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On the some properties of the orthogonal polynomials over a contour with general Jacobi weight

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Abstract. In present work, we continue the study the growth of the orthogonal polynomials over a contour with weight function in the weighted Lebesgue space, when the contour and the weight function having some singularities. We study case, when interference of weight and contour is not satisfied, for piecewise smooth contour with interior zero angles. Also we investigated case of more general contours.

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1. Introduction

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $L \subset \mathbb{C}$ be a closed rectifiable Jordan curve, $G := intL$, with $0 \in G$, $\Omega := extL$. Let $h(z)$ nonnegative, summable on a L and nonzero except possible on a set of measure zero function. The systems of polynomials $\{K_n(z)\}$, $K_n(z) = a_n z^n + \ldots$, $\deg K_n = n$, $n = 0, 1, 2, \ldots$, satisfying the condition

$$\int_L h(z) K_n(z) \overline{K_m(z)}\, |dz| = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}$$

are called orthonormal polynomials for the pair (L, h). These polynomials are determined uniquely if the coefficient $a_n > 0$.

These polynomials were first studied by G. Szegö [33, 34]. V. I. Smirnov [32], P. P. Korovkin [19] and Ya. L. Geronimus [16] was investigated these polynomials under the various conditions on the weight function $h(z)$ and contour L. In [37], P. K. Suetin was investigated many properties of the polynomials $\{K_n(z)\}_{n=0}^{\infty}$ for sufficiently smooth contour and weight

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function $h(z)$ wich is zero or infinite at finite number points on contour L. A. L. Kuz'mina [20] and G. Fauth [14] have considered some properties of the polynomials $\{K_n(z)\}_{n=0}^{\infty}$ for piecewise analytic contour L with finite number corners. In [38], P. K. Suetin obtain several estimates for the rate of growth of the polynomials $\{K_n(z)\}_{n=0}^{\infty}$ on the contour L, depending of the singularites of the weight function $h(z)$ on L and of the contour L.

Let a rectifiable Jordan curve L, has a natural parametrization $z = z(s)$, $0 \leq s \leq |L| := mesL$. It is said to be $L \in C(1, \alpha)$, $0 < \alpha < 1$, if $z(s)$ is continuously differentiable and $z'(s) \in Lip\,\alpha$. Let L belong to $C(1, \alpha)$ everywhere except for a single point $z_1 \in L$, i.e., the derivative $z'(s)$ satisfies the Lipschitz condition on the $[0, |L|]$ and $z(0) = z(|L|) = z_1$, but $z'(0) \neq z'(|L|)$. Assume that L has a corner at z_1 with exterior angle $\omega_1\pi,\ 0 < \omega_1 \leq 2$, and denote the set of such curves by $C(1, \alpha, \omega_1)$.

Denoted by $w = \Phi(z)$, the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$. For $t \geq 1$, we set

$$L_t := \{z : \ |\Phi(z)| = t\}\,,\ L_1 \equiv L,\ G_t := intL_t,\ \Omega_t := extL_t.$$

Let $\{z_j\}_{j=1}^{m}$ be the fixed system of distinct points on curve L. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0} \backslash G$, consider generalized Jacobi weight function $h\,(z)$, which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}\,, \tag{1.1}$$

where $\gamma_j > -1$, for all $j = 1, 2, \ldots, m$, and h_0 is uniformly separated from zero in L, i.e. there exists a constant $c_0(L) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0(L) > 0.$$

P. K. Suetin [38] investigated this problem for $K_n(z)$ with the weight function $h(z)$ defined as in (1.1) and for the curve $L \in C(1, \alpha, \omega_1)$. He showed that the condition of "pay off" singularity curve and weight function at the points z_1 can be given as following:

$$(1 + \gamma_1)\,\omega_1 = 1, \tag{1.2}$$

and, under this conditions, for $K_n(z)$ provided the following estimation:

$$|K_n(z)| \leq c(L)\sqrt{n+1},\ z \in L, \tag{1.3}$$

where $c(L) > 0$ is a constant independent on n.

In [38], author also investigated the case, where $(1+\gamma_1)\,\omega_1 \neq 1$. It is shown, if the singularity of a curve and weight function at the points z_1 satisfy the following condition:

$$(1+\gamma_1)\,\omega_1 < 1. \tag{1.4}$$

Then for $|K_n(z)|$, the following estimation is true

$$|K_n(z)| \leq c_1(L)\left[(n+1)^{s_1} + |z-z_1|^{\sigma_1}\sqrt{n+1}\right],\ z \in L, \tag{1.5}$$

where

$$s_1 = \frac{1}{2}(1+\gamma_1)\,\omega_1,\ \sigma_1 = \frac{1}{2}\left(\frac{1}{\omega_1} - 1 - \gamma_1\right),$$

and $c_1(L) > 0$ is the constant independent on n.

In this work we study the estimations of the (1.5)-type under the condition (1.4), for more general contours of the complex plane and we obtain the analog of the estimation (1.5) for more general case. In parallel, we also study the growth of arbitrary algebraic polynomials with respect to their seminorm in the weighted Lebesgue space, under the condition of (1.4)-type.

2. Definitions and Main Results

Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most n, $n \in \mathbb{N} := \{1,2,\ldots\} \cup \{0\}$.

Without loss of generality, the number R_0 in the definition of the weight function, we can take $R_0 = 2$. Otherwise the number n can be choosen $n \geq \left[\frac{\varepsilon_0}{R_0-1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we denote

$$\begin{aligned}
\|P_n\|_{\mathcal{L}_p} &: = \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z)\,|P_n(z)|^p\,|dz|\right)^{1/p},\ 0 < p < \infty, \\
\|P_n\|_{\mathcal{L}_\infty} &: = \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z\in L}|P_n(z)|,\ p = \infty.
\end{aligned}$$

Clearly, $\|\cdot\|_{\mathcal{L}_p}$ is a quasinorm (i.e. a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$).

Throughout this work, notation $i = \overline{k,m}$ means $i = k, k+1, \ldots, m$, for any $k \geq 0$ and $m > k$.

Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w : |w| < 1\}$ onto the G normalized by $\psi(0) = 0$, $\psi'(0) > 0$. By [28, p. 286–294], we say a bounded Jordan region G is called *κ-quasidisk*, $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on plane $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a *κ-quasicircle*. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is *κ-quasidisk (κ-quasicircle)* for some $0 \leq \kappa < 1$.

We denoted the class of *κ-quasicircle* by $Q(\kappa)$, $0 \leq \kappa < 1$, and denote by $L \in Q$, if $L \in Q(\kappa)$, for some $0 \leq \kappa < 1$. It is well-known that the quasicircle may not even be locally rectifiable ([21, p. 104]).

Definition 2.1. *We say that $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $L \in Q(\kappa)$ and L is rectifiable. Analogously, $L \in \widetilde{Q}$, if $L \in \widetilde{Q}(\kappa)$, for some $0 \leq \kappa < 1$.*

In [8] was obtained the following result for $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$:

Theorem A. *Let $p > 0$. Suppose that $L \in \widetilde{Q}(\kappa)$, for some $0 \leq \kappa < 1$ and $h(z)$ defined in* (1.1) *for $\gamma_j = 0$, for all $j = \overline{1,m}$. Then, for any $n \in \mathbb{N}$, there exists $c_1 = c_1(L,p) > 0$ such that:*

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_1(n+1)^{\frac{1+\kappa}{p}} \|P_n\|_{\mathcal{L}_p(h_0,L)} . \tag{2.1}$$

Thus, Theorem A provides an opportunity to observe the growth of $|P_n(z)|$ on the curve L. Note that, Theorem A for $L := \{z : |z| = 1\}$ (i.e. $\kappa = 0$) provided in [18]. The other classical results are similar to (2.1) we can find in [35]. The evaluations of (2.1)-type for $0 < p < \infty$, $h(z) \equiv 1$ (or $h(z) \neq 1$) was also investigated in [23, 24, 30, 36], [26, p. 122–133], [13, Theorem 6], [2–8] and others (see also the references cited therein), for different Jordan curves having special properties. There are more references regarding the inequality of (2.1)-type, we can find in Milovanovic et al. [25, Sect. 5.3].

Now, we will define a more general class of curves with another characteristic.

Definition 2.2. *We say that $L \in Q_\alpha$, $0 < \alpha \leq 1$, if $L \in Q$ and $\Phi \in Lip\,\alpha$, $z \in \overline{\Omega}$.*

We note that the class Q_α is sufficiently wide. A detailed account on it and the related topics are contained in [22, 29, 39] and the references cited therein. We consider only some cases:

Remark 2.1. a) If $L = \partial G$ is a Dini-smooth curve [29, p. 48], then $L \in Q_1$.

b) If $L=\partial G$ is a piecewise Dini-smooth curve and largest exterior angle at L has opening $\alpha\pi, 0<\alpha\leq 1$, [29, p. 52], then $L\in Q_{\alpha}$.

c) If $L=\partial G$ is a smooth curve having continuous tangent line then $L\in Q_{\alpha}$ for all $0<\alpha<1$.

d) If L is quasismooth (in the sense of Lavrentiev), that is, for every pair $z_1,\ z_2\in L$, if $s(z_1,z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on L,there exists a constant $c>1$ such that $s(z_1,z_2)\leq c\left|z_1-z_2\right|$, then $\Phi\in Lip\,\alpha$ for $\alpha=\frac{1}{2}(1-\frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$ [39].

e) If L is "c-quasiconformal" (see, for example, [22]), then $\Phi\in Lip\ \alpha$ for $\alpha=\frac{\pi}{2(\pi-\arcsin\frac{1}{c})}$. Also, if L is an asymptotic conformal curve, then $\Phi\in Lip\ \alpha$ for all $0<\alpha<1$ [22].

Definition 2.3. *It is said that $L\in\widetilde{Q}_{\alpha}$, $0<\alpha\leq 1$, if $L\in Q_{\alpha}$ and L is rectifiable.*

If the weight function h, has "singularities" at the points $\{z_i\}_{i=1}^{m}$, i.e., $\gamma_i\neq 0$ for all $i=\overline{1,m}$, then we have the following:

Theorem B. ([27]) *Let $p>0$. Suppose that $L\in\widetilde{Q}_{\alpha}$, for some $\frac{1}{2}\leq\alpha\leq 1$ and $h(z)$ defined as in* (1.1). *Then, for any $\gamma_i>-1$, $i=\overline{1,m}$, and $P_n\in\wp_n$, $n\in\mathbb{N}$, there exists $c_2=c_2(L,p,\gamma_i,\alpha)>0$ such that*

$$\left\|P_n\right\|_{\mathcal{L}_\infty}\leq c_4 n^{\frac{\widetilde{\gamma}+1}{p\beta_i}}\left\|P_n\right\|_{\mathcal{L}_p(h,L)},\quad where\ \ \widetilde{\gamma}:=\max\{0,\ \gamma_i,\ i=\overline{1,m}\}. \tag{2.2}$$

Therefore, according to 2.1, we can calculate α in the right parts of estimations (2.2) for each case.

Now, let's introduce "special" corners on the curve L, thus spoiling its "smoothness".

Definition 2.4. *We say that $L\in Q\left[\nu\right]$, $0<\nu<2$, if*

a) $L\in Q$,

b) For $\forall z\in L$, there exists a $r:=r(L,z)>0$ and $\nu:=\nu(L,z)$, $0<\nu<2$, such that for some $0\leq\theta_0<2$ a closed maximal circular sector $S(z;r,\nu):=\left\{\zeta:\zeta=z+re^{i\theta\pi},\ \theta_0<\theta<\theta_0+\nu\right\}$ of radius r and opening $\nu\pi$ lies in $\overline{G}$ with vetrex at z.

It is well known that each quasicircle satisfies the condition b). Nevertheless, this condition imposed on L gives a new geometric characterization of the curve. For example, if the contour L^* defined by

$$L^*:=[0,i]\cup\left\{z:z=e^{i\theta\pi},\ \frac{1}{2}<\theta<2\right\}\cup\left[1,0\right],$$

then the coefficient of quasiconformality k of the L^* does not obtained so easily, whereas $L^* \in Q\left[\frac{3}{2}\right]$.

Analogously, we say that $L \in \widetilde{Q}[\nu]$, $0 < \nu < 2$, if $L \in Q[\nu]$, $0 < \nu < 2$, and L is rectifiable.

Definition 2.5. *We say that $L \in \widetilde{Q}_{\alpha}[\nu_1, \ldots, \nu_m]$, $0 < \nu_1, \ldots, \nu_m < 2$, $0 < \alpha \leq 1$, if there exists a system of points $\{\zeta_i\}_{i=1}^m \in L$, such that $L \in \widetilde{Q}[\nu_i]$ for any points $\zeta_i \in L$, $i = \overline{1,m}$, and $\Phi \in Lip\alpha$, $0 < \alpha \leq 1$, for $z \in \overline{\Omega} \setminus \{\zeta_i\}$.*

It is clear from Definition 2.5, that each contour $L \in \widetilde{Q}_{\alpha}[\nu_1, \ldots, \nu_m]$, $0 < \nu_1, \ldots, \nu_m < 2$, $0 < \alpha \leq 1$, $i = \overline{1,m}$, may have singularities (corners) at the points $\{\zeta_i\}_{i=1}^m \in L$. If a contour L does not have such singularities, i.e. if $\nu_i = 1$, $i = \overline{1,m}$, then it is written as $L \in \widetilde{Q}_{\alpha}[1] := \widetilde{Q}_{\alpha}$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^m \in L$ are defined in (1.1) and $\{\zeta_i\}_{i=1}^m \in L$ are defined in Definitions 2.3 coincides. Without the loss of generality, we also will assume that the points $\{z_i\}_{i=1}^m$ are ordered in the positive direction on the curve L.

In [9] we prove the following:

Theorem C. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_{\alpha}[\nu_1, \ldots, \nu_m]$, for some $0 < \nu_1, \ldots, \nu_m < 1$, $\frac{1}{2-\nu_i} \leq \alpha \leq 1$; $h(z)$ defined as in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_3 = c_3(L, p, \gamma_i, \nu_i, \alpha) > 0$ such that*

$$|P_n(z_i)| \leq c_3(n+1)^{\frac{\gamma_i+1}{p}(2-\nu_i)} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.3}$$

and

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_3(n+1)^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.4}$$

if

$$\gamma_i + 1 = \frac{1}{\alpha(2-\nu_i)} \tag{2.5}$$

satisfies for each points $\{z_i\}_{i=1}^m$.

Corollary C. *Suppose that $L \in \widetilde{Q}_{\alpha}[\nu_1, \ldots, \nu_m]$, for some $0 < \nu_1, \ldots, \nu_m < 1$, $\frac{1}{2-\nu_i} \leq \alpha \leq 1$; $h(z)$ defined as in (1.1). Then, for any $K_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_4 = c_4(L, \gamma_i, \nu_i, \alpha) > 0$ such that*

$$|K_n(z_i)| \leq c_4(n+1)^{\frac{\gamma_i+1}{2}(2-\nu_i)}, \tag{2.6}$$

and

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_4(n+1)^{\frac{1}{2\alpha}}, \tag{2.7}$$

if (2.5) satisfies for each points $\{z_i\}_{i=1}^m$.

Comparing Theorem C with Theorem B (for case $\gamma_i = 0,\ i = \overline{1,m}$), it is seen that, if the equality (2.5) is satisfied, then the growth of rate of the polynomials $P_n(z)$ (consequently $K_n(z)$) on L does not depend on whether the weight function $h(z)$ and the boundary contour L have singularity or not. The condition (2.5) is called the condition of "interference of singularities" of weight function h and contour L at the "singular" points $\{z_i\}_{i=1}^m$.

Now, we begin state our new results. Our first results is related to the general case. For simplify our calculations, we take $i = 1$.

Theorem 2.1. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1]$, for some $0 < \nu_1 < 1$ and $\frac{1}{2-\nu_1} \le \alpha \le 1$; $h(z)$ defined as in* (1.1) *and*

$$\gamma_1 + 1 < \frac{1}{\alpha(2-\nu_1)}, \tag{2.8}$$

at the point z_1. Then, for every $z \in L$ and $P_n \in \wp_n,\ n \in \mathbb{N}$, there exists $c_5 = c_5(L, p, \gamma_1, \nu_1, \alpha) > 0$, such that

$$|P_n(z)| \le c_5 \left[(n+1)^{s_1} + |z - z_1|^{\sigma_1}(n+1)^{1/p\alpha}\right] \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.9}$$

and

$$|P_n(z_1)| \le c_5 (n+1)^{s_1} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.10}$$

where

$$s_1 = \frac{1+\gamma_1}{p}(2-\nu_1),\ \ \sigma_1 = \frac{1}{2\alpha(2-\nu_1)} - \frac{1+\gamma_1}{2}. \tag{2.11}$$

Corollary 2.2. *Under the comditions of Theorem 2.1, we have:*

$$|K_n(z)| \le c_5 \left[(n+1)^{s_1} + |z - z_1|^{\sigma_1}(n+1)^{1/2\alpha}\right], \tag{2.12}$$

where s_1 (for $p = 2$) and σ_1 denines as in (2.11)

Since $\alpha \ge \frac{1}{2-\nu_1}$, (2.8) will be satisfied when $-1 < \gamma_1 < 0$. Here and from (2.9), (2.12), we see that, the order of the height of P_n (K_n) in point z_1 and points $z \in L$, $z \ne z_1$, where $h(z) \to \infty$ and curve L does not have singularity, acts itself identically. Thus, the conditions (2.8) can be called "algebraic pole" conditions of the order $\lambda_1 = 1 - \alpha(2-\nu_1)(1+\gamma_1)$.

In case, if L and $h(z)$ have two singular points simultaneously, then (2.12) can be written as:

$$|K_n(z)| \le c_6 |z - z_1|^{\sigma_1}(n+1)^{s_2} + c_7 |z - z_2|^{\sigma_2}(n+1)^{s_1} \tag{2.13}$$
$$+ c_8 |z - z_1|^{\sigma_1}|z - z_2|^{\sigma_2}(n+1)^{1/2\alpha},\ z \in L,$$

for some constants $c_j = c_j(L, \gamma_i, \nu_i, \alpha)$, $j = 6, 7, 8$, where s_i, σ_i, $i = 1, 2$, are defined as it is in (2.11) for $p = 2$, respectively.

Corollary 2.3. *If $L \in C(1, \alpha, \omega_1)$, $1 \leq \omega_1 \leq 2$, and the condition*

$$(\gamma_1 + 1)\,\omega_1 < 1$$

satisfies at the point z_1, then we have:

$$|K_n(z)| \leq c_5 \left[(n+1)^{s_1} + |z - z_1|^{\sigma_1} \sqrt{n+1} \right], \tag{2.14}$$

where

$$s_1 = \frac{1}{2}\,(1+\gamma_1)\,\omega_1, \;\; \sigma_1 = \frac{1}{2}\left(\frac{1}{\omega_1} - 1 - \gamma_1 \right). \tag{2.15}$$

Estimation (2.14) coincides from the result by P. K. Suetin [38, Theorem 2]. So, the Corollary 2.2 generalizes the result [38, Theorem 2] for $1 \leq \nu_1 \leq 2$ and extends this result to the more general curves of the complex plane.

Theorem 2.1 is true under the condition $0 < \nu_1 < 1$. On the other hand, from $\frac{1}{2-\nu_1} \leq \alpha \leq 1$ we see that the $\nu_1 = 1 - \varepsilon$ true only for $\alpha \geq 1 - \varepsilon, \forall \varepsilon > 0$. Therefore, for the value $1 \leq \nu_1 \leq 2$, we can consider only curves L such that $\Phi(z) \in Lip\,(1-\varepsilon)\,,\ \forall \varepsilon > 0,\ z \in \overline{\Omega}$. For this purpose, let's give a following definition.

Let S be rectifiable Jordan curve or arc and let $z = z(s),\ s \in [0,\ |S|]\,,\ |S| := mes\ S$, denote the natural representation of S.

Definition 2.6. *We say that a Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$.*

When we consider the arc, at the endpoints we will understand the existence sided tangents.

Now, we will define a new class of curves L, which have a exterior corners (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^m \in L$.

Definition 2.7. *We say that a Jordan curve $L \in PC_\theta(\lambda_1, \lambda_2, \ldots, \lambda_m)$, $0 < \lambda_i \leq 2$, $i = \overline{1,m}$, if L consists of the union number of finite C_θ (smooth)-arcs $\{L_i\}_{i=1}^m$, where they have exterior (with respect to $\overline{G}$) $\lambda_i\pi-$angles, $0 < \lambda_i \leq 2$, at the corner points $\{z_i\}_{i=1}^m \in L$, where two arcs meet.*

According to the "three-point" criterion [1, p. 100], every piecewise smooth curve (without cusps) is quasicircle.

In this case, we have the following:

Theorem 2.4. *Let $p > 0$. Suppose that $L \in PC_\theta(\lambda_1)$, for some $0 < \lambda_1 \leq 2$; $h(z)$ defined as in (1.1). Then, for any $P_n,\ n \in \mathbb{N}$, there exists $c_9 = c_9(L, p, \gamma_1, \lambda_1, \varepsilon) > 0$ such that*

$$|P_n(z_1)| \leq c_9 (n+1)^{\frac{\gamma_1+1}{p}\widetilde{\lambda}_1} \|P_n\|_{\mathcal{L}_p(h,L)}\,, \tag{2.16}$$

and

$$|P_n(z)| \leq c_9 \left[(n+1)^{s_1} + |z - z_1|^{\sigma_1} (n+1)^{\frac{1}{p}+\varepsilon} \right] \|P_n\|_{\mathcal{L}_p(h,L)}, \; z \in L, \tag{2.17}$$

if

$$\gamma_1 + 1 < \frac{1}{\lambda_1}, \tag{2.18}$$

where

$$s_1 = \frac{1+\gamma_1}{p} \widetilde{\lambda}_1, \quad \sigma_1 = \frac{1}{2\lambda_1} - \frac{1+\gamma_1}{2}. \tag{2.19}$$

and $\widetilde{\lambda}_1 := \begin{cases} \lambda_1 + \varepsilon, & \textit{if } \; 0 < \lambda_1 < 2, \\ 2, & \textit{if } \; \lambda_1 = 2, \end{cases}$ *for arbitrary small* $\varepsilon > 0$.

Corollary 2.5. *Under the conditions of Theorem 2.4,*

$$|K_n(z_1)| \leq c_9 (n+1)^{\frac{\gamma_1+1}{2}\widetilde{\lambda}_1}, \tag{2.20}$$

and

$$|K_n(z)| \leq c_9 \left[(n+1)^{s_1} + |z - z_1|^{\sigma_1} (n+1)^{\frac{1}{2}+\varepsilon} \right], \; z \in L, \tag{2.21}$$

if

$$\gamma_1 + 1 < \frac{1}{\lambda_1}, \tag{2.22}$$

where s_1 *(for* $p = 2$*) and* σ_1 *defines as in* (2.11).

The number $\varepsilon > 0$ on the right sides of the estimations (2.16), (2.17) and, consequently, (2.20), (2.22) can be removed. For this, we introduce the following definitions:

Definition 2.8. *([29, p. 48] (see also [12])) We say that a Jordan curve or arc* S *called Dini-smooth (DS), if it has a parametrization* $z = z(s)$, $0 \leq s \leq |S|$, *such that* $z^{'}(s) \neq 0$, $0 \leq s \leq |S|$ *and* $\left| z^{'}(s_2) - z^{'}(s_1) \right| < g(s_2 - s_1)$, $s_1 < s_2$, *where* g *is an increasing function for which*

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

Now, we shall define a new class of curves, which at the finite number points have exterior corners and interior cusps simultaneously.

Definition 2.9. *We say that a Jordan curve* $L \in PDS(\lambda_1, \lambda_2, \ldots, \lambda_m)$, $0 < \lambda_i \le 2$, $i = \overline{1,m}$, *if* $L = \partial G$ *consists of a union of finite number of Dini-smooth arcs* $\{L_j\}_{j=0}^{m}$, *connecting at the points* $\{z_j\}_{j=0}^{m} \in L$ *such that for every* $z_i \in L$, $i = \overline{1,m}$, *they have exterior (with respect to* $\overline{G}$*) angles* $\lambda_i \pi$, $0 < \lambda_i \le 2$, *at the corner* z_i.

In this case, we have the following:

Theorem 2.6. *Let* $p > 0$. *Suppose that* $L \in PDS(\lambda_1)$, *for some* $0 < \lambda_1 \le 2$; $h(z)$ *defined as in* (1.1). *Then, for any* P_n, $n \in \mathbb{N}$, *there exists* $c_{10} = c_{10}(L, p, \gamma_i) > 0$ *such that*

$$|P_n(z_1)| \le c_{10}(n+1)^{\frac{\gamma_1+1}{p\alpha}\lambda_1} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.23}$$

and

$$|P_n(z)| \le c_{10}\left[(n+1)^{s_1} + |z - z_1|^{\sigma_1}(n+1)^{1/p\alpha}\right] \|P_n\|_{\mathcal{L}_p(h,L)}, \ z \in L, \tag{2.24}$$

if

$$\gamma_1 + 1 < \frac{1}{\lambda_1}, \tag{2.25}$$

where

$$s_1 = \frac{1+\gamma_1}{p}\lambda_1, \qquad \sigma_1 = \frac{1}{2\lambda_1} - \frac{1+\gamma_1}{2}. \tag{2.26}$$

Corollary 2.7. *Under the conditions of Theorem 2.6*

$$|K_n(z_1)| \le c_{10}(n+1)^{\frac{\gamma_1+1}{2}\widehat{\lambda}}, \tag{2.27}$$

and

$$|K_n(z)| \le c_{10}\left[(n+1)^{s_1} + |z - z_1|^{\sigma_1}\sqrt{n+1}\right], \ z \in L, \tag{2.28}$$

if

$$\gamma_i + 1 < \frac{1}{\lambda_1}, \tag{2.29}$$

where s_1 *(for* $p = 2$*) and* σ_1 *defines as in* (2.26).

Note that, $C(1, \alpha, \lambda_1) \subset PDS(\lambda_1) \subset PC_\theta(\lambda_1)$ for each fixed $0 < \lambda_1 \le 2$ and $PC_\theta(\lambda_1) \subset \widetilde{Q}_\alpha[\lambda_1]$, for each fixed $0 < \lambda_1 < 1$. In this, (2.28) and (2.29) coincides with (1.2) and (1.3). Thus, the Corollary 2.7 generalizes the corresponding result in [38].

The sharpness of the estimations (2.1), (2.2), (2.3), (2.4) and others estimations for $P_n(z)$ for some special cases can be discussed by comparing them with the following results:

Remark 2.2. For any $n \in \mathbb{N}$, there exists a polynomials $P_n^* \in \wp_n$, weight functions h^* and the constants $c_{10} = c_{10}(L) > 0$ such that, for $L := \{z : |z| = 1\}$ we have:

$$\|P_n^*\|_{\mathcal{L}_\infty} \geq c_{10}(n+1)^{\frac{1}{p}} \|P_n^*\|_{\mathcal{L}_p(h^*,\ L)} .$$

3. Some auxiliary results

For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well-known (see, for example, [1], [21, p. 97] and [31]):

Definition 3.1. *The Jordan arc (or curve) L is called K-quasiconformal ($K \geq 1$), if there is a K-quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).*

Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and lets define

$$K_L := \inf \{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a K-quasiconformal curve, if $K_L \leq K$.

According to [31], we have the following:

Corollary 3.1. *If $S \in C_\theta$, then S is $(1+\varepsilon)$ — quasiconformal for arbitrary small $\varepsilon > 0$.*

Remark 3.1. It is well-known that, if we are not interested with the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are identical. However, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider that if the curve L is K-quasiconformal, then it is κ-quasicircle with $\kappa = \frac{K^2-1}{K^2+1}$.

According the this Remark 3.1, for simplicity, we will use both terms, depending on the situation.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set

$$d(z, M) = dist(z, M) := \inf \{|z - \zeta| : \ \zeta \in M\}.$$

For $\delta > 0$ and $z \in \mathbb{C}$ let us set: $B(z, \delta) := \{\zeta : |\zeta - z| < \delta\}$, $\Omega(z, \delta) := \Omega \cap B(z, \delta)$.

Lemma 3.2. ([1]) *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap B(z_1, d(z_1, L_{r_0}))$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent, and similarly so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon_1 = \varepsilon_1(L) < 1$, $c = c(L) > 1$, $1 < r_0 < 1$ are constants, depending on L and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Corollary 3.3. *Under the assumptions of Lemma 3.2, if $z_3 \in L_{R_0}$, $R_0 > 1$, then*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}.$$

Corollary 3.4. *If $L \in C_\theta$, then*

$$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon},$$

for all $\varepsilon > 0$.

Let $\{z_j\}_{j=1}^m$ be a fixed the system of the points on L and the weight function $h(z)$ defined as (1.1).

Recall that for

$$0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_i - z_j| : i, j = 1, 2, \ldots, m, i \neq j\},$$

we put $\Omega(z_j, \ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min\limits_{1 \leq j \leq m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{m} \Omega(z_j, \ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \ \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j, \ \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$.

Throughout this work, we will take $R = 1 + \frac{\varepsilon_0}{n+1}$, for some fixed $0 < \varepsilon_0 < 1$. Further, we introduce:

$$w_j := \Phi(z_j), \varphi_j := \arg w_j, \ L^j := L \cap \overline{\Omega}^j, \ L_R^j := L_R \cap \Omega^j, \ j = \overline{1, m}, \quad (3.1)$$

where $\Omega^j := \Psi(\Delta_j^{'})$ and

$$\Delta_1^{'} \ : \ = \left\{t = Re^{i\theta}, \ \frac{\varphi_m + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},$$

$$\Delta_m^{'} \ : \ = \left\{t = Re^{i\theta}, \ \frac{\varphi_{m-1} + \varphi_m}{2} \leq \theta < \frac{\varphi_m + \varphi_1}{2}\right\},$$

and, for $j = \overline{2, m-1}$

$$\Delta_j^{'} := \left\{ t = Re^{i\theta},\ \frac{\varphi_{j-1} + \varphi_j}{2} \ \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\};$$

$L = \bigcup\limits_{j=1}^{m} L^j$; $L_R = \bigcup\limits_{j=1}^{m} L_R^j$.

The estimation for the $\Psi^{'}$ (see, for example, [11, Th.2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi\left(\tau\right), L)}{|\tau| - 1}. \tag{3.2}$$

The following lemma is a consequence of the results given in [15, 39].

Lemma 3.5. *Let* $L \in C_\theta(\lambda_1, \ldots, \lambda_m),\ 0 < \lambda_j < 2,\ j = \overline{1, m}$. *Then*
i) for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j + \varepsilon} \preceq |\Psi(w) - \Psi(w_j)| \preceq |w - w_j|^{\lambda_j - \varepsilon}$, $|w - w_j|^{\lambda_j - 1 + \varepsilon} \preceq |\Psi^{'}(w)| \preceq |w - w_j|^{\lambda_j - 1 - \varepsilon}$,
ii) for any $w \in \overline{\Delta} \backslash \Delta_j$, $(|w| - 1)^{1+\varepsilon} \preceq d(\Psi(w), L) \preceq (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \preceq |\Psi^{'}(w)| \preceq (|w| - 1)^{-\varepsilon}$, *for arbitrary small* $\varepsilon > 0$.

The following lemma is a consequence of the results given in [29], [12, p. 32–36], and estimation (3.2) (see, for example, [11, Th. 2.8]):

Lemma 3.6. *Let a Jordan curve* $L \in PDS(\lambda_1, \ldots, \lambda_m),\ 0 < \lambda_j \leq 2$, $j = \overline{1, m}$. *Then,*

i) for any $w \in \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|^{\lambda_j}$,

$$\left|\Psi'(w)\right| \asymp |w - w_j|^{\lambda_j - 1};$$

ii) for any $w \in \overline{\Delta} \backslash \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|$, $|\Psi'(w)| \asymp 1$.

Lemma 3.7. ([6]) *Let L be a rectifiable Jordan curve, $h(z)$ defined as in* (1.1). *Then, for arbitrary* $P_n(z) \in \wp_n$, *any* $R > 1$ *and* $n \in \mathbb{N}$, *we have*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \leq R^{n + \frac{1+\gamma*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \quad \gamma* = \max\{\gamma_j,\ j = \overline{1, m}\},\ p > 0. \tag{3.3}$$

Remark 3.2. In case of $h(z) \equiv 1$, the estimation (3.3) has been proved in [17].

4. Proof of Theorems

Throughout proofs of all theorems, we will take $n \geq \left[\frac{\varepsilon_0}{R_0 - 1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant. In addition, in case when $n = 0$, the number n, participating in the all inequalities below will be changed to $(n + 1)$.

4.1. Proof of Theorem 2.1.

Proof. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1]$, for some $0 < \nu_1 < 1$, $\frac{1}{2-\nu_1} \leq \alpha \leq 1$, and $h(z)$ defined as in (1.1). For each $R > 1$, let $w = \varphi_R(z)$ denotes be a univalent conformal mapping G_R onto the B, normalized by $\varphi_R(0) = 0$, $\varphi'_R(0) > 0$, and let $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, be a zeros of $P_n(z)$ lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} \widetilde{B}_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)}, \tag{4.1}$$

denotes a Blashke function with respect to zeros $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$ ([40]). Clearly,

$$|B_{m,R}(z)| \equiv 1, \ z \in L_R, \tag{4.2}$$

and

$$|B_{m,R}(z)| < 1, \ z \in G_R. \tag{4.3}$$

For any $p > 0$ and $z \in G_R$, let us set

$$T_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2} . \tag{4.4}$$

The function $T_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_n(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i} \int\limits_{L_R} T_n(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in G_R. \tag{4.5}$$

Putting $z = z_1$, and subtracting from (4.5), we obtain:

$$T_n(z) - T_n(z_1) = \frac{1}{2\pi i} \int\limits_{L_R} T_n(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - z_1}\right] d\zeta \tag{4.6}$$

$$= \frac{1}{2\pi i} \int\limits_{L_R} \left[\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right]^{p/2} \left[\frac{z - z_1}{(\zeta - z)(\zeta - z_1)}\right] d\zeta.$$

For arbitrary $z \in L$, $z \neq z_1$, multiplying both sides of the equality by $(z - z_1)^{-\sigma_1}$, we obtain

$$\left|\frac{T_n(z) - T_n(z_1)}{(z - z_1)^{\sigma_1}}\right| \leq \frac{1}{2\pi} \int\limits_{L_R} |P_n(\zeta)|^{p/2} \left|\frac{(z - z_1)^{1-\sigma_1}}{(\zeta - z)(\zeta - z_1)}\right| |d\zeta|, \tag{4.7}$$

since $|B_{m,R}(\zeta)| = 1$, for $\zeta \in L_R$. By multiplying the numerator and the denominator of the integrand by $h^{1/2}(\zeta)$ and by applying the Hölder inequality, we obtain

$$\left|\frac{T_n(z) - T_n(z_1)}{(z-z_1)^{\sigma_1}}\right| \leq \frac{1}{2\pi}\left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2} \tag{4.8}$$

$$\times\left(\int\limits_{L_R} \frac{|z-z_1|^{2-2\sigma_1}}{|\zeta - z|^2\,|\zeta - z_1|^{2+\gamma_1}}\,|d\zeta|\right)^{1/2} =: \frac{1}{2\pi}\left(J_{n,1}\times J_{n,2}\right)^{1/2},$$

where

$$J_{n,1} \ : \ = \int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta| = \|P_n\|^p_{\mathcal{L}_p(h,L_R)}, \tag{4.9}$$

$$J_{n,2} \ : \ = \int\limits_{L_R} \frac{|z-z_1|^{2-2\sigma_1}}{|\zeta - z|^2\,|\zeta - z_1|^{2+\gamma_1}}\,|d\zeta|.$$

Then, for $z \in L$, from Lemma 3.7, we have

$$\left|\frac{T_n(z) - T_n(z_1)}{(z-z_1)^{\sigma_1}}\right| \preceq \|P_n\|^{\frac{p}{2}}_{\mathcal{L}_p}\cdot (J_{n,2})^{1/2},\ z \in L\setminus\{z_1\}.$$

From (4.4), we obtain

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2} \leq c_9\left|\frac{P_n(z_1)}{B_{m,R}(z_1)}\right|^{p/2} + c_{10}\,\|P_n\|^{\frac{p}{2}}_{\mathcal{L}_p}\cdot (J_{n,2})^{1/2}. \tag{4.10}$$

According to well-known inequalities in [40, p. 121],

$$\begin{aligned} |A+B|^p &\leq 2^{p-1}(|A|^p + |B|^p),\ p>1, \\ |A+B|^p &\leq |A|^p + |B|^p,\ 0<p\leq 1,\ A>0,\ B>0, \end{aligned} \tag{4.11}$$

from (4.10), we obtain

$$|P_n(z)| \leq c_{11}\left|\frac{B_{m,R}(z)}{B_{m,R}(z_1)}\right| |P_n(z_1)| + c_{12}\,\|P_n\|_{\mathcal{L}_p}\cdot (J_{n,2})^{1/p}. \tag{4.12}$$

Since $|B_{m,R}(z)| < 1$, for $z \in L$ and $|B_{m,R}(\zeta)| = 1$, for $\zeta \in L_R$, then there exists ε_1, where $0 < \varepsilon_1 < \frac{1}{n}$, such that fulfilled the following:

$$|B_{m,R}(z_1)| > 1 - \varepsilon_1. \tag{4.13}$$

Then, from (4.12) and (4.13), for each $z \in L\backslash \{z_1\}$, we have

$$|P_n(z)| \le c_{13} |P_n(z_1)| + c_{12} \|P_n\|_{\mathcal{L}_p} \cdot (J_{n,2})^{1/p}, \ p > 0. \tag{4.14}$$

By (2.3), we know the estimation for $|P_n(z_1)|$. Therefore, for completion, we need to find an estimate of $J_{n,2}$

$$J_{n,2} = \int\limits_{L_R} \frac{|z-z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta|. \tag{4.15}$$

Since $L \in \widetilde{Q}_\alpha[\nu_1]$, for some $0 < \nu_1 < 1$, $0 < \alpha \le 1$, then, according to [22], $\psi \in Lip\nu_1$ and there exists the number δ, $0 < \delta_1 < \delta_0 < diam\, \overline{G}$, such that

$$\Phi \in Lip\frac{1}{2-\nu_1}, \ z \in \overline{\Omega(z_1, \delta)}. \tag{4.16}$$

We set:

$$\begin{aligned} L^1_{R,1} &:= L^1_R \cap \Omega(z_1, \delta_1), \ L^1_{R,2} := L_R \backslash L^1_{R,1}; \ F^1_{R,i} := \Phi(L^1_{R,i}); \quad (4.17) \\ F^{1,1}_{R,i} &:= \left\{ \tau \in F^1_{R,i} : |\tau - w_1| \ge |\tau - w| \right\}, F^{1,2}_{R,i} := F^1_{R,i} \backslash F^{1,1}_{R,i}; \\ L^1_1 &:= L^1 \cap B(z_1, \delta_1), \ L^1_2 := \ L^1 \backslash L^1_1; \ F^1_i := \Phi(L^1_i), \ i = 1, 2. \end{aligned}$$

Then, from (4.9), we get:

$$J_{n,2} = J_{n,2}(L^1_{R,1}) + J_{n,2}(L^1_{R,2}), \tag{4.18}$$

where

$$J_{n,2}(l) := \int\limits_{l} \frac{|z-z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta| \ , \tag{4.19}$$

for $l \subset L$. There are two possible cases: the point z may lie on L^1 or L^2. Suppose first, $z \in L^1$. If $z \in L^1_i$, then $w \in F^1_i$, for $i = 1, 2$. Consider the individual cases:

1) Let $z \in L^1_1$.

1.1) By applying (4.11), we have

$$\begin{aligned} J_{n,2}(L^1_{R,1}) &= \int\limits_{L^1_{R,1}} \frac{|z-z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta| \quad (4.20) \\ &\preceq \int\limits_{L^1_{R,1}} \frac{[|\zeta - z| + |\zeta - z_1|]^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta| \\ &= \int\limits_{L^1_{R,1}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1} |\zeta - z_1|^{2+\gamma_1}} + \int\limits_{L^1_{R,1}} \frac{|d\zeta|}{|\zeta - z|^2 |\zeta - z_1|^{2\sigma_1+\gamma_1}}. \end{aligned}$$

Lets

$$\begin{aligned} L^{1,1}_{R,j} &: = \left\{\zeta \in L^1_{R,j} : |\zeta - z_1| \geq |\zeta - z|\right\}, \\ L^{1,2}_{R,j} &: = L^1_{R,j} \backslash L^{1,1}_{R,j},\ F^{1,i}_{R,j} := \Phi(l^{1,i}_{R,j}), i,\ j = 1, 2. \end{aligned}$$

Then, from (4.20), we get

$$J_{n,2}(L^1_{R,1}) \preceq \int\limits_{L^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{L^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \tag{4.21}$$

$$\begin{aligned} &\preceq \int\limits_{F^{1,1}_{R,2}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^{2\sigma_1+2+\gamma_1}\,(|\tau| - 1)} \\ &+ \int\limits_{F^{1,2}_{R,2}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^{2\sigma_1+2+\gamma_1}\,(|\tau| - 1)} \\ &\preceq n \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau - w|^{(2\sigma_1+1+\gamma_1)(2-\nu_1)}} + n \int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\tau - w|^{(2\sigma_1+1+\gamma_1)(2-\nu_1)}} \\ &\preceq n^{(2\sigma_1+1+\gamma_1)(2-\nu_1)}. \end{aligned}$$

1.2) For any $\zeta \in L^1_{R,2}$ and $z \in L^1_1$, $|\zeta - z_1| \geq \delta_1$ and by (4.11), (4.16), we obtain:

$$J_{n,2}(L^1_{R,2}) \preceq \int\limits_{L^1_{R,2}} \frac{[|\zeta - z| + |\zeta - z_1|]^{2-2\sigma_1}}{|\zeta - z|^2\,|\zeta - z_1|^{2+\gamma_1}}\,|d\zeta| \tag{4.22}$$

$$\begin{aligned} &= \int\limits_{L^1_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1}\,|\zeta - z_1|^{2+\gamma_1}} + \int\limits_{L^1_{R,2}} \frac{|d\zeta|}{|\zeta - z|^2\,|\zeta - z_1|^{2\sigma_1+\gamma_1}} \\ &\leq \frac{1}{\delta_1^{2+\gamma_1}} \int\limits_{L^1_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1}} + \frac{1}{\delta_1^{2\sigma_1+\gamma_1}} \int\limits_{L^1_{R,2}} \frac{|d\zeta|}{|\zeta - z|^2} \\ &\preceq \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^{2\sigma_1}\,(|\tau| - 1)} + \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^2\,(|\tau| - 1)}. \end{aligned}$$

If $z \in L^1_1 \cap \overline{B(z_1, \frac{\delta}{2})}$, then according to $|\zeta - z| \geq |\zeta - z_1| - |z - z_1| \geq$

$\delta_1 - \frac{\delta_1}{2} = \frac{\delta_1}{2}$, from (4.22), we obtain:

$$\begin{aligned} J_{n,2}(L^1_{R,2}) &\preceq n \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{2\sigma_1}} + n \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{2}} \qquad (4.23) \\ &\leq n \cdot \left(\frac{2}{\delta_1}\right)^{2\sigma_1-1} \int\limits_{F^1_{R,2}} |d\tau| + n \cdot \left(\frac{2}{\delta_1}\right) \int\limits_{F^1_{R,2}} |d\tau| \preceq n \cdot \left|F^1_{R,2}\right| \preceq n. \end{aligned}$$

If $z \in L^1_1 \setminus \overline{B(z_1, \frac{\delta_1}{2})}$, then $|\zeta - z| \geq |\tau - w|^{\frac{1}{\alpha}}$, since $\Phi \in Lip\alpha$, $z \in \overline{\Omega} \setminus \{z_1\}$, and from (4.22), we get:

$$\begin{aligned} J_{n,2}(L^1_{R,2}) &\preceq n \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{2\sigma_1}} + n \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{2}} \qquad (4.24) \\ &\preceq n \int\limits_{F^1_{R,1}} \frac{|d\tau|}{|\tau-w|^{\frac{2\sigma_1-1}{\alpha}}} + n \int\limits_{F^1_{R,1}} \frac{|d\tau|}{|\tau-w|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}. \end{aligned}$$

2) Let $z \in L^1_2$.

2.1) According to $|\zeta - z_1| < |z - z_1|$, by (4.16), we obtain:

$$J_{n,2}(L^1_{R,1}) = \int\limits_{L^1_{R,1}} \frac{|z-z_1|^{2-2\sigma_1}}{|\zeta-z|^2\,|\zeta-z_1|^{2+\gamma_1}}\,|d\zeta|$$

$$\begin{aligned} &\preceq \int\limits_{L^1_{R,1}} \frac{[|\zeta-z|+|\zeta-z_1|]^{2-2\sigma_1}}{|\zeta-z|^2\,|\zeta-z_1|^{2+\gamma_1}}\,|d\zeta| \\ &= \int\limits_{L^1_{R,1}} \frac{|d\zeta|}{|\zeta-z|^{2\sigma_1}\,|\zeta-z_1|^{2+\gamma_1}} + \int\limits_{L^1_{R,1}} \frac{|d\zeta|}{|\zeta-z|^{2}\,|\zeta-z_1|^{2\sigma_1+\gamma_1}} \\ &\preceq \int\limits_{L^{1,1}_{R,1}} \frac{|d\zeta|}{|\zeta-z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{L^{1,2}_{R,1}} \frac{|d\zeta|}{|\zeta-z_1|^{2\sigma_1+2+\gamma_1}} \\ &\preceq \int\limits_{F^{1,1}_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{2\sigma_1+2+\gamma_1}\,(|\tau|-1)} \\ &+ \int\limits_{F^{1,2}_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{2\sigma_1+2+\gamma_1}\,(|\tau|-1)} \end{aligned}$$

$$\preceq n \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau - w|^{\frac{(2\sigma_1+1+\gamma_1)}{\alpha}}} + n \int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\tau - w_1|^{(2\sigma_1+1+\gamma_1)(2-\nu_1)}}$$
$$\preceq n^{(2\sigma_1+1+\gamma_1)(2-\nu_1)}.$$

Therefore, in this case we have

$$J_{n,2}(L^1_{R,1}) \preceq n^{(2\sigma_1+1+\gamma_1)(2-\nu_1)}. \tag{4.25}$$

2.2) For any $\zeta \in L^1_{R,2}$ and $z \in L^1_2$, $|\zeta - z_1| \geq \delta_1$ and analogously to the case 1.2), in this case, we obtain:

$$J_{n,2}(L^1_{R,2}) = \int\limits_{L^1_{R,2}} \frac{|z - z_1|^{2-2\sigma_1}}{|\zeta - z|^2 \, |\zeta - z_1|^{2+\gamma_1}} \, |d\zeta|$$

$$\preceq \frac{(diamL)^{2-2\sigma_1}}{(\delta_1)^{2+\gamma_1}} \int\limits_{L^1_{R,2}} \frac{|d\zeta|}{|\zeta - z|^2} \tag{4.26}$$
$$\preceq \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau), L) \, |d\tau|}{|\Psi(\tau) - \Psi(w)|^2 \, (|\tau| - 1)} \preceq n \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|}$$
$$\preceq n \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\tau - w|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}.$$

By combining the estimations (4.12), (4.14)–(4.26), finally we obtain

$$|P_n(z)| \leq c_{13} \, |P_n(z_1)| + c_{12} \, \|P_n\|_{\mathcal{L}_p} \cdot n^{\frac{1}{\alpha p}}, \; p > 0.$$

and, then the proof of (2.9) is completed. □

4.2. Proof of Theorem 2.4

Proof. Analogously to beginning of proof of the Theorem 2.1, in this case, from (4.1)–(4.14), we obtain:

$$|P_n(z_1)| \preceq \|P_n\|_{\mathcal{L}_p} \cdot I_{n,2}, \quad z \in L. \tag{4.27}$$

where

$$(I_{n,2})^p = \int\limits_{L_R} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}}. \tag{4.28}$$

By using the notations (3.1) and setting $\delta := c_1 d_{1,R}$ for some $c_1 > 1$, where $d_{1,R} := d(z_1, L_R^1)$, $\left| L_{R,i}^1 \right| := mes L_{R,i}^1$, $i = 1, 2$, we have:

$$I_{n,2}^1 := \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int\limits_{L_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} + \int\limits_{L_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}},$$

where

$$\int\limits_{L_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} \preceq \int\limits_{d_{1,R}}^{c_1 d_{1,R}} \frac{ds}{s^{2+\gamma_1}} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}};$$

$$\int\limits_{L_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} \preceq \int\limits_{c_1 d_{1,R}}^{|L_{R,2}^1|} \frac{ds}{s^{2+\gamma_1}} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}.$$

According these estimations, from (4.27) and (4.28), we get:

$$|P_n(z_1)| \preceq \frac{1}{d_{1,R}^{1+\gamma_1}} \|P_n\|_{\mathcal{L}_p}. \tag{4.29}$$

On the other hand, by Lemma 3.5, for $0 < \lambda_1 < 2$, and [11], for arbitrary continuum with simple connected complement, we have:

$$d_{1,R} \succeq \frac{1}{n^{\widetilde{\lambda}_1}}, \tag{4.30}$$

where $\widetilde{\lambda}_1 := \begin{cases} \lambda_1 + \varepsilon, & \text{if } 0 < \lambda_1 < 2, \\ 2, & \text{if } \lambda_1 = 2, \end{cases}$ for arbitrary small $\varepsilon > 0$.

Now, we will begin to proof of (2.18). Analogously to beginning of proof of the Theorem 2.1, in this case, from (4.1)–(4.14), we get:

$$|P_n(z)| \leq c_{13} |P_n(z_1)| + c_{12} \|P_n\|_{\mathcal{L}_p} \cdot (J_{n,2})^{1/p}, \; p > 0, \quad z \in L, \tag{4.31}$$

where

$$J_{n,2} = \int\limits_{L_R} \frac{|z - z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta|. \tag{4.32}$$

Therefore, for the proof of (2.18), sufficiently to evaluate the integral (4.32).

Lets denote by:

$$\begin{aligned}
l^1_{R,1} \quad &: \quad = L^1_R \cap \Omega(z_1, c_1 d_{1,R}),\ c_1 > 1, \\
l^1_{R,2} \quad &: \quad = L^1_R \cap (\Omega(z_1, \delta_1) \setminus \Omega(z_1, c_1 d_{1,R})), \\
l^1_{R,3} \quad &: \quad = L^1_R \backslash (l^1_{R,1} \cup l^1_{R,2});\ F^1_{R,j} := \Phi(l^1_{R,j}), \\
l^1_1 \quad &: \quad = L^1 \cap B(z_1, c_1 d_{1,R}),\ l^1_2 := L^1 \cap (B(z_1, \delta_1) \backslash B(z_1, c_1 d_{1,R})), \\
l^1_3 \quad &: \quad = \quad L^1 \backslash \left(l^1_1 \cup l^1_2\right);\ F^1_j := \Phi(l^1_j),\ j = 1, 2, 3.
\end{aligned}$$

Then

$$J_{n,2} = \sum_{j=1}^{3} \int\limits_{l_{R,j}} \frac{|z - z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta| = \sum_{j=1}^{3} J_{n,2}(l^1_{R,j}),$$

where

$$J_{n,2}(l^1_{R,j}) := \int\limits_{l^1_{R,j}} \frac{|z - z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta|, \quad j = 1, 2, 3. \tag{4.33}$$

Lets

$$\begin{aligned}
l^{1,1}_{R,j} \quad &: \quad = \left\{\zeta \in l^1_{R,j} : |\zeta - z_1| \geq |\zeta - z|\right\}, \\
l^{1,2}_{R,j} \quad &: \quad = l^1_{R,j} \backslash l^{1,1}_{R,j},\ F^{1,i}_{R,j} := \Phi(l^{1,i}_{R,j}),\ i = 1, 2,\ j = 1, 2, 3.
\end{aligned}$$

Then, according to (4.11), for each $j = 1, 2, 3$, we get:

$$J_{n,2}(l^1_{R,j}) = \int\limits_{l^1_{R,j}} \frac{|z - z_1|^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta| \tag{4.34}$$

$$\begin{aligned}
&\leq \int\limits_{l^1_{R,j}} \frac{[|\zeta - z| + |\zeta - z_1|]^{2-2\sigma_1}}{|\zeta - z|^2 |\zeta - z_1|^{2+\gamma_1}} |d\zeta| \\
&\preceq \int\limits_{l^1_{R,j}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1} |\zeta - z_1|^{2+\gamma_1}} + \int\limits_{l^1_{R,j}} \frac{|d\zeta|}{|\zeta - z|^2 |\zeta - z_1|^{2\sigma_1+\gamma_1}} \\
&\preceq \int\limits_{l^{1,1}_{R,j}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,j}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}}.
\end{aligned}$$

So, we need to evaluate the integrals $J_{n,2}(l^{1,i}_{R,j})$ for any $z \in L$ and for each $i = 1, 2, \quad j = 1, 2, 3$. There are two possible cases: the point z

may lie on L^1 or L^2. Suppose first, $z \in L^1$. If $z \in l_j^1$, then $w \in F_j^1$, for $j = 1, 2, 3$. Consider the individual cases:

1) Suppose first that $z \in l_1^1$. According Lemma 3.5, from (4.34) we get:

$$J_{n,2}(l^1_{R,1} \cup l^1_{R,2}) \preceq \int\limits_{l^{1,1}_{R,1} \cup l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,1} \cup l^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \tag{4.35}$$

$$= \int\limits_{l^{1,1}_{R,1} \cup l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^{1,2}_{R,1} \cup l^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{d(z,\ l^1_{R,1})}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d_{1,R}}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \preceq \frac{1}{d^{\frac{1}{\lambda_1}}(z,\ l^1_{R,1})} + \frac{1}{d^{\frac{1}{\lambda_1}}_{1,R}} \preceq n^{1+\varepsilon};$$

and

$$J_{n,2}(l^1_{R,3}) \preceq \int\limits_{l^{1,1}_{R,3}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,3}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \tag{4.36}$$

$$\preceq \frac{1}{(\delta_1 - c_1 d_{1,R})^{2\sigma_1+2+\gamma_1}} \int\limits_{l^{1,1}_{R,3}} |d\zeta| + \frac{1}{\delta_1^{2\sigma_1+2+\gamma_1}} \int\limits_{l^{1,2}_{R,3}} |d\zeta| \preceq \left| l^1_{R,3} \right| \preceq 1.$$

2) Suppose first that $z \in l_2^1$. Analogously, we get:

$$\begin{aligned} J_{n,2}(l^1_{R,1} \cup l^1_{R,2}) \quad &\preceq \int\limits_{l^{1,1}_{R,1} \cup l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,1} \cup l^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \\ &= \int\limits_{l^{1,1}_{R,1} \cup l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^{1,2}_{R,1} \cup l^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \qquad (4.37) \\ &\preceq \int\limits_{d(z,\ l^1_{R,1})}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d_{1,R}}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \\ &\preceq \frac{1}{d^{\frac{1}{\lambda_1}}(z,\ l^1_{R,1})} + \frac{1}{d^{\frac{1}{\lambda_1}}_{1,R}} \preceq n^{1+\varepsilon}, \end{aligned}$$

and

$$
\begin{aligned}
J_{n,2}(l^1_{R,3}) &\preceq \int\limits_{l^{1,1}_{R,3}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,3}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \\
&\preceq \int\limits_{d(z,\ l^1_R)}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \frac{1}{\delta_1^{2\sigma_1+2+\gamma_1}} \int\limits_{l^{1,2}_{R,3}} |d\zeta| \qquad (4.38) \\
&\preceq \frac{1}{d^{\frac{1}{\lambda_1}}(z,\ l^1_{R,1})} + \left|l^1_{R,3}\right| \preceq n^{1+\varepsilon}.
\end{aligned}
$$

3) Suppose first that $z \in l^1_3$. In this case, according Lemma 3.5, we obtain:

$$
\begin{aligned}
J_{n,2}(l^1_{R,1}) &\preceq \int\limits_{l^{1,1}_{R,1}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \\
&= \int\limits_{l^{1,1}_{R,1}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^{1,2}_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \qquad (4.39) \\
&\preceq \frac{1}{(\delta_1 - c_1 d_{1,R})^{2\sigma_1+2+\gamma_1}} \int\limits_{l^{1,1}_{R,1}} |d\zeta| + \int\limits_{d_{1,R}}^{c_1 d_{1,R}} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \\
&\preceq \frac{1}{d_{1,R}^{\frac{1}{\lambda_1}}} \preceq n^{1+\varepsilon},
\end{aligned}
$$

$$
J_{n,2}(l^1_{R,2}) \preceq \int\limits_{l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{2\sigma_1+2+\gamma_1}} + \int\limits_{l^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{2\sigma_1+2+\gamma_1}} \qquad (4.40)
$$

$$
\begin{aligned}
&= \int\limits_{l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^{1,2}_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \preceq \int\limits_{l^{1,1}_{R,2}} \frac{|d\zeta|}{|\zeta - \widetilde{z}|^{\frac{1}{\lambda_1}+1}} + \int\limits_{d_{1,R}}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \\
&\preceq \int\limits_{d(\widetilde{z},\ l^1_{R,2})}^{c_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d_{1,R}}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \preceq \frac{1}{d^{\frac{1}{\lambda_1}}(\widetilde{z},\ l^1_{R,2})} + \frac{1}{d_{1,R}^{\frac{1}{\lambda_1}}} \preceq n^{1+\varepsilon},
\end{aligned}
$$

where $\widetilde{z} \in l_2^1$ fixed point such that $|\zeta - z| \succeq |\zeta - \widetilde{z}|$ for $\zeta \in l_{R,2}^{1,1}$.

$$J_{n,2}(l_{R,3}^1) \preceq \int\limits_{l_{R,3}^1} \frac{|z - z_1|^{2-2\sigma_1}}{|\zeta - z|^2 \, |\zeta - z_1|^{2+\gamma_1}} \, |d\zeta| \tag{4.41}$$

$$\leq \frac{(diamL)^{2-2\sigma_1}}{\delta_1^{2+\gamma_1}} \int\limits_{l_{R,3}^1} \frac{|d\zeta|}{|\zeta - z|^2} \preceq \int\limits_{d(z,\ l_{R,3}^1)}^{c_1} \frac{ds}{s^2} \preceq \frac{1}{d(z,\ l_{R,3}^1)} \preceq n^{1+\varepsilon}.$$

Combining the estimations (4.32)–(4.41), we get:

$$J_{n,2} \preceq n^{1+\varepsilon}, \ \forall \varepsilon > 0. \tag{4.42}$$

From estimations (4.29), (4.30), (4.31) and (4.42) we complete the proof of (2.18). $\square$

4.3. Proof of Theorem 2.6.

The proof of the this Theorem it is follows from of the proof of Theorem 2.4, by using Lemma 3.6, instead Lemma 3.5.

4.4. Proof of Remark 2.2.

Proof. a) Let's $L := \{z : |z| = 1\}$, $h^*(z) \equiv 1$ and $P_n^*(z) = \sum\limits_{j=0}^{n} (j+1) z^j$.

Then, $L \in \widetilde{Q}_1$;

$$|P_n^*(z)| \leq \sum_{j=0}^{n} \left| (j+1) z^j \right| = \frac{(n+1)(n+2)}{2}, |z| = 1.$$

On the other hand,

$$|P_n^*(1)| = \frac{(n+1)(n+2)}{2}.$$

Therefore,

$$\|P_n^*\|_{\mathcal{L}_\infty} = \frac{(n+1)(n+2)}{2}; \ \|P_n^*\|_{\mathcal{L}_2(1,L)} = \sqrt{\frac{(n+1)(n+2)(2n+3)}{3}\pi}.$$

Then,

$$\|P_n^*\|_{\mathcal{L}_\infty} = \sqrt{\frac{3(n+1)(n+2)}{4\pi(2n+3)}} \, \|P_n^*\|_{\mathcal{L}_2(1,L)} \geq \sqrt{\frac{3}{8\pi}} \cdot \sqrt{n} \, \|P_n^*\|_{\mathcal{L}_2(1,L)} \, .$$

$\square$

4.5. Proof of Corollary 2.3

Proof. If $L \in C(1, \alpha, \omega_1)$, $1 \leq \omega_1 < 2$, then the curve $L = \partial G$ has a interior (with respect to $\overline{G}$) $(2-\omega_1)-$ angle at the z_1. Therefore $L \in \widetilde{Q}_{\alpha}\left[\frac{1}{2-\omega_1}\right]$ for $\alpha = 1$ by Remark (2.1). Then, according to [22], $\psi \in Lip(2-\omega_1)$, and so, by [22], $\Phi \in Lip\frac{1}{\omega_1}$. Therefore, In this case, for $p = 2$ from (2.9) and (2.12), we get the proof. □

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