3.6}$$

In view of Lemma 1.4, adopting the similar procedure as Theorem 2.2 and putting $u:=c_1$ and using $t=(4-u^2)$ in (3.6), we get

$$H_{3,1,\frac{3}{2}}(f)=\frac{4-u^2}{552960}\Big(36u^4\mu+12u^2\mu^2+108u^4\mu^3-75u^4\mu^2$$
$$+(4-u^2)\Big(56u^2\mu^3+12u^2\mu^4+304\mu^3\Big)$$
$$+\Big(144u^3(1-3\mu)-24(4-u^2)\mu(11+2\mu)\Big)(1-|\mu|^2)\rho$$
$$-48\big(9u^2\bar{\mu}+(4-u^2)(5+|\mu|^2)\big)1-|\mu|^2)\rho^2$$
$$+144(3u^2+2(4-u^2)\mu)\big(1-|\mu|^2\big)\big(1-|\rho|^2\big)\psi\Big). \tag{3.7}$$

Taking modulus on both sides of (3.7), using $|\mu| = v \in [0,1],\ |\rho| = w \in [0,1],\ c_1 = u \in [0,2]$ and $|\psi| \le 1,$ we have

$$\left|H_{3,1,\frac{3}{2}}(f)\right| \le \frac{\vartheta_2(u,v,w)}{552960}, \tag{3.8}$$

where $\vartheta_2 : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\begin{aligned}\vartheta_2(u,v,w) = (4-u^2)\big(&36u^4v + 12u^2v^2 + 108u^4v^3 + 75u^4v^2\\ &+ (4-u^2)\big(56u^2v^3 + 12u^2v^4 + 304v^3\big)\\ &+ \Big(144u^3(1+3v) + 24uv(4-u^2)(11+2v)\Big)(1-v^2)w\\ &+ 48\Big(9u^2v + (4-u^2)(5+v^2)\Big)(1-v^2)w^2\\ &+ 144(3u^2 + 2(4-u^2)v)\big(1-v^2\big)\big(1-w^2\big)\big).\end{aligned} \tag{3.9}$$

It remains to maximize $\vartheta_1$ on $\Omega_2 : [0,2]\times[0,1]\times[0,1].$

**A.** On the vertices of $\Omega_2,$ from (3.9), we obtain

$$\vartheta_2(0,0,0) = \vartheta_2(2,0,0) = \vartheta_2(2,1,0) = \vartheta_2(2,1,1) = \vartheta_2(2,0,0) = 0,$$

$$\vartheta_2(0,0,1) = 3840, \quad \vartheta_2(0,1,0) = \vartheta_2(0,1,1) = 4864.$$

**B.** On the edges of $\Omega_2,$ in view of (3.9) as follows:

(i) $u = 0,\ v = 0,\ 0 < w < 1,$ then

$$\vartheta_2(0,0,w) = 3840w^2 < 3840;$$

(ii) $u = 0,\ v = 1,\ 0 < w < 1,$ then

$$\vartheta_2(0,1,w) = 4864;$$

(iii) $u = 0,\ w = 0,\ 0 < v < 1,$ then

$$\vartheta_2(0,v,0) = 4608v + 256v^3 < 4864;$$

(iv) $u = 0,\ w = 1,\ 0 < v < 1,$ then

$$\begin{aligned}\vartheta_2(0,v,1) &= 3840 - 3072v^2 + 4864v^3 - 768v^4\\ &= 3840 - 3072v^2 + 1024v^3 + 3840v^3 - 768v^4\\ &< 4864 - 3072v^2 + 3840v^3 - 768v^4\\ &< 4864 - 768(4-v)(1-v)v^2 < 4864;\end{aligned}$$

(v) $v = 0,\ w = 0,\ 0 < u < 2,$ then

$$\vartheta_2(u,0,0) = 432u^2(4-u^2) \le 1728 \text{ for } u = \sqrt{2};$$

(vi) $v = 0,\ w = 1,\ 0 < u < 2$, then

$$\begin{aligned}\vartheta_2(u,0,1) &= 3840 - 1920u^2 + 576u^3 + 240u^4 - 144u^5 \\ &= 3840 - 960(2-u)u^2 - 192(2-u)u^3 + 48u^4 - 144u^5 \\ &< 3840 + 48u^4 < 4608;\end{aligned}$$

(vii) $v = 1,\ w = 0,\ 0 < u < 2$ and $v = 1,\ w = 1,\ 0 < u < 2$, then

$$\begin{aligned}\vartheta_2(u,1,w) &= 4864 - 1296u^2 + 624u^4 - 151u^6 \\ &= 4864 - 3(4-u^2)(108-25u^2)u^2 - 76u^6 < 4860;\end{aligned}$$

(viii) $u = 2,\ v = 0,\ 0 < w < 1$, or $u = 2,\ v = 1,\ 0 < w < 1$, or $u = 2,\ w = 0,\ 0 < v < 1$, or $u = 2,\ w = 1,\ 0 < v < 1$, then

$$\vartheta_2(2,v,w) = 0.$$

**C.** On the faces $\Omega_2$, from (3.9), we have:
(i) $u = 2,\ 0 < v < 1,\ 0 < w < 1$, then

$$\vartheta_2(2,v,w) = 0;$$

(ii) $u = 0,\ 0 < v < 1,\ 0 < w < 1$, then

$$\begin{aligned}\vartheta_2(0,v,w) &= 4608v + 256v^3 + (3840 - 4608v - 3072v^2 + 4608v^3 - 768v^4)w^2 \\ &= 4608v + 256v^3 + 768(5-v)(-1+v)^2(1+v)w^2 \\ &< 4608v + 256v^3 + 768(5-v)(-1+v)^2(1+v) \\ &= 3840 - 3072v^2 + 4864v^3 - 768v^4 < 4860\ \ (\because \text{B(iv)});\end{aligned}$$

(iii) $v = 0,\ 0 < u < 2,\ 0 < w < 1$, then

$$\begin{aligned}\vartheta_1(u,0,w) &= 1728u^2 - 432u^4 + (576u^3 - 144u^5)w + (3840 - 3648u^2 + 672u^4)w^2 \\ &= 1728u^2 - 432u^4 + (576u^3 - 144u^5)w + 96(4-u^2)(10-7u^2)w^2 \\ &< 1728u^2 - 432u^4 + (576u^3 - 144u^5)w + 96(4-u^2)(10-2u^2)w^2 \\ &< 1728u^2 - 432u^4 + (576u^3 - 144u^5) + 96(4-u^2)(10-2u^2) \\ &= 3840 - 500(2-u)u^2 + 76u^3 - 240u^4 - 144u^5 \\ &< 3840 + 76u^3 < 4448;\end{aligned}$$

(iv) $v = 1,\ u \in (0,2),\ w \in (0,1)$, then, we observe that the function $\vartheta_2(u,1,w)$ is independent of $w$, from B(vii), we have

$$\vartheta_2(u,1,w) < 4864;$$

(v) $w = 0$ for $(u, v) \in (0, 2) \times (0, 1)$, then

$$\begin{aligned}
\vartheta_2(u, v, 0) &= (4 - u^2)\big[432u^2 + (1152 - 288u^2 + 36u^4)v - (420u^2 - 75u^4)v^2 \\
&\qquad + (64 + 208u^2 + 52u^4)v^3 + (48u^2 - 12u^4)v^4\big] \\
&< (4 - u^2)\big[432u^2 + (1152 - 288u^2 + 36u^4)v - (420u^2 - 75u^4)v^2 \\
&\qquad + (64 + 208u^2 + 52u^4)v^2 + (48u^2 - 12u^4)v^2\big] \\
&= (4 - u^2)\big[432u^2 + (1152 - 288u^2 + 36u^4)v + (64 - 164u^2 + 115u^4)v^2\big] \\
&= (4 - u^2)\big[432u^2 + (1152 - 288u^2 + 36u^4) + (64 - 164u^2 + 115u^4)\big] \\
&= 4864 - 1296u^2 + 624u^4 - 151u^6 < 4864 \ \ (\because \mathrm{B(vii)});
\end{aligned}$$

(vi) $w = 1$ for $(u, v) \in (0, 2) \times (0, 1)$, then

$$\begin{aligned}
\vartheta_2(u, v, 1) = (4 - u^2)\Big(&36u^4v + 12u^2v^2 + 108u^4v^3 + 75u^4v^2 \\
&+ (4 - u^2)\big(56u^2v^3 + 12u^2v^4 + 304v^3\big) \\
&+ \big(144u^3(1 + 3v) + 24uv(4 - u^2)(11 + 2v)\big)(1 - v^2) \\
&+ 48\big(9u^2v + (4 - u^2)(5 + v^2)\big)(1 - v^2)\Big) \\
:= g_2(u, v) \quad &\text{with} \quad u \in (0, 2) \quad \text{and} \quad v \in (0, 1).
\end{aligned}$$

By a numerical computation, it observe that all the real solutions $(u, v) \in (0, 2) \times (0, 1)$ of the system of equation

$$\begin{aligned}
\frac{\partial g_2}{\partial u} = 6\Big(&-64v(-11 - 2v + 11v^2 + 2v^3) + u^5v(-36 - 75v - 52v^2 + 12v^3) \\
&- 20u^4(6 + 7v - 8v^2 - 7v^3 + 2v^4) + 96u^2(3 - 2v - 5v^2 + 2v^3 + 2v^4) \\
&- 32u^3(-5 + 6v - 2v^2 - 15v^3 + 3v^4) \\
&+ 16u(-40 + 36v + 33v^2 - 68v^3 + 12v^4)\Big) = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g_2}{\partial v} = 6(-4 + u^2)\Big(&32v(8 - 19v + 4v^2) + u^4(-6 - 25v - 26v^2 + 8v^3) \\
&- 4u^3(7 - 16v - 21v^2 + 8v^3) + 16u(-11 - 4v + 33v^2 + 8v^3) \\
&- 4u^2(18 + 17v - 64v^2 + 16v^3)\Big) = 0
\end{aligned}$$

are the following:

$$(u_4, v_4) \approx (1.06135, 0.690822), \qquad (u_5, v_5) \approx (0.539801, 0.888371)$$

$$\text{and} \qquad (u_6, v_6) \approx (0.0690861, 0.0570909).$$

But $\vartheta_2(u_4, v_4, 1) \approx 4769.23$, $\vartheta_2(u_5, v_5, 1) \approx 4565.82$, $\vartheta_2(u_6, v_6, 1) \approx 3839.18$.

Therefore, $\vartheta_2(u,v,1) \le 4769.23$ on the face $w=1$.

**D.** Now we consider the interior $\Omega_2$.

We can rewrite $\vartheta_2(u,v,w)$ as

$$\begin{aligned}\vartheta_2(u,v,w) = (4-u^2)\Big(&36u^4v + 12u^2v^2 + 108u^4v^3 + 75u^4v^2 \\ &+ (4-u^2)\big(56u^2v^3 + 12u^2v^4 + 304v^3\big) \\ &+ \Big(144u^3(1+3v) + 24uv(4-u^2)(11+2v)\Big)(1-v^2)w \\ &+ 48\Big[9u^2v + (4-u^2)(5+v^2) \\ &- 144(3u^2 + 2(4-u^2)v)\Big](1-v^2)w^2 \\ &+ 144(3u^2 + 2(4-u^2)v)\big(1-v^2\big)\Big).\end{aligned}$$

*Case* 1. If $9u^2v + (4-u^2)(5+v^2) - 144(3u^2 + 2(4-u^2)v) > 0$, then $\vartheta_2(u,v,w)$ is an increase function of $w$. Hence, $\vartheta_2(u,v,w) < \vartheta_2(u,v,1)$.

*Case* 2. If $9u^2v + (4-u^2)(5+v^2) - 144(3u^2 + 2(4-u^2)v) \le 0$, then

$$\begin{aligned}\vartheta_1(u,v,w) \le (4-u^2)\Big(&36u^4v + 12u^2v^2 + 108u^4v^3 + 75u^4v^2 \\ &+ (4-u^2)\big(56u^2v^3 + 12u^2v^4 + 304v^3\big) \\ &+ \Big(144u^3(1+3v) + 24uv(4-u^2)(11+2v)\Big)(1-v^2)w \\ &+ 144(3u^2 + 2(4-u^2)v)\big(1-v^2\big)\Big) \\ \le (4-u^2)\Big(&36u^4v + 12u^2v^2 + 108u^4v^3 + 75u^4v^2 \\ &+ (4-u^2)\big(56u^2v^3 + 12u^2v^4 + 304v^3\big) \\ &+ \Big(144u^3(1+3v) + 24uv(4-u^2)(11+2v)\Big)(1-v^2) \\ &+ 144(3u^2 + 2(4-u^2)v)\big(1-v^2\big)\Big) \\ := h_2(u,v) &\text{ with } u \in (0,2) \text{ and } v \in (0,1).\end{aligned}$$

Note that all the real solution $(u,v) \in (0,2) \times (0,1)$ of the system of equation

$$\begin{aligned}\frac{\partial h_2}{\partial u} = 3456u &+ 1728u^2 - 1728u^3 - 720u^4 + 4224v - 4608uv \\ &- 1152u^2v + 1728u^3v - 840u^4v - 216u^5v + 768v^2\end{aligned}$$

$$
\begin{aligned}
&- 3360uv^2 - 2880u^2v^2 + 2880u^3v^2 + 960u^4v^2 \\
&- 450u^5v^2 - 4224v^3 + 1536uv^3 + 1152u^2v^3 + 840u^4v^3 \\
&- 312u^5v^3 - 768v^4 + 384uv^4 + 1152u^2v^4 - 384u^3v^4 \\
&- 240u^4v^4 + 72u^5v^4 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial h_2}{\partial v} = {} & 4608 + 4224u - 2304u^2 - 384u^3 + 432u^4 - 168u^5 - 36u^6 \\
&+ 1536uv - 3360u^2v - 1920u^3v + 1440u^4v + 384u^5v \\
&- 150u^6v + 768v^2 - 12672uv^2 + 2304u^2v^2 + 1152u^3v^2 \\
&+ 504u^5v^2 - 156u^6v^2 - 3072uv^3 + 768u^2v^3 \\
&+ 1536u^3v^3 - 384u^4v^3 - 192u^5v^3 + 48u^6v^3 = 0
\end{aligned}
$$

by a numerical computation are the following:

$$
(u_7, v_7) \approx (1.21171, 0.674712) \qquad \text{and} \qquad (u_8, v_8) \approx (0.647547, 0.887607).
$$

But $h_2(u_7, v_7) \approx 4772.04$ and $h_1(u_8, v_8) \approx 4516.37$. Therefore, from cases 1 and 2, we have $\vartheta_2(u, v, w) < 4864$.

In review of cases **A**, **B**, **C** and **D**, we obtain

$$
\max\Big\{\vartheta_2(u, v, w) \leq 4864 \colon u \in [0, 2], v \in [0, 1] \text{ and } w \in [0, 1]\Big\}. \tag{3.10}
$$

Symplifying the expressions (3.9) and (3.10), we get

$$
\big|H_{3,1}(f)\big| \leq \frac{19}{2160}. \tag{3.11}
$$

For $p_0 \in \mathcal{P}$, we obtain $a_2 = a_4 = 0$, $a_3 = -1/6$ and $a_5 = -1/40$, which follows the equality in (3.11).

Theorem 3.2 is proved.

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## References

1. M. Arif, Mohsan Raza, Huo Tang, Shehzad Hussain, Hassan Khan, *Hankel determinant of order three for familiar subsets of analytic functions related with sine function*, Open Math., **17**, № 1, 1615 – 1630 (2019).
2. K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequal. Theory and Appl., Nova Sci. Publ., New York (2010), p. 1 – 7.

3. S. Banga, S. Sivaprasad Kumar, *The sharp bounds of the second and third Hankel determinants for the class $\mathcal{SL}^*$*, Math. Slovaca, **70**, № 4, 849 – 862 (2020); DOI: 10.1515/ms-2017-0398.
4. D. Breaz, A. Cătaş, L. Cotîrlă, *On the upper bound of the third Hankel determinant for certain class of analytic functions related with exponential function*, An. Ştiinţ. Univ. “Ovidius” Constanţa Ser. Mat., **30**, 75 – 89 (2022); https://doi.org/10.2478/auom-2022-0005.
5. P. L. Duren, *Univalent functions*, vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York (1983).
6. T. Hayami, S. Owa, *Generalized Hankel determinant for certain classes*, Int. J. Math. Anal., **4**, № 52, 2573 – 2585 (2010).
7. B. Kowalczyk, A. Lecko, Y. J. Sim, *The sharp bound for the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc., **97**, № 3, 435 – 445 (2018).
8. B. Kowalczyk, A. Lecko, M. Lecko, Y. J. Sim, *The sharp bound of the third Hankel deternimant for some classes of analytic functions*, Bull. Korean Math. Soc., **55**, № 6, 1859 – 1868 (2018); https://doi.org/10.4134/BKMS.b171122.
9. O. S. Kwon, A. Lecko , Y. J. Sim, *The bound of the Hankel determinant of the third kind for starlike functions*, Bull. Malays. Math. Sci. Soc., **42**, № 2, 767 – 780 (2019).
10. O. S. Kwon, Y. J. Sim, *The sharp bound of the Hankel determinant of the third kind for starlike functions with real coefficients*, Mathematics (2019); DOI:10.3390/math7080721.
11. R. J. Libera, E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathscr{P}$*, Proc. Amer. Math. Soc., **87**, № 2, 251 – 257 (1983).
12. S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, I. Ali, *Upper bound of the third Hankel determinant for a subclass of $q$-starlike functions*, Symmetry, **11**, № 3, Article ID 347 (2019).
13. J. Nishiwaki, S. Owa, *Coefficient inequalities for certain analytic functions*, Int. J. Math. and Math. Sci., **29**, № 5, 285 – 290 (2002).
14. H. Orhan, M. Çağlar, L. Cotîrlă, *Third Hankel determinant for a subfamily of holomorphic functions related with Lemniscate of Bernoulli*, Mathematics, **11**, № 5 (2023); https://doi.org/10.3390/math11051147.
15. Ch. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen (1975).
16. Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc., **41**, № 1, 111 – 122 (1966).
17. J. K. Prajapat, D. Bansal, S. Maharana, *Bounds on third Hankel determinant for certain classes of analytic functions*, Stud. Univ. Babeş-Bolyai Math., **62**, № 2, 183 – 195 (2017); DOI:10.24193/subbmath.2017.2.05.
18. B. Rath, K. S. Kumar, D. V. Krishna, G. K. S. Viswanadh, *The sharp bound for the third Hankel determinant of the inverse of functions associated with lemniscate of Bernoulli*, Asian-Eur. J. Math., https://doi.org/10.1142/S1793557123501267.
19. K. Sanjay Kumar, B. Rath, D. V. Krishna, *The sharp bound of the third Hankel determinant for the inverse of bounded turning functions*, Contemp. Math. Internet., **4**, № 1, 30 – 41 (2023); https://doi.org/10.37256/cm.4120232183.
20. L. Shi, M. Shutaywi, N. Alreshidi, M. Arif, M. Ghufran, *The sharp bounds of the third-order Hankel determinant for certain analytic functions associated with an eight-shaped domain*, Fractal and Fract., **6**, № 4 (2022); https://doi.org/10.3390/fractalfract6040223.
21. B. Rath, K. S. Kumar, D. V. Krishna, G. K. S. Viswanadh, *The sharp bound of the third Hankel determinants for inverse of starlike functions with respect to symmetric points*, Mat. Stud., **58**, 45 – 50 (2022).
22. B. Rath, K. S. Kumar, D. V. Krishna, A. Lecko, *The sharp bound of the third Hankel determinant for starlike functions of order 1/2*, Complex Anal. and Oper. Theory (2022); https://doi.org/10.1007/s11785-022-01241-8.
23. B. A. Uralegaddi, M. D. Ganigi, S. M. Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math., **25**, № 3, 225 – 230 (1994); DOI: 10.5556/j.tkjm.25.1994.444.
24. Virendra Kumar, Sushil Kumar, V. Ravichandran, *Third Hankel determinant for certain classes of analytic functions*, Mathematical Analysis I: Approximation Theory (2020); DOI: 10.1007/978-981-15-1153-019.

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