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Spectral and pseudospectral functions of various dimensions for symmetric systems

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(*Presented by M.M. Malamud*)

Abstract. The main object of the paper is a symmetric system $Jy' - B(t)y = \lambda\Delta(t)y$ defined on an interval $\mathcal{I} = [a, b)$ with the regular endpoint a. Let $\varphi(\cdot, \lambda)$ be a matrix solution $\varphi(\cdot, \lambda)$ of this system of an arbitrary dimension and let $(Vf)(s) = \int_{\mathcal{I}} \varphi^*(t, s)\Delta(t)f(t)\, dt$ be the Fourier transform of the function $f(\cdot) \in L^2_\Delta(\mathcal{I})$. We define a pseudospectral function of the system as a matrix-valued distribution function $\sigma(\cdot)$ of the dimension n_σ such that V is a partial isometry from $L^2_\Delta(\mathcal{I})$ to $L^2(\sigma; \mathbb{C}^{n_\sigma})$ with the minimally possible kernel. Moreover, we find the minimally possible value of n_σ and parameterize all spectral and pseudospectral functions of every possible dimensions n_σ by means of a Nevanlinna boundary parameter. The obtained results develop the results by Arov and Dym; A. Sakhnovich, L. Sakhnovich and Roitberg; Langer and Textorius.

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1. Introduction

Let H and $\widehat{H}$ be finite dimensional Hilbert spaces, let $\mathbb{H} = H \oplus \widehat{H} \oplus H$ and let $[\mathbb{H}]$ be the set of linear operators in $[\mathbb{H}]$. Recall that a non-decreasing left continuous operator (matrix) function $\sigma(\cdot) : \mathbb{R} \to [\mathbb{H}]$ with $\sigma(0) = 0$ is called a distribution function of the dimension $n_\sigma := \dim \mathbb{H}$.

We consider symmetric differential system [3, 11]

$$Jy' - B(t)y = \lambda\Delta(t)y, \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

where $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ are $[\mathbb{H}]$-valued functions defined on an interval $\mathcal{I} = [a, b)$, $b \leq \infty$, and integrable on each compact subinterval $[a, \beta] \subset \mathcal{I}$ and

$$J = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & iI_{\widehat{H}} & 0 \\ I_H & 0 & 0 \end{pmatrix} : H \oplus \widehat{H} \oplus H \to H \oplus \widehat{H} \oplus H. \qquad (1.2)$$

System (1.1) is called a Hamiltonian system if $\widehat{H} = \{0\}$ and hence $\mathbb{H} = H \oplus H$,

$$J = \begin{pmatrix} 0 & -I_H \\ I_H & 0 \end{pmatrix} : H \oplus H \to H \oplus H.$$

The system is called regular if $b < \infty$ and $\int_{\mathcal{I}} ||B(t)|| dt < \infty$, $\int_{\mathcal{I}} ||\Delta(t)|| dt < \infty$.

As is known a spectral function is a basic concept in the theory of eigenfunction expansions of differential operators (see e.g. [10, 29] and references therein). In the case of a symmetric system definition of the spectral function requires a certain modification. Namely, let $\mathfrak{H} = L^2_\Delta(\mathcal{I})$ be the Hilbert space of functions $f : \mathcal{I} \to \mathbb{H}$ satisfying

$$\int_{\mathcal{I}} (\Delta(t)f(t), f(t)) \, dt < \infty.$$

Assume that system (1.1) is Hamiltonian and $\varphi(\cdot, \lambda)$ is an $[H, H \oplus H]$-valued operator solution of (1.1) such that $\varphi(0, \lambda) = (0, I_H)^\top$. An $[H]$-valued distribution function $\sigma(\cdot)$ is called a spectral function of the system if the (generalized) Fourier transform $V_\sigma : \mathfrak{H} \to L^2(\sigma; H)$ defined by

$$(V_\sigma f)(s) = \widehat{f}_0(s) := \int_{\mathcal{I}} \varphi^*(t, s)\Delta(t)f(t) \, dt, \quad f(\cdot) \in \mathfrak{H} \qquad (1.3)$$

is an isometry. If $\sigma(\cdot)$ is a spectral function, then the inverse Fourier transform is defined for each $f \in \mathfrak{H}$ by

$$f(t) = \int_{\mathcal{I}} \varphi(t, s) \, d\sigma(s) \widehat{f}_0(s) \qquad (1.4)$$

(the integrals in (1.3) and (1.4) converge in the norm of $L^2(\sigma; H)$ and $\mathfrak{H}$ respectively). If the operator $\Delta(t)$ is invertible a.e. on $\mathcal{I}$, then spectral functions exist. Otherwise the Fourier transform may have a nontrivial kernel $\ker V_\sigma$ and hence the set of spectral functions may be empty. The natural generalization of a spectral function to this case is an $[H]$-valued distribution function $\sigma(\cdot)$ such that the Fourier transform V_σ of the form (1.3) is a partial isometry. If $\sigma(\cdot)$ is such a function, then the inverse Fourier transform (1.4) is valid for each $f \in \mathfrak{H} \ominus \ker V_\sigma$. Therefore an

interesting problem is a characterization of $[H]$-valued distribution functions $\sigma(\cdot)$ such that the Fourier transform V_σ is a partial isometry with the minimally possible kernel $\ker V_\sigma$. This problem was solved in [2,31,32] for regular systems and in [27] for general systems. The results of [27] was obtained in the framework of the extension theory of symmetric linear relations. As is known [14, 18, 22, 30] system (1.1) generates the minimal (symmetric) linear relation $T_{\min}$ and the maximal relation $T_{\max}(= T_{\min}^*)$ in $\mathfrak{H}$. Let $T \supset T_{\min}$ be a symmetric relation in $\mathfrak{H}$ given by

$$T = \{\{y, f\} \in T_{\max} : (I_H, 0)y(a) = 0$$
$$\text{and } \lim_{t \to b}(Jy(t), z(t)) = 0, \quad z \in \operatorname{dom} T_{\max}\}$$

and let $\operatorname{mul} T$ be the multivalued part of T. It was shown in [27] that for each $[H]$-valued distribution function $\sigma(\cdot)$ such that V_σ is a partial isometry the inclusion $\operatorname{mul} T \subset \ker V_\sigma$ is valid. This fact makes natural the following definition.

Definition 1.1. [27] An $[H]$-valued distribution function $\sigma(\cdot)$ is called a pseudospectral function of the system (1.1) (with respect to $\varphi(\cdot, \lambda)$) if the Fourier transform V_σ is a partial isometry with the minimally possible kernel $\ker V_\sigma = \operatorname{mul} T$.

If the Hamiltonian system is regular, then $\ker V_\sigma = \{f \in \mathfrak{H} : \widehat{f}_0(s) = 0, \ s \in \mathbb{R}\}$ and therefore Definition 1.1 turns into the definition of the pseudospectral function from the monographes [2, 32]. In these monographes all $[H]$-valued pseudospectral functions of the regular system are parameterized in the form of a linear fractional transform of a Nevanlinna parameter. Similar result for singular systems was obtained in our paper [27]. Observe also that an existence of $[H]$-valued pseudospectral functions of the singular Hamiltonian system in the case $\dim H = 1$ was proved in [14].

Assume now that system (1.1) is not necessarily Hamiltonian. Let $Y(\cdot, \lambda)$ be the $[\mathbb{H}]$-valued operator solution of (1.1) with $Y(a, \lambda) = I_{\mathbb{H}}$ and let $\Sigma(\cdot)$ be an $[\mathbb{H}]$-valued distribution function such that the Fourier transform $V_\Sigma : \mathfrak{H} \to L^2(\Sigma; \mathbb{H})$ defined by

$$(V_\Sigma f)(s) = \widehat{f}(s) := \int_{\mathcal{I}} Y^*(t, s)\Delta(t)f(t)\, dt, \quad f(\cdot) \in \mathfrak{H} \tag{1.5}$$

is a partial isometry. Moreover, let $\operatorname{mul} T_{\min}$ be the multivalued part of $T_{\min}$. Then according to [26] $\operatorname{mul} T_{\min} \subset \ker V_\Sigma$ and the same arguments as for transform (1.3) make natural the following definition.

Definition 1.2. [26] An $[\mathbb{H}]$-valued distribution function $\Sigma(\cdot)$ is called a pseudospectral function of the system (1.1) (with respect to $Y(\cdot, \lambda)$) if the Fourier transform V_Σ is a partial isometry with the minimally possible kernel $\ker V_\Sigma = \operatorname{mul} T_{\min}$.

Existence of pseudospectral functions $\Sigma(\cdot)$ follows from the results of [8,9,16,17]. In [16,17] a parametrization of all pseudospectral functions $\Sigma(\cdot)$ of the regular system (1.1) is given. This parametrization is closed to that of the $[H]$-valued pseudospectral functions $\sigma(\cdot)$ in [2, 32]. Similar result for singular systems is obtained in [26].

In the present paper we continue our investigations of pseudospectral and spectral functions of symmetric systems contained in [1, 26, 27].

According to Definitions 1.1 and 1.2 the dimensions of pseudospectral functions $\Sigma(\cdot)$ and $\sigma(\cdot)$ are $n_\Sigma = \dim \mathbb{H}$ and $n_\sigma = \dim H(< n_\Sigma)$. In this connection the following problems seems to be interesting:

- To define naturally a spectral and pseudospectral function of an arbitrary dimension for the system (1.1) and describe all such functions by analogy with [26, 27].

- To characterize spectral functions of the minimally possible dimension

  The paper is devoted to the solution of these problems.

Let $\mathbb{H}_0$ and θ be subspaces in $\mathbb{H}$, $K_\theta \in [\mathbb{H}_0, \mathbb{H}]$ be an operator isomorphically mapping $\mathbb{H}_0$ onto θ and $\varphi(\cdot, \lambda)$ be the $[\mathbb{H}_0, \mathbb{H}]$-valued operator solution of (1.1) with $\varphi(a, \lambda) = K_\theta$. Moreover, let $\sigma(\cdot)$ be an $[\mathbb{H}_0]$-valued distribution function such that the Fourier transform $V_\sigma : \mathbb{H} \to L^2(\sigma; \mathbb{H}_0)$ defined by (1.3) is a partial isometry. It turns out that $\operatorname{mul} T \subset \ker V_\sigma$, where $\operatorname{mul} T$ is the multivalued part of a symmetric relation $T \supset T_{\min}$ in $\mathfrak{H}$ given by

$$T = \{\{y, f\} \in T_{\max} : y(a) \in \theta \text{ and } \lim_{t \to b}(Jy(t), z(t)) = 0, z \in \operatorname{dom} T_{\max}\}.$$

This statement makes natural the following most general definition of pseudospectral and spectral functions.

Definition 1.3. An $[\mathbb{H}_0]$-valued distribution function $\sigma(\cdot)$ is called a pseudospectral function of the system (1.1) (with respect to the operator K_θ) if the Fourier transform V_σ is a partial isometry with the minimally possible kernel $\ker V_\sigma = \operatorname{mul} T$.

A pseudospectral function $\sigma(\cdot)$ with $\ker V_\sigma = \{0\}$ is called a spectral function.

It turns out that actually a pseudospectral function with respect to the operator K_θ is uniquely characterized by the subspace $\theta \subset \mathbb{H}$.

We parametrize all pseudospectral (spectral) functions for a given θ and find a lower bound of the dimension of all spectral functions $\sigma(\cdot)$ corresponding to various θ. More precisely the following three theorems are the main results of the paper.

Theorem 1.4. *Assume that system* (1.1) *is definite (see Definition 3.15) and deficiency indices* $n_\pm(T_{\min})$ *of* $T_{\min}$ *satisfy* $n_-(T_{\min}) \leq n_+(T_{\min})$. *Moreover, let* θ *be a subspace in* $\mathbb{H}$ *and let* $\theta^\times := \mathbb{H} \ominus J\theta$. *Then a pseudospectral function* $\sigma(\cdot)$ *(with respect to* K_θ*) exists if and only if* $\theta^\times \subset \theta$.

Theorem 1.5. *Assume that* θ *is a subspace in* $\mathbb{H}$ *such that* $\theta^\times \subset \theta$ *and there exists only a trivial solution* $y = 0$ *of the system* (1.1) *such that* $\Delta(t)y(t) = 0$ *(a.e. on* $\mathcal{I}$*) and* $y(a) \in \theta$ *(the last condition is fulfilled for definite systems). Moreover, let for simplicity* $n_+(T_{\min}) = n_-(T_{\min})$. *Then:*

(1) *There exist auxiliary finite-dimensional Hilbert spaces* $\mathbb{H}_0 \subset \mathbb{H}$ *and* $\dot{\mathcal{H}}$, *an operator* $U = U_\theta \in [\mathbb{H}_0, \mathbb{H}]$ *isomorphically mapping* $\mathbb{H}_0$ *onto* θ, *Nevanlinna operator functions* $m_0(\lambda)(\in [\mathbb{H}_0])$, $\dot{M}(\lambda)(\in [\dot{\mathcal{H}}])$ *and an operator function* $S(\lambda)(\in [\dot{\mathcal{H}}, \mathbb{H}_0])$ *such that the equalities*

$$m_\tau(\lambda) = m_0(\lambda) + S(\lambda)(C_0(\lambda) - C_1(\lambda)\dot{M}(\lambda))^{-1}C_1(\lambda)S^*(\overline{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{1.6}$$

$$\sigma_\tau(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} m_\tau(u + i\varepsilon) \, du \tag{1.7}$$

establish a bijective correspondence $\sigma(s) = \sigma_\tau(s)$ *between all Nevanlinna operator pairs* $\tau = \{C_0(\lambda), C_1(\lambda)\}$, $C_j(\lambda) \in [\dot{\mathcal{H}}]$, $j \in \{0, 1\}$, *satisfying the admissibility conditions*

$$\lim_{y \to \infty} \tfrac{1}{iy}(C_0(iy) - C_1(iy)\dot{M}(iy))^{-1}C_1(iy) = 0 \tag{1.8}$$

$$\lim_{y \to \infty} \tfrac{1}{iy}\dot{M}(iy)(C_0(iy) - C_1(iy)\dot{M}(iy))^{-1}C_0(iy) = 0 \tag{1.9}$$

and all pseudospectral functions $\sigma(\cdot)$ *of the system (with respect to* U*). Moreover, each pair* τ *is admissible (and hence the conditions* (1.8) *and* (1.9) *may be omitted) if and only if* $\lim_{y \to \infty} \tfrac{1}{iy}\dot{M}(iy) = 0$ *and* $\lim_{y \to \infty} y \cdot \operatorname{Im}(\dot{M}(iy)h, h) = +\infty$, $0 \neq h \in \dot{\mathcal{H}}$.

(2) *The set of spectral functions (with respect to* U*) is not empty if and only if* $\operatorname{mul} T = \{0\}$. *If this condition id fulfilled, then the sets of spectral and spectral function (with respect to* U*) coincide and hence statement* (1) *holds for spectral functions.*

Theorem 1.6. *Let system* (1.1) *be definite and let* $n_-(T_{\min}) \leq n_+(T_{\min})$. *Then the set of spectral functions of the system is not empty if and only if* $\operatorname{mul} T_{\min} = \{0\}$. *If this condition is fulfilled, then the dimension* n_σ *of each spectral function* $\sigma(\cdot)$ *satisfies* $\dim(H \oplus \widehat{H}) \leq n_\sigma \leq \dim \mathbb{H}$ *and there exists a subspace* $\theta \subset \mathbb{H}$ *and a spectral function* $\sigma(\cdot)$ *(with respect to* K_θ) *such that the dimension* n_σ *of* $\sigma(\cdot)$ *has the minimally possible value* $n_\sigma = \dim(H \oplus \widehat{H})$.

Note that the coefficients $m_0(\lambda)$, $S(\lambda)$ and $\dot{M}(\lambda)$ in (1.6) are defined in terms of the boundary values of respective operator solutions of (1.1) at the endpoints a and b. Observe also that $m_\tau(\lambda)$ in (1.6) is an $[\mathbb{H}_0]$-valued Nevanlinna function (the m-function of the system) and (1.7) is the Stieltjes formula for $m_\tau(\cdot)$. If the system is Hamiltonian, θ is a self-adjoint linear relation in $H \oplus H$ and $\tau = \tau^*$, then $m_\tau(\lambda)$ is the Titchmarsh–Weyl function of the system corresponding to self-adjoint separated boundary conditions [13]. In the case of a non-Hamiltonian system such conditions do not exist [22] and $m_\tau(\lambda)$ corresponds to special mixed boundary conditions (see Definition 4.16).

For pseudospectral functions $\sigma(\cdot)$ of the minimal dimension $n_\sigma = \dim(H \oplus \widehat{H})$ formulas similar to (1.6) and (1.7) were obtained in [1]. These formulas are proved in [1] only for a parameter τ of a special form; therefore not all pseudospectral functions $\sigma(\cdot)$ are parametrize in this paper.

As is known [15, 28] the set of spectral functions of a symmetric differential operator $l[y]$ of an order m coincides with the set of spectral functions of a special definite symmetric system corresponding to $l[y]$. Moreover, this system is Hamiltonian if and only if m is even. According to the classical monograph by N. Dunford and J. T. Schwartz [10, ch. 13.21] an important problem of the spectral theory of differential operators is a characterization of their spectral functions $\sigma_{\min}(\cdot)$ with the minimally possible dimension $n_{\min}$. It follows from Theorem 1.6 that $n_{\min} = k + 1$ in the case $m = 2k + 1$ and $n_{\min} = k$ in the case $m = 2k$. Moreover, by using Theorem 1.5 one may obtain a parametrization of $\sigma_{\min}(\cdot)$. In more details this results will be specified elsewhere.

For a differential operator $l[y]$ of an even order m formulas similar to (1.6) and (1.7) were proved in our paper [23]. These formulas enable one to calculate spectral functions $\sigma(\cdot)$ of an arbitrary dimension n_σ ($\frac{m}{2} \leq n_\sigma \leq m$) corresponding to a special parameter τ; hence they do not parametrize all spectral functions of $l[y]$.

In conclusion note that our approach is based on the theory of boundary triplets (boundary pairs) for symmetric linear relations and their Weyl function (see [4, 6, 7, 12, 19, 22] and references therein).

2. Preliminaries

2.1. Notations

The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plane of the complex plane. If $\mathcal{H}$ is a subspace in $\widetilde{\mathcal{H}}$, then $P_{\mathcal{H}}(\in [\widetilde{\mathcal{H}}])$ denote the orthoprojection in $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$ and $P_{\widetilde{\mathcal{H}},\mathcal{H}}(\in [\widetilde{\mathcal{H}}, \mathcal{H}])$ denote the same orthoprojection considered as an operator from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$.

Recall that a linear relation $T : \mathcal{H}_0 \to \mathcal{H}_1$ from a Hilbert space $\mathcal{H}_0$ to a Hilbert space $\mathcal{H}_1$ is a linear manifold in the Hilbert $\mathcal{H}_0 \oplus \mathcal{H}_1$. If $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ one speaks of a linear relation T in $\mathcal{H}$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ $(\widetilde{\mathcal{C}}(\mathcal{H}))$. A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\mathrm{gr}\, T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\mathrm{dom}\, T$, $\mathrm{ran}\, T$, $\ker T$ and $\mathrm{mul}\, T$ the domain, range, kernel and the multivalued part of T respectively. Recall that $\mathrm{mul}\, T$ ia a subspace in $\mathcal{H}_1$ defined by

$$\mathrm{mul}\, T := \{h_1 \in \mathcal{H}_1 : \{0, h_1\} \in T\}. \tag{2.1}$$

Clearly, $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is an operator if and only if $\mathrm{mul}\, T = \{0\}$. For $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we will denote by $T^{-1}(\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ and $T^*(\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ the inverse and adjoint linear relations of T respectively.

Recall that an operator function $\Phi(\cdot) : \mathbb{C}_+ \to [\mathcal{H}]$ is called a Nevanlinna function (and referred to the class $R[\mathcal{H}]$) if it is holomorphic and $\mathrm{Im}\,\Phi(\lambda) \geq 0$, $\lambda \in \mathbb{C}_+$.

2.2. Symmetric relations and generalized resolvents

As is known a linear relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is called symmetric (self-adjoint) if $A \subset A^*$ (resp. $A = A^*$). For each symmetric relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ the following decompositions hold

$$\mathfrak{H} = \mathfrak{H}' \oplus \mathrm{mul}\, A, \qquad A = \mathrm{gr}\, A' \oplus \widehat{\mathrm{mul}\, A}, \tag{2.2}$$

where $\widehat{\mathrm{mul}\, A} = \{0\} \oplus \mathrm{mul}\, A$ and A' is a closed symmetric not necessarily densely defined operator in $\mathfrak{H}'$ (the operator part of A). Moreover, $A = A^*$ if and only if $A' = (A')^*$.

Let $A = A^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$, let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$ and let $E_0(\cdot) : \mathcal{B} \to [\mathfrak{H}_0]$ be the orthogonal spectral measure of A_0. Then the spectral measure $E_A(\cdot) : \mathcal{B} \to [\mathfrak{H}]$ of A is defined as $E_A(B) = E_0(B) P_{\mathfrak{H}'}$, $B \in \mathcal{B}$.

Definition 2.1. Let $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ and let $\mathfrak{H}$ be a subspace in $\widetilde{\mathfrak{H}}$. The relation $\widetilde{A}$ is called $\mathfrak{H}$-minimal if there is no a nontrivial subspace $\mathfrak{H}' \subset \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ such that $E_{\widetilde{A}}(\delta)\mathfrak{H}' \subset \mathfrak{H}'$ for each bounded interval $\delta = [\alpha, \beta) \subset \mathbb{R}$.

Definition 2.2. The relations $T_j \in \widetilde{\mathcal{C}}(\mathfrak{H}_j)$, $j \in \{1, 2\}$, are said to be unitarily equivalent (by means of a unitary operator $U \in [\mathfrak{H}_1, \mathfrak{H}_2]$) if $T_2 = \widetilde{U}T_1$ with $\widetilde{U} = U \oplus U \in [\mathfrak{H}_1^2, \mathfrak{H}_2^2]$.

Let $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be a symmetric relation. Recall the following definitions and results.

Definition 2.3. A relation $\widetilde{A} = \widetilde{A}^*$ in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ satisfying $A \subset \widetilde{A}$ is called an exit space self-adjoint extension of A. Moreover, such an extension $\widetilde{A}$ is called minimal if it is $\mathfrak{H}$-minimal.

In what follows we denote by $\widetilde{\mathrm{Self}}(A)$ the set of all minimal exit space self-adjoint extensions of A. Moreover, we denote by $\mathrm{Self}(A)$ the set of all extensions $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$ of A (such an extension is called canonical). As is known, for each A one has $\widetilde{\mathrm{Self}}(A) \neq \emptyset$. Moreover, $\mathrm{Self}(A) \neq \emptyset$ if and only if A has equal deficiency indices, in which case $\mathrm{Self}(A) \subset \widetilde{\mathrm{Self}}(A)$.

Definition 2.4. Exit space extensions $\widetilde{A}_j = \widetilde{A}_j^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}_j)$, $j \in \{1, 2\}$, of A are called equivalent (with respect to $\mathfrak{H}$) if there exists a unitary operator $V \in [\widetilde{\mathfrak{H}}_1 \ominus \mathfrak{H}, \widetilde{\mathfrak{H}}_2 \ominus \mathfrak{H}]$ such that $\widetilde{A}_1$ and $\widetilde{A}_2$ are unitarily equivalent by means of $U = I_{\mathfrak{H}} \oplus V$.

Definition 2.5. The operator functions $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathfrak{H}]$ and $F(\cdot) : \mathbb{R} \to [\mathfrak{H}]$ are called a generalized resolvent and a spectral function of A respectively if there exists an exit space self-adjoint extension $\widetilde{A}$ of A (in a certain Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$) such that

$$R(\lambda) = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.3}$$

$$F(t) = P_{\widetilde{\mathfrak{H}}, \mathfrak{H}} E_{\widetilde{A}}((-\infty, t)) \upharpoonright \mathfrak{H}, \quad t \in \mathbb{R}. \tag{2.4}$$

Proposition 2.6. *Each generalized resolvent $R(\lambda)$ of A is generated by some (minimal) extension $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$. Moreover, the extensions $\widetilde{A}_1$, $\widetilde{A}_2 \in \widetilde{\mathrm{Self}}(A)$ inducing the same generalized resolvent $R(\cdot)$ are equivalent.*

In the sequel we suppose that a generalized resolvent $R(\cdot)$ and a spectral function $F(\cdot)$ are generated by an extension $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$. Moreover, we identify equivalent extensions. Then by Proposition 2.6 the equality (2.3) gives a bijective correspondence between generalized resolvents $R(\lambda)$ and extensions $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$, so that each $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ is uniquely defined by the corresponding generalized resolvent (2.3) (spectral function (2.4)).

Definition 2.7. An extension $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ ($\widetilde{A} \in \mathrm{Self}(A)$) belongs to the class $\widetilde{\mathrm{Self}}_0(A)$ (resp. $\mathrm{Self}_0(A)$) if $\mathrm{mul}\,\widetilde{A} = \mathrm{mul}\,A$.

It follows from (2.2) that the operator A' is densely defined if and only if $\mathrm{mul}\,A = \mathrm{mul}\,A^*$. This yields the equivalence

$$\widetilde{\mathrm{Self}}(A) = \widetilde{\mathrm{Self}}_0(A) \iff \mathrm{mul}\,A = \mathrm{mul}\,A^* \tag{2.5}$$

2.3. The classes $\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and $\widetilde{R}(\mathcal{H})$

In the following $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, $P_1 := P_{\mathcal{H}_0, \mathcal{H}_1}$ and $P_2 = P_{\mathcal{H}_2}$.

Definition 2.8. [24] A function $\tau(\cdot) : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is referred to the class $\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ if:

(i) $2\mathrm{Im}(h_1, h_0) - \|P_2 h_0\|^2 \geq 0$, $\{h_0, h_1\} \in \tau(\lambda)$, $\lambda \in \mathbb{C}_+$;

(ii) $(\tau(\lambda) + iP_1)^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$, $\lambda \in \mathbb{C}_+$, and the operator-function $(\tau(\lambda) + iP_1)^{-1}$ is holomorphic on $\mathbb{C}_+$.

According to [24] the equality

$$\tau(\lambda) = \{C_0(\lambda), C_1(\lambda)\}$$
$$:= \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0(\lambda)h_0 + C_1(\lambda)h_1 = 0\}, \quad \lambda \in \mathbb{C}_+ \tag{2.6}$$

establishes a bijective correspondence between all functions $\tau(\cdot) \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and all pairs of holomorphic operator-functions $C_j(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_j, \mathcal{H}_0]$, $j \in \{0, 1\}$, satisfying

$$2\,\mathrm{Im}(C_1(\lambda)P_1 C_0^*(\lambda)) + C_0(\lambda)P_2 C_0^*(\lambda) \geq 0,$$
$$(C_0(\lambda) - iC_1(\lambda)P_1)^{-1} \in [\mathcal{H}_0], \quad \lambda \in \mathbb{C}_+. \tag{2.7}$$

This fact enables one to identify a function $\tau(\cdot) \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and the corresponding pair of operator-functions $C_j(\cdot)$ (more precisely the equivalence class of such pairs [24]).

If $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$, then the class $\widetilde{R}(\mathcal{H}, \mathcal{H})$ coincides with the well-known class $\widetilde{R}(\mathcal{H})$ of Nevanlinna $\widetilde{\mathcal{C}}(\mathcal{H})$-valued functions (Nevanlinna operator pairs) $\tau(\lambda) = \{C_0(\lambda), C_1(\lambda)\}$, $\lambda \in \mathbb{C}_+$. In this case the class $\widetilde{R}^0(\mathcal{H})$ is defined as the set of all $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$ such that

$$\tau(\lambda) \equiv \theta = \{C_0, C_1\}, \quad \lambda \in \mathbb{C}_+, \tag{2.8}$$

with $\theta = \theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $C_j \in [\mathcal{H}]$ satisfying $\mathrm{Im}(C_1 C_0^*) = 0$ and $(C_0 \pm iC_1)^{-1} \in [\mathcal{H}]$.

2.4.   Boundary triplets and boundary pairs

Here we recall some facts about boundary triplets and boundary pairs following [4, 6, 7, 12, 19, 21, 22].

Assume that A is a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$, $\mathfrak{N}_\lambda(A) = \ker(A^* - \lambda)$ ($\lambda \in \mathbb{C}$) is a defect subspace of A, $\widehat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$ and $n_\pm(A) := \dim \mathfrak{N}_\lambda(A) \leq \infty$, $\lambda \in \mathbb{C}_\pm$, are deficiency indices of A.

Definition 2.9. A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0, 1\}$, are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma : \widehat{f} \to \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$, $\widehat{f} \in A^*$, from A^* into $\mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green's identity

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}_0} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}_0} + i(P_2 \Gamma_0 \widehat{f}, P_2 \Gamma_0 \widehat{g})_{\mathcal{H}_2} \quad (2.9)$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in A^*$.

A boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* exists if and only if $n_-(A) \leq n_+(A)$, in which case $\dim \mathcal{H}_1 = n_-(A)$ and $\dim \mathcal{H}_0 = n_+(A)$.

Proposition 2.10. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let π_1 be the orthoprojection in $\mathfrak{H} \oplus \mathfrak{H}$ onto $\mathfrak{H} \oplus \{0\}$. Then the equalities*

$$\gamma_+(\lambda) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \lambda \in \mathbb{C}_+; \gamma_-(\lambda) = \pi_1(P_1 \Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \lambda \in \mathbb{C}_-$$
$$(2.10)$$

$$M_+(\lambda) h_0 = \Gamma_1\{\gamma_+(\lambda) h_0, \lambda \gamma_+(\lambda) h_0\}, \ h_0 \in \mathcal{H}_0, \ \lambda \in \mathbb{C}_+ \qquad (2.11)$$

correctly define holomorphic operator functions $\gamma_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathfrak{H}]$, $\gamma_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathfrak{H}]$ (γ-fields of Π) and $M_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$ (the Weyl function of Π).

γ-field $\gamma_+(\cdot)$ ($\gamma_-(\cdot)$) can be also defined as a unique $[\mathcal{H}_0, \mathfrak{H}]$-valued (resp. $[\mathcal{H}_1, \mathfrak{H}]$-valued) operator function such that $\gamma_+(\lambda)\mathcal{H}_0 \subset \mathfrak{N}_\lambda(A)$ (resp. $\gamma_-(\lambda)\mathcal{H}_1 \subset \mathfrak{N}_\lambda(A)$) and

$$\Gamma_0\{\gamma_+(\lambda) h_0, \lambda \gamma_+(\lambda) h_0\} = h_0, \ h_0 \in \mathcal{H}_0, \ \lambda \in \mathbb{C}_+ \qquad (2.12)$$
$$P_1 \Gamma_0\{\gamma_-(\lambda) h_1, \lambda \gamma_-(\lambda) h_1\} = h_1, \ h_1 \in \mathcal{H}_1, \ \lambda \in \mathbb{C}_-. \qquad (2.13)$$

A boundary pair for A^* is a generalization of a boundary triplet. Namely, a pair $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ with a linear relation $\Gamma : \mathfrak{H} \oplus \mathfrak{H} \to \mathcal{H}_0 \oplus \mathcal{H}_1$ is called a boundary pair for A^* if $\overline{\operatorname{dom} \Gamma} = A^*$, the identity

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h_1, x_0)_{\mathcal{H}_0} - (h_0, x_1)_{\mathcal{H}_0} + i(P_2 h_0, P_2 x_0)_{\mathcal{H}_0} \quad (2.14)$$

holds for every $\{f \oplus f', h_0 \oplus h_1\}$, $\{g \oplus g', x_0 \oplus x_1\} \in \Gamma$ and a certain maximality condition is satisfied [6, 22]. The following proposition is immediate from [22, Section 3].

Proposition 2.11. *Let $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ be a boundary pair for A^* with $\dim \mathcal{H}_0 < \infty$ and let $\Gamma_0 : \mathfrak{H} \oplus \mathfrak{H} \to \mathcal{H}_0$ be the linear relations, given by $\Gamma_0 = P_{\mathcal{H}_0 \oplus \{0\}} \Gamma$. Moreover, let*

$$\mathcal{K}_\Gamma = \mathrm{mul}\,(\mathrm{mul}\,\Gamma) = \{h_1 \in \mathcal{H}_1 : \{0 \oplus 0, 0 \oplus h_1\} \in \Gamma\}, \qquad \mathcal{K}_\Gamma \subset \mathcal{H}_1. \tag{2.15}$$

Then: (1) $\mathrm{dom}\,\Gamma = A^*$;
(2) If $\mathcal{K}_\Gamma = \{0\}$, then $\mathrm{ran}\,\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A) = \mathcal{H}_0$, $\lambda \in \mathbb{C}_+$; $\mathrm{ran}\,P_1\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A) = \mathcal{H}_1$, $\lambda \in \mathbb{C}_-$, and the equality

$$\mathrm{gr}\,M_+(\lambda) = \{h_0 \oplus h_1 : \{f \oplus \lambda f, h_0 \oplus h_1\} \in \Gamma \ \text{ with some } \ f \in \mathfrak{N}_\lambda(A)\},$$
$$\lambda \in \mathbb{C}_+ \tag{2.16}$$

defines the operator function $M_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$ (the Weyl function of the pair $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$). Moreover,

$$\mathrm{gr}\,M_+^*(\overline{\lambda})$$
$$= \{\{P_1 h_0 \oplus (h_1 + iP_2 h_0\} : \{f \oplus \lambda f, h_0 \oplus h_1\} \in \Gamma \ \text{ with some } \ f \in \mathfrak{N}_\lambda(A)\},$$
$$\lambda \in \mathbb{C}_-. \tag{2.17}$$

3.   Pseudospectral and spectral functions of symmetric systems

3.1.   Notations

For an interval $\mathcal{I} = [a, b\rangle \subset \mathbb{R}$ and a finite-dimensional Hilbert space $\mathbb{H}$ we denote by $AC(\mathcal{I}; \mathbb{H})$ the set of all functions $f(\cdot) : \mathcal{I} \to \mathbb{H}$, which are absolutely continuous on each segment $[\alpha, \beta] \subset \mathcal{I}$.

Assume that $\Delta(\cdot) : \mathcal{I} \to [\mathbb{H}]$ is a locally integrable function such that $\Delta(t) \geq 0$ a.e. on $\mathcal{I}$. Denote by $\mathcal{L}_\Delta^2(\mathcal{I})$ the semi-Hilbert space of Borel measurable functions $f(\cdot) : \mathcal{I} \to \mathbb{H}$ satisfying $\int_\mathcal{I} (\Delta(t)f(t), f(t))_\mathbb{H}\, dt < \infty$ (see e.g. [10, Chapter 13.5]). The semi-definite inner product in $\mathcal{L}_\Delta^2(\mathcal{I})$ will be denoted $(\cdot, \cdot)_\Delta$. Moreover, let $L_\Delta^2(\mathcal{I})$ be the Hilbert space of equivalence classes in $\mathcal{L}_\Delta^2(\mathcal{I})$ with respect to the semi-norm in $\mathcal{L}_\Delta^2(\mathcal{I})$, π_Δ be the quotient map from $\mathcal{L}_\Delta^2(\mathcal{I})$ onto $L_\Delta^2(\mathcal{I})$ and $\widetilde{\pi}_\Delta\{f, g\} := \{\pi_\Delta f, \pi_\Delta g\}$, $\{f, g\} \in (\mathcal{L}_\Delta^2(\mathcal{I}))^2$.

For a given finite-dimensional Hilbert space $\mathcal{K}$ we denote by $\mathcal{L}_\Delta^2[\mathcal{K}, \mathbb{H}]$ the set of all Borel measurable operator-functions $F(\cdot) : \mathcal{I} \to [\mathcal{K}, \mathbb{H}]$ such that $F(t)h \in \mathcal{L}_\Delta^2(\mathcal{I})$, $h \in \mathcal{K}$.

In the following for a distribution function $\sigma(\cdot) : \mathbb{R} \to [\mathbb{H}]$ we denote by $\mathcal{L}^2(\sigma; \mathbb{H})$ the semi-Hilbert space of Borel-measurable functions $g(\cdot) : \mathbb{R} \to \mathbb{H}$ such that $\int_{\mathbb{R}} (d\sigma(s)g(s), g(s))(s) < \infty$ and by $L^2(\sigma; \mathbb{H})$ the a Hilbert space of all equivalence classes in $\mathcal{L}^2(\sigma; \mathbb{H})$ with respect to the seminorm $\| \cdot \|_{\mathcal{L}^2(\sigma;\mathbb{H})}$ (see e.g. [10, Chapter 13.5]). Moreover, we denote by π_σ the quotient map from $\mathcal{L}^2(\sigma; \mathbb{H})$ onto $L^2(\sigma; \mathbb{H})$.

3.2. Symmetric systems

Let H and $\widehat{H}$ be finite dimensional Hilbert spaces and let

$$\mathbb{H} := H \oplus \widehat{H} \oplus H \tag{3.1}$$

$$\nu = \dim H, \qquad \widehat{\nu} = \dim \widehat{H}, \qquad n = \dim \mathbb{H} = 2\nu + \widehat{\nu}. \tag{3.2}$$

A first order symmetric system of differential equations on an interval $\mathcal{I} = [a, b\rangle, -\infty < a < b \leq \infty$, (with the regular endpoint a) is of the form

$$Jy' - B(t)y = \lambda \Delta(t)y, \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}, \tag{3.3}$$

where J is the operator (1.2) and $B(\cdot)$ and $\Delta(\cdot)$ are locally integrable $[\mathbb{H}]$-valued functions on $\mathcal{I}$ such that $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ (a.e. on $\mathcal{I}$).

A function $y \in AC(\mathcal{I}; \mathbb{H})$ is a solution of system (3.3) if equality (3.3) holds a.e. on $\mathcal{I}$. An operator function $Y(\cdot, \lambda) : \mathcal{I} \to [\mathcal{K}, \mathbb{H}]$ is an operator solution of (3.3) if $y(t) = Y(t, \lambda)h$ is a solution of (3.3) for every $h \in \mathcal{K}$ (here $\mathcal{K}$ is a Hilbert space with $\dim \mathcal{K} < \infty$).

In the sequel we denote by $\mathcal{N}_\lambda, \ \lambda \in \mathbb{C}$, the linear space of all solutions of the system (3.3) belonging to $\mathcal{L}^2_\Delta(\mathcal{I})$. According to [15, 18] the numbers $N_\pm = \dim \mathcal{N}_\lambda, \ \lambda \in \mathbb{C}_\pm$, do not depend on λ in either $\mathbb{C}_+$ or $\mathbb{C}_-$. These numbers are called the formal deficiency indices of the system [15]. Clearly $N_\pm \leq n$.

In the following for each operator solution $Y(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{K}, \mathbb{H}]$ we denote by $Y(\lambda)$ the linear operator from $\mathcal{K}$ to $\mathcal{N}_\lambda$ given by $(Y(\lambda)h)(t) = Y(t, \lambda)h, \ h \in \mathcal{K}$.

Clearly, for any $\lambda \in \mathbb{C}$ the space $\mathcal{N}$ of all solutions y of (3.3) with $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) is a subspace of $\mathcal{N}_\lambda$; moreover, $\mathcal{N}$ does depend on λ. The space $\mathcal{N}$ is called the null manifold of the system [15]. Denote by $\theta_\mathcal{N}$ the subspace in $\mathbb{H}$ given by

$$\theta_\mathcal{N} = \{y(a) : y \in \mathcal{N}\}. \tag{3.4}$$

As is known [14, 18, 30] system (3.3) gives rise to the maximal linear relations $\mathcal{T}_{\max}$ and $T_{\max}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and $L^2_\Delta(\mathcal{I})$ respectively. They are

given by

$$\mathcal{T}_{\max} = \{\{y, f\} \in (\mathcal{L}_\Delta^2(\mathcal{I}))^2 : y \in AC(\mathcal{I}; \mathbb{H})$$
$$\text{and } Jy'(t) - B(t)y(t) = \Delta(t)f(t) \text{ a.e. on } \mathcal{I}\}$$

and $T_{\max} = \widetilde{\pi}_\Delta \mathcal{T}_{\max}$. Moreover the Lagrange's identity

$$(f, z)_\Delta - (y, g)_\Delta = [y, z]_b - (Jy(a), z(a)), \quad \{y, f\}, \{z, g\} \in \mathcal{T}_{\max} \quad (3.5)$$

holds with

$$[y, z]_b := \lim_{t \uparrow b}(Jy(t), z(t)), \quad y, z \in \operatorname{dom} \mathcal{T}_{\max}.$$

Let $\mathcal{D}_b$ be the set of all $y \in \operatorname{dom} \mathcal{T}_{\max}$ such that $[y, z]_b = 0$ for all $z \in \operatorname{dom} \mathcal{T}_{\max}$. The minimal relation $T_{\min}$ in $L_\Delta^2(\mathcal{I})$ is defined via $T_{\min} = \widetilde{\pi}_\Delta \mathcal{T}_a$, where

$$\mathcal{T}_a = \{\{y, f\} \in \mathcal{T}_{\max} : y \in \mathcal{D}_b, \ y(a) = 0\}. \quad (3.6)$$

As was shown in [14, 18, 22, 30] $T_{\min}$ is a closed symmetric linear relation in $L_\Delta^2(\mathcal{I})$, $T_{\min}^* = T_{\max}$ and

$$n_+(T_{\min}) = N_+ - \dim \mathcal{N}, \qquad n_-(T_{\min}) = N_- - \dim \mathcal{N}. \quad (3.7)$$

With each subspace $\theta \subset \mathbb{H}$ we associate the subspace $\theta^\times \subset \mathbb{H}$ given by

$$\theta^\times = \mathbb{H} \ominus J\theta = \{h \in \mathbb{H} : (Jh, k) = 0, \ k \in \theta\}.$$

Clearly $\theta^{\times\times} = \theta$. Moreover, by [22, Proposition 4.19]

$$\theta_\mathcal{N}^\times = \{y(a) : y \in \mathcal{D}_b\}. \quad (3.8)$$

Denote by $Sym(\mathbb{H})$ the set of all subspaces θ in $\mathbb{H}$ satisfying $\theta \subset \theta^\times$ or, equivalently, $(Jh, k) = 0$, $h, k \in \theta$.

The following three lemmas will be useful in the sequel.

Lemma 3.1. (1) *If* $\eta \in Sym(\mathbb{H})$, *then* $\dim \eta \leq \nu$ *and* $\dim \eta^\times \geq \nu + \widehat{\nu}$.

(2) *For every* $\eta \in Sym(\mathbb{H})$ *there exists a subspace* $\theta \subset \mathbb{H}$ *such that* $\theta^\times \in Sym(\mathbb{H})$, $\dim \theta = \nu + \widehat{\nu}$ *(i.e., the dimension of* θ *is minimally possible) and* $\theta^\times \cap \eta = \{0\}$.

(3) *Let* θ *be a subspace in* $\mathbb{H}$ *and* $\theta^\times \in Sym(\mathbb{H})$. *Then there exist an operator* $\widetilde{U} \in [\mathbb{H}]$ *and a subspace* $H_1 \subset H$ *such that* $\widetilde{U}^* J \widetilde{U} = J$ *and* $\widetilde{U}\mathbb{H}_0 = \theta$, *where*

$$\mathbb{H}_0 = H \oplus \widehat{H} \oplus H_1. \quad (3.9)$$

Proof. (1) Let $\widehat{J}$ and X be operators in $\mathbb{H}$ given by the block representations

$$\widehat{J} = i \begin{pmatrix} I_H & 0 & 0 \\ 0 & I_{\widehat{H}} & 0 \\ 0 & 0 & -I_H \end{pmatrix}, \qquad X = \frac{1}{\sqrt{2}} \begin{pmatrix} -iI_H & 0 & I_H \\ 0 & \sqrt{2}I_{\widehat{H}} & 0 \\ iI_H & 0 & I_H \end{pmatrix}$$

with respect to decomposition (3.1) of $\mathbb{H}$. One can easily verify that

$$X^*\widehat{J}X = J, \qquad X^*X = XX^* = I_{\mathbb{H}}. \tag{3.10}$$

and the equality $\operatorname{gr} V_\eta = X\eta$ gives a bijective correspondence between all $\eta \in Sym(\mathbb{H})$ and all isometries $V_\eta \in [\operatorname{dom} V_\eta, H]$ with $\operatorname{dom} V_\eta \subset H \oplus \widehat{H}$. Hence for every $\eta \in Sym(\mathbb{H})$ one has $\dim \eta = \dim \operatorname{ran} V_\eta \leq \nu$ and, consequently, $\dim \eta^\times \geq \nu + \widehat{\nu}$.

(2) Assume that $\eta \in Sym(\mathbb{H})$ and let $U \in [\operatorname{dom} U, H]$ be an isometry such that $\operatorname{dom} U \in H \oplus \widehat{H}$, $-V_\eta \subset U$ and $\operatorname{ran} U = H$. Then $U = U_{\theta_0}$ with some $\theta_0 \in Sym(\mathbb{H})$ and the obvious equality $\operatorname{gr} V_\eta \cap \operatorname{gr} U = \{0\}$ yields $\eta \cap \theta_0 = \{0\}$. Moreover, $\dim \theta_0 = \dim \operatorname{ran} U = \nu$ and hence $\theta := \theta_0^\times$ possesses the required properties.

(3) Let H_1 be a subspace in H with $\operatorname{codim} H_1 = \dim \theta^\times$, let $H_1^\perp = H \ominus H_1$ and let $\mathbb{H}_0 \subset \mathbb{H}$ be subspace (3.9). Then $\mathbb{H}_0^\times = H_1^\perp \oplus \{0\} \oplus \{0\}$ and therefore $\mathbb{H}_0^\times \in Sym(\mathbb{H})$. Let $V_1 = V_{\mathbb{H}_0^\times}$ and $V_2 = V_{\theta^\times}$. Since $\dim \mathbb{H}_0^\times = \dim \theta^\times$, one has $\dim \operatorname{dom} V_1 = \dim \operatorname{dom} V_2$. Therefore there exist unitary operators $U_1 \in [H \oplus \widehat{H}]$ and $U_2 \in [H]$ such that $U_1 \operatorname{dom} V_1 = \operatorname{dom} V_2$ and $V_2 U_1 \upharpoonright \operatorname{dom} V_1 = U_2 V_1$. Letting $\widehat{U} = \operatorname{diag}(U_1, U_2)$ one obtains the operator $\widehat{U} \in [\mathbb{H}]$ such that $\widehat{U}^*\widehat{J}\widehat{U} = \widehat{J}$ and $\widehat{U}\operatorname{gr} V_1 = \operatorname{gr} V_2$. This and (3.10) imply that the operator $\widetilde{U} := X^*\widehat{U}X$ satisfies $\widetilde{U}^*J\widetilde{U} = J$ and $\widetilde{U}\mathbb{H}_0^\times = \theta^\times$. Therefore $\widetilde{U}\mathbb{H}_0 = \theta$. $\qquad\square$

Remark 3.2. If $H_1 \subset H$ is a subspace from Lemma 3.1 (3), $H_1^\perp = H \ominus H_1$ and $\mathbb{H}_0$ is given by (3.9), then the following decompositions are obvious:

$$\mathbb{H}_0 = \underbrace{H_1^\perp \oplus H_1}_{H} \oplus \widehat{H} \oplus H_1, \qquad \mathbb{H} = \underbrace{H_1^\perp \oplus H_1}_{H} \oplus \widehat{H} \oplus \underbrace{H_1 \oplus H_1^\perp}_{H} = \mathbb{H}_0 \oplus H_1^\perp.$$

$$\tag{3.11}$$

Lemma 3.3. *Let θ be a subspace in $\mathbb{H}$. Then:*
   (1) *The equalities*

$$T = T_{\theta^\times} := \{\widetilde{\pi}_\Delta\{y, f\} : \{y, f\} \in \mathcal{T}_{\max}, \ y \in \mathcal{D}_b, \ y(a) \in \theta^\times\} \tag{3.12}$$

$$T^* = \{\widetilde{\pi}_\Delta\{y, f\} : \{y, f\} \in \mathcal{T}_{\max}, \ y(a) \in \theta\} \tag{3.13}$$

defines a relation $T \in \widetilde{\mathcal{C}}(L^2_\Delta(\mathcal{I}))$ and its adjoint T^*. Moreover, $T_{\min} \subset T \subset T_{\max}$

(2) The multivalued part $\operatorname{mul} T$ of T is the set of all $\widetilde{f} \in \mathfrak{H}$ such that for some (and hence for all) $f(\cdot) \in \widetilde{f}$ there exists a solution y of the system

$$Jy' - B(t)y = \Delta(t)f(t), \quad t \in \mathcal{I} \tag{3.14}$$

satisfying $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$), $y(a) \in \theta^\times$ and $y \in \mathcal{D}_b$.

(3) The relation T is symmetric if and only if $\theta^\times \cap \theta^\times_{\mathcal{N}} \in Sym(\mathbb{H})$.

Proof. (1) The inclusions $T_{\min} \subset T \subset T_{\max}$ directly follow from (3.12) and definitions of $T_{\min}$ and $T_{\max}$. Next we show that the relation T^* adjoint to T is of the form (3.13). In view of the Lagrange's identity (3.5) for every $\{y, f\} \in \mathcal{T}_{\max}$ with $y(a) \in \theta$ one has $\widetilde{\pi}_\Delta\{y, f\} \in T^*$. Conversely, assume that $\{\widetilde{y}, \widetilde{f}\} \in T^*$ and prove the existence of $\{y, f\} \in \mathcal{T}_{\max}$ such that $y(a) \in \theta$ and $\widetilde{\pi}_\Delta\{y, f\} = \{\widetilde{y}, \widetilde{f}\}$. Since $T_{\min} \subset T$, it follows that $T^* \subset T_{\max}$ and hence there is $\{y_1, f\} \in \mathcal{T}_{\max}$ such that $\widetilde{\pi}_\Delta\{y_1, f\} = \{\widetilde{y}, \widetilde{f}\}$. Let $h \in \theta^\times \cap \theta^\times_{\mathcal{N}}$. Then in view of (3.8) there exists $\{z, g\} \in \mathcal{T}_{\max}$ such that $z \in \mathcal{D}_b$, $z(a) = h$ and hence $\{\widetilde{z}, \widetilde{g}\} := \widetilde{\pi}_\Delta\{z, g\} \in T$. Applying the Lagrange's identity (3.5) to $\{y_1, f\}$ and $\{z, g\}$ one obtains

$$(Jy_1(a), h) = (y_1, g)_\Delta - (f, z)_\Delta = (\widetilde{y}, \widetilde{g}) - (\widetilde{f}, \widetilde{z}) = 0, \quad h \in \theta^\times \cap \theta^\times_{\mathcal{N}}.$$

Therefore $y_1(a) \in (\theta^\times \cap \theta^\times_{\mathcal{N}})^\times$. Obviously $(\theta^\times \cap \theta^\times_{\mathcal{N}})^\times = \theta + \theta_{\mathcal{N}}$ and hence $y_1(a) = h + y_2(a)$ with some $h \in \theta$ and $y_2 \in \mathcal{N}$. Let $y = y_1 - y_2$. Since $\{y_2, 0\} \in \mathcal{T}_{\max}$, it follows that a pair $\{y, f\} := \{y_1, f\} - \{y_2, 0\}$ belongs to $\mathcal{T}_{\max}$. Moreover, $y(a) = y_1(a) - y_2(a) = h$ and hence $y(a) \in \theta$. Finally, $\pi_\Delta y_2 = 0$ and therefore $\widetilde{\pi}_\Delta\{y, f\} = \widetilde{\pi}_\Delta\{y_1, f\} = \{\widetilde{y}, \widetilde{f}\}$. This completes the proof of (3.13).

Statement (2) directly follows from (2.1).

(3) It follows from (3.8) that $T = T_{\theta^\times \cap \theta^\times_{\mathcal{N}}}$. Therefore to prove statement (3) it is sufficient to prove the following equivalent statement: if $\theta^\times \subset \theta^\times_{\mathcal{N}}$, then the equivalence $T \subset T^* \iff \theta^\times \subset \theta$ is valid.

So assume that $\theta^\times \subset \theta^\times_{\mathcal{N}}$ and let $T \subset T^*$. If $h, k \in \theta^\times$, then by (3.8) there exist $\{y, f\}, \{z, g\} \in \mathcal{T}_{\max}$ such that $y, z \in \mathcal{D}_b$ and $y(a) = h$, $z(a) = k$. Therefore $\widetilde{\pi}_\Delta\{y, f\}, \widetilde{\pi}_\Delta\{z, g\} \in T$ and hence

$$(f, z)_\Delta - (y, g)_\Delta = 0.$$

This and the Lagrange's identity (3.5) imply that $(Jh, k) = 0$. Therefore $\theta^\times \subset \theta$. If conversely $\theta^\times \subset \theta$, then the inclusion $T \subset T^*$ directly follows from (3.12) and (3.13). $\qquad\square$

Lemma 3.4. *There exists a subspace* $\theta \subset \mathbb{H}$ *such that* $\theta^\times \in Sym(\mathbb{H})$, $\dim \theta = \nu + \widehat{\nu}$ *and the symmetric extension* $T = T_{\theta^\times}$ *of* $T_{\min}$ *defined by* (3.12) *satisfies* $\operatorname{mul} T = \operatorname{mul} T_{\min}$.

Proof. Let η be a subspace in $\mathbb{H}$ defined by

$$\eta = \{y(a) : y \in \mathcal{D}_b, \ \Delta(t)y(t) = 0 \ (\text{a.e. on } \mathcal{I})\}. \tag{3.15}$$

If $h, k \in \eta$, then there exist $\{y, f\}, \{z, g\} \in \mathcal{T}_{\max}$ such that $y, z \in \mathcal{D}_b$, $y(a) = h$, $z(a) = k$ and $\Delta(t)y(t) = \Delta(t)z(t) = 0$ (a.e. on $\mathcal{I}$). Application of the Lagrange's identity (3.5) to such $\{y, f\}$ and $\{z, g\}$ gives $(Jh, k) = 0$, which implies that $\eta \in Sym(\mathbb{H})$. Therefore by Lemma 3.1, (2) there exists a subspace $\theta \subset \mathbb{H}$ such that $\theta^\times \in Sym(\mathbb{H})$, $\dim \theta = \nu + \widehat{\nu}$ and $\theta^\times \cap \eta = \{0\}$. Let $T = T_{\theta^\times}$ be given by (3.12) and let $\widetilde{f} \in \operatorname{mul} T$. Then according to Lemma 3.3, (2) there exists $y \in \mathcal{D}_b$ such that $y(a) \in \theta^\times$, $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) and $\{y, f\} \in \mathcal{T}_{\max}$ for each $f(\cdot) \in \widetilde{f}$. Since by (3.15) $y(a) \in \theta^\times \cap \eta$, it follows that $y(a) = 0$ and hence $\{y, f\} \in \mathcal{T}_a$. Hence $\{\pi_\Delta y, \widetilde{f}\} \in T_{\min}$ and the equality $\pi_\Delta y = 0$ yields $\widetilde{f} \in \operatorname{mul} T_{\min}$. Thus $\operatorname{mul} T \subset \operatorname{mul} T_{\min}$ and in view of the obvious inclusion $\operatorname{mul} T_{\min} \subset \operatorname{mul} T$ one has $\operatorname{mul} T = \operatorname{mul} T_{\min}$. $\square$

3.3. q-pseudospectral and spectral functions

In what follows we put $\mathfrak{H} := L^2_\Delta(\mathcal{I})$ and denote by $\mathfrak{H}_b$ the set of all $\widetilde{f} \in \mathfrak{H}$ with the following property: there exists $\beta_{\widetilde{f}} \in \mathcal{I}$ such that for some (and hence for all) function $f \in \widetilde{f}$ the equality $\Delta(t)f(t) = 0$ holds a.e. on $(\beta_{\widetilde{f}}, b)$.

Let θ and $\mathbb{H}'_0$ be subspaces in $\mathbb{H}$, let $K = K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$ be an operator such that $\ker K_\theta = \{0\}$ and $K_\theta \mathbb{H}'_0 = \theta$ and let $\varphi_K(\cdot, \lambda)(\in [\mathbb{H}'_0, \mathbb{H}])$ be an operator solution of (3.3) satisfying $\varphi_K(a, \lambda) = K$, $\lambda \in \mathbb{C}$. With each $\widetilde{f} \in \mathfrak{H}_b$ we associate the function $\widehat{f}(\cdot) : \mathbb{R} \to \mathbb{H}'_0$ given by

$$\widehat{f}(s) = \int_\mathcal{I} \varphi_K^*(t, s)\Delta(t)f(t)\, dt, \quad f(\cdot) \in \widetilde{f}. \tag{3.16}$$

One can easily prove that $\widehat{f}(\cdot)$ is a continuous function on $\mathbb{R}$.

Recall that an operator $V \in [\mathfrak{H}_1, \mathfrak{H}_2]$ is a partial isometry if $||Vf|| = ||f||$ for all $f \in \mathfrak{H}_1 \ominus \ker V$.

Definition 3.5. A distribution function $\sigma(\cdot) : \mathbb{R} \to [\mathbb{H}'_0]$ is called a q-pseudospectral function of the system (3.3) (with respect to the operator $K = K_\theta$) if $\widehat{f} \in \mathcal{L}^2(\sigma; \mathbb{H}'_0)$ for all $\widetilde{f} \in \mathfrak{H}_b$ and the operator $V\widetilde{f} := \pi_\sigma \widehat{f}$, $\widetilde{f} \in \mathfrak{H}_b$, admits a continuation to a partial isometry $V_\sigma \in [\mathfrak{H}, L^2(\sigma; \mathbb{H}'_0)]$. The operator V_σ is called the (generalized) Fourier transform corresponding to $\sigma(\cdot)$.

Clearly, if $\sigma(\cdot)$ is a q-pseudospectral function, then for each $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ there exists a unique (up to the seminorm in $\mathcal{L}^2(\sigma; \mathbb{H}'_0)$) function $\widehat{f}(\cdot) \in \mathcal{L}^2(\sigma; \mathbb{H}'_0)$ such that

$$\lim_{\beta \uparrow b} \left\| \widehat{f}(\cdot) - \int\limits_{[a,\beta)} \varphi^*_K(t, \cdot) \Delta(t) f(t)\, dt \right\|_{\mathcal{L}^2(\sigma; \mathbb{H}'_0)} = 0.$$

The function $\widehat{f}(\cdot)$ is called the Fourier transform of the function $f(\cdot)$.

Remark 3.6. Similarly to [10, 33] (see also [26, Proposition 3.4]) one proves that for each q-pseudospectral function $\sigma(\cdot)$

$$V^*_\sigma \widetilde{g} = \pi_\Delta \left(\int_{\mathbb{R}} \varphi_K(\cdot, s)\, d\sigma(s) g(s) \right), \quad \widetilde{g} \in L^2(\sigma; \mathbb{H}'_0), \quad g(\cdot) \in \widetilde{g}, \quad (3.17)$$

where the integral converges in the seminorm of $\mathcal{L}^2_\Delta(\mathcal{I})$. Hence for each function $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ with $\pi_\Delta f \in \mathfrak{H} \ominus \ker V_\sigma$ the equality (the inverse Fourier transform)

$$f(t) = \int_{\mathbb{R}} \varphi_K(t, s)\, d\sigma(s) \widehat{f}(s)$$

is valid. Therefore the natural problem is a characterization of q-pseudospectral functions $\sigma(\cdot)$ with the minimally possible kernel of V_σ.

The following lemma can be proved in the same way as Lemma 3.7 in [27].

Lemma 3.7. *Assume that θ and $\mathbb{H}'_0$ are subspaces in $\mathbb{H}$, $\sigma(\cdot)$ is a q-pseudospectral function (with respect to $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$), V_σ is the corresponding Fourier transform and $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is given by (3.12). Then there exist a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and a self-adjoint operator $\widetilde{T}_0$ in $\widetilde{\mathfrak{H}}_0 := \widetilde{\mathfrak{H}} \ominus \ker V_\sigma$ such that $\widetilde{T}_0 \subset T^*_{\widetilde{\mathfrak{H}}}$ (here $T^*_{\widetilde{\mathfrak{H}}} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ is the linear relation adjoint to T in $\widetilde{\mathfrak{H}}$).*

By using Lemma 3.7 one can prove similarly to [27, Proposition 3.8] the following theorem.

Theorem 3.8. *Let the assumptions of Lemma 3.7 be satisfied and let $\mathrm{mul}\, T$ be the multivalued part of T (see Lemma 3.3, (2)). Then*

$$\mathrm{mul}\, T \subset \ker V_\sigma \qquad (3.18)$$

Definition 3.9. Under the assumptions of Theorem 3.8 a q-pseudospectral function $\sigma(\cdot)$ of the system (3.3) with $\ker V_\sigma = \mathrm{mul}\, T$ is called a pseudospectral function .

Definition 3.10. Let θ and $\mathbb{H}'_0$ be subspaces in $\mathbb{H}$. A distribution function $\sigma(\cdot) : \mathbb{R} \to [\mathbb{H}'_0]$ is called a spectral function of the system (3.3) (with respect to $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$) if for every $\widetilde{f} \in \mathfrak{H}_b$ the inclusion $\widehat{f} \in \mathcal{L}^2(\sigma; \mathbb{H}'_0)$ holds and the Parseval equality $||\widehat{f}||_{\mathcal{L}^2(\sigma;\mathbb{H}'_0)} = ||\widetilde{f}||_{\mathfrak{H}}$ is valid (for $\widehat{f}$ see (3.16)).

The number $n_\sigma := \dim \mathbb{H}'_0 (= \dim \theta)$ is called a dimension of the spectral function $\sigma(\cdot)$.

If for a given $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$ a pseudospectral function $\sigma(\cdot)$ exists, then in view of (3.18) it is a q-pseudospectral function with the minimally possible $\ker V_\sigma$ (see the problem posted in Remark 3.6). Moreover, (3.18) yields the following proposition.

Proposition 3.11. *Let θ and $\mathbb{H}'_0$ be subspaces in $\mathbb{H}$ and let $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be given by (3.12). If $\operatorname{mul} T \neq \{0\}$, then the set of spectral functions (with respect to $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$) is empty. If $\operatorname{mul} T = \{0\}$, then the sets of spectral and pseudospectral functions (with respect to K_θ) coincide.*

A connection between different q-pseudospectral functions corresponding to the same subspace $\theta \subset \mathbb{H}$ is specified in the following proposition.

Proposition 3.12. *Assume that θ and $\mathbb{H}'_{0j}$ are subspaces in $\mathbb{H}$ and $K_j = K_{j\theta} \in [\mathbb{H}'_{0j}, \mathbb{H}]$ are operators such that $\ker K_j = \{0\}$ and $K_j \mathbb{H}'_{0j} = \theta$, $j \in \{1, 2\}$. Then: (1) there exists a unique isomorphism $X \in [\mathbb{H}'_{01}, \mathbb{H}'_{02}]$ such that $K_1 = K_2 X$; (2) the equality $\sigma_2(s) = X\sigma_1(s)X^*$ gives a bijective correspondence between all q-pseudospectral functions $\sigma_1(\cdot)$ (with respect to K_1) and $\sigma_2(\cdot)$ (with respect to K_2) of the system (3.3). Moreover $\sigma_2(\cdot)$ is a pseudospectral or spectral function if and only if so is $\sigma_1(\cdot)$.*

Proof. Statement (1) is obvious To prove statement (2) assume that $\sigma_1(\cdot)$ is a q-pseudospectral function (with respect to K_1) and $\sigma_2(\cdot)$ is an $[\mathbb{H}'_{02}]$ -valued distribution function given by $\sigma_2(s) = X\sigma_1(s)X^*$. One can easily verify that the equality $(\mathcal{U}g)(s) = X^{-1*}g(s)$, $g = g(\cdot) \in \mathcal{L}^2(\sigma_1; \mathbb{H}'_{01})$, defines a surjective linear operator $\mathcal{U} : \mathcal{L}^2(\sigma_1; \mathbb{H}'_{01}) \to \mathcal{L}^2(\sigma_2; \mathbb{H}'_{02})$ satisfying $||\mathcal{U}g||_{\mathcal{L}^2(\sigma_2;\mathbb{H}'_{02})} = ||g||_{\mathcal{L}^2(\sigma_1;\mathbb{H}'_{01})}$. Therefore the equality $U\widetilde{g} = \pi_{\sigma_2}\mathcal{U}g$, $\widetilde{g} \in L^2(\sigma_1; \mathbb{H}'_{01}), g \in \widetilde{g}$, defines a unitary operator $U \in [L^2(\sigma_1; \mathbb{H}'_{01}), L^2(\sigma_2; \mathbb{H}'_{02})]$. Next assume that $\widetilde{f} \in \mathfrak{H}_b$, $f(\cdot) \in \widetilde{f}$ and $\widehat{f}_j(\cdot)$ is the Fourier transform of $f(\cdot)$ given by (3.16) with $\varphi_K(\cdot, \lambda) = \varphi_{K_j}(\cdot, \lambda)$, $j \in \{1, 2\}$. Since obviously $\varphi_{K_1}(t, s) = \varphi_{K_2}(t, s)X$, it follows that $\widehat{f}_2(s) = X^{-1*}\widehat{f}_1(s)$. Hence $\widehat{f}_2 = \mathcal{U}\widehat{f}_1 \in \mathcal{L}^2(\sigma_2; \mathbb{H}'_{02})$ and $\pi_{\sigma_2}\widehat{f}_2 = U\pi_{\sigma_1}\widehat{f}_1 = UV_{\sigma_1}\widetilde{f}$. This implies that the operator $V_2\widetilde{f} := \pi_{\sigma_2}\widehat{f}_2$, $\widetilde{f} \in \mathfrak{H}_b$, admits a continuation to the partial isometry $V_{\sigma_2} = UV_{\sigma_1}(\in [\mathfrak{H}, L^2(\sigma_2; \mathbb{H}'_{02})])$ with $\ker V_{\sigma_2} = \ker V_{\sigma_1}$. Therefore $\sigma_2(\cdot)$ is a q-pseudospectral function (with respect to K_2); moreover, $\sigma_2(\cdot)$ is a pseudospectral or spectral function if and only if so is $\sigma_1(\cdot)$. $\qquad\square$

Remark 3.13. It follows from Proposition 3.12 that a q-pseudospectral (in particular pseudospectral) function $\sigma(\cdot)$ with respect to the operator $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$ is uniquely characterized by the subspace $\theta \subset \mathbb{H}$.

Under the assumptions of Theorem 3.8 we let $\mathfrak{H}_0 := \mathfrak{H} \ominus \operatorname{mul} T$, so that
$$\mathfrak{H} = \operatorname{mul} T \oplus \mathfrak{H}_0.$$

Moreover, for a pseudospectral function $\sigma(\cdot)$ we denote by $V_0 = V_{0,\sigma}$ the isometry from $\mathfrak{H}_0$ to $L^2(\sigma; \mathbb{H}'_0)$ given by $V_{0,\sigma} := V_\sigma \upharpoonright \mathfrak{H}_0$. Next assume that $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ is a Hilbert space and $\widetilde{T} = \widetilde{T}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ with $\operatorname{mul} \widetilde{T} = \operatorname{mul} T$. In the following we put $\widetilde{\mathfrak{H}}_0 := \widetilde{\mathfrak{H}} \ominus \operatorname{mul} T$, so that $\mathfrak{H}_0 \subset \widetilde{\mathfrak{H}}_0$ and
$$\widetilde{\mathfrak{H}} = \operatorname{mul} T \oplus \widetilde{\mathfrak{H}}_0.$$

Denote also by $\widetilde{T}_0$ the operator part of $\widetilde{T}$. Clearly, $\widetilde{T}_0$ is a self-adjoint operator in $\widetilde{\mathfrak{H}}_0$.

Proposition 3.14. *Assume that θ and $\mathbb{H}'_0$ are subspaces in $\mathbb{H}$, $\sigma(\cdot)$ is a pseudospectral function (with respect to $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$) and $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is given by (3.12). Moreover, let $L_0 = V_\sigma \mathfrak{H}$ and let $\Lambda_\sigma = \Lambda_\sigma^*$ be the multiplication operator in $L^2(\sigma; \mathbb{H}'_0)$ defined by*
$$\operatorname{dom} \Lambda_\sigma = \{\widetilde{f} \in L^2(\sigma; \mathbb{H}'_0) : sf(s) \in \mathcal{L}^2(\sigma; \mathbb{H}'_0)$$
$$\text{for some (and hence for all) } f(\cdot) \in \widetilde{f}\}$$
$$\Lambda_\sigma \widetilde{f} = \pi_\sigma(sf(s)), \quad \widetilde{f} \in \operatorname{dom} \Lambda_\sigma, \quad f(\cdot) \in \widetilde{f}.$$

Then T is a symmetric extension of $T_{\min}$ and there exist a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and an exit space self-adjoint extension $\widetilde{T} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ of T such that $\operatorname{mul} \widetilde{T} = \operatorname{mul} T$ and the relative spectral function
$$F(t) = P_{\widetilde{\mathfrak{H}},\mathfrak{H}} E_{\widetilde{T}}((-\infty, t)) \upharpoonright \mathfrak{H}$$

of T satisfies
$$((F(\beta)-F(\alpha))\widetilde{f}, \widetilde{f}) = \int_{[\alpha,\beta)} (d\sigma(s)\widehat{f}(s), \widehat{f}(s)), \widetilde{f} \in \mathfrak{H}_b, -\infty < \alpha < \beta < \infty.$$
$$(3.19)$$
Moreover, there exists a unitary operator $\widetilde{V} \in [\widetilde{\mathfrak{H}}_0, L^2(\sigma; \mathbb{H}'_0)]$ such that $\widetilde{V} \upharpoonright \mathfrak{H}_0 = V_{0,\sigma}$ and the operators $\widetilde{T}_0$ and Λ_σ are unitarily equivalent by means of $\widetilde{V}$.

If in addition the operator Λ_σ is L_0-minimal, then the extension $\widetilde{T}$ is unique(up to the equivalence) and $\widetilde{T} \in \widetilde{\operatorname{Self}}_0(T)$ (that is, $\widetilde{T}$ is $\mathfrak{H}$-minimal).

Proof. By using Lemma 3.7 one proves as in [27, Proposition 5.6] the following statement:

(S) There is a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and a relation $\widetilde{T} = \widetilde{T}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ such that $\mathrm{mul}\, \widetilde{T} = \mathrm{mul}\, T$, $T \subset \widetilde{T}$ and (3.19) holds with $F(t) = P_{\widetilde{\mathfrak{H}},\mathfrak{H}} E_{\widetilde{T}}((-\infty, t)) \upharpoonright \mathfrak{H}$.

Moreover, by Lemma 3.3, (1) $T_{\min} \subset T$. Therefore T is a symmetric extension $T_{\min}$ and $F(\cdot)$ is a spectral function of T. Other statements of the proposition can be proved as in [27, Proposition 5.6]. $\qquad\square$

Definition 3.15. [3,11] System (3.3) is called definite if $\mathcal{N} = \{0\}$ or, equivalently, if for some (and hence for all) $\lambda \in \mathbb{C}$ there exists only a trivial solution $y = 0$ of this system satisfying $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$).

Proposition 3.16. *Let θ be a subspaces in $\mathbb{H}$ and let $\sigma(\cdot)$ be a pseudospectral function (with respect to $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$). Then $\theta^\times \cap \theta^\times_\mathcal{N} \in Sym(\mathbb{H})$. If in addition the system is definite, then $\theta^\times \in Sym(\mathbb{H})$.*

Proof. The first statement is immediate from Proposition 3.14 and Lemma 3.3, (3). For a definite system $\theta_\mathcal{N} = \{0\}$ and hence $\theta^\times_\mathcal{N} = \mathbb{H}$. This yields the second statement. $\qquad\square$

Remark 3.17. Proposition 3.16 shows that a necessary condition for existence of a pseudospectral function for a given θ is $\theta^\times \cap \theta^\times_\mathcal{N} \in Sym(\mathbb{H})$. Clearly this condition is satisfied if $\theta^\times \in Sym(\mathbb{H})$.

4. m-functions of symmetric systems

4.1. Boundary pairs and boundary triplets for symmetric systems

Definition 4.1. Let θ be a subspace in $\mathbb{H}$. System (3.3) will be called θ-definite if the conditions $y \in \mathcal{N}$ and $y(a) \in \theta$ yield $y = 0$.

Remark 4.2. If system is definite then obviously it is θ-definite for any $\theta \in \mathbb{H}$. Hence θ-definiteness is generally speaking a weaker condition then definiteness. At the same time in the case $\theta = \mathbb{H}(\Leftrightarrow \theta^\times = \{0\})$ definiteness of the system is the same as θ-definiteness.

The following assertion directly follows from definition of $T_{\min}$ and (3.13), (2.1).

Assertion 4.3. (1) *The equality $\mathrm{mul}\, T_{\min} = \{0\}$ is equivalent to the following condition:*

(C0) If $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ and there exists a solution $y(\cdot)$ of (3.14) such that $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$), $y(a) = 0$ and $y \in \mathcal{D}_b$, then $\Delta(t)f(t) = 0$ (a.e. on $\mathcal{I}$).

(2) Let $\theta^\times \in Sym(\mathbb{H})$, let system (3.3) be θ-definite and let T be the relation (3.12). Then the equalities $\mathrm{mul}\, T = \{0\}$, $\mathrm{mul}\, T = \mathrm{mul}\, T^*$ and $\mathrm{mul}\, T^* = \{0\}$ are equivalent to the following conditions (C1), (C2) and (C3) respectively:

(C1) If $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ and there exists a solution $y(\cdot)$ of the system (3.14) such that $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$), $y(a) \in \theta^\times$ and $y \in \mathcal{D}_b$, then $\Delta(t)f(t) = 0$ (a.e. on $\mathcal{I}$).

(C2) If $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ and $y(\cdot)$ is a solution of (3.14) such that $y(a) \in \theta$ and $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$), then $y(\cdot) \in \mathcal{D}_b$ and $y(a) \in \theta^\times$.

(C3) If $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ and there exists a solution $y(\cdot)$ of (3.14) satisfying $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) and $y(a) \in \theta$, then $\Delta(t)f(t) = 0$ (a.e. on $\mathcal{I}$).

The following proposition can be proved in the same way as Proposition 5.5 in [27].

Proposition 4.4. *Assume that θ and $\mathbb{H}'_0$ are subspaces in $\mathbb{H}$, $\sigma(\cdot)$ is a q-pseudospectral function (with respect to $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$)and $L_0 := V_\sigma \mathfrak{H}$. If system is θ-definite, then the multiplication operator Λ_σ is L_0-minimal.*

Below within this section we suppose the following assumptions:

(A1) θ is a subspace in $\mathbb{H}$ and $\theta^\times \in Sym(\mathbb{H})$. Moreover, system (3.3) is θ-definite and satisfies $N_- \leq N_+$.

(A2) H_1 is a subspace in H, $\mathbb{H}_0 \subset \mathbb{H}$ is the subspace (3.9), $\widetilde{U} \in [\mathbb{H}]$ is an operator satisfying $\widetilde{U}^* J \widetilde{U} = J$ and $\widetilde{U} \mathbb{H}_0 = \theta$, $\Gamma_a : \mathrm{dom}\, \mathcal{T}_{\max} \to \mathbb{H}$ is the linear operator given by $\Gamma_a y = \widetilde{U}^{-1} y(a)$, $y \in \mathrm{dom}\, \mathcal{T}_{\max}$, and

$$\Gamma_a = (\Gamma^1_{0a}, \Gamma^2_{0a}, \widehat{\Gamma}_a, \Gamma^2_{1a}, \Gamma^1_{1a})^\top : \mathrm{dom}\, \mathcal{T}_{\max} \to H^\perp_1 \oplus H_1 \oplus \widehat{H} \oplus H_1 \oplus H^\perp_1 \tag{4.1}$$

is the block representation of Γ_a in accordance with the decomposition (3.11) of $\mathbb{H}$.

(A3) $\widetilde{\mathcal{H}}_b$ and $\mathcal{H}_b \subset \widetilde{\mathcal{H}}_b$ are finite dimensional Hilbert spaces and

$$\Gamma_b = (\Gamma_{0b}, \widehat{\Gamma}_b, \Gamma_{1b})^\top : \mathrm{dom}\, \mathcal{T}_{\max} \to \widetilde{\mathcal{H}}_b \oplus \widehat{H} \oplus \mathcal{H}_b \tag{4.2}$$

is a surjective linear operator satisfying for all $y, z \in \mathrm{dom}\, \mathcal{T}_{\max}$ the following identity

$$[y, z]_b = (\Gamma_{0b} y, \Gamma_{1b} z) - (\Gamma_{1b} y, \Gamma_{0b} z) + i(P_{\mathcal{H}^\perp_b} \Gamma_{0b} y, P_{\mathcal{H}^\perp_b} \Gamma_{0b} z) + i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z) \tag{4.3}$$

(here $\mathcal{H}^\perp_b = \widetilde{\mathcal{H}}_b \ominus \mathcal{H}_b$).

Remark 4.5. Existence of the operators $\widetilde{U}$ in assumption (A2) and Γ_b in assumption (A3) follows from Lemma 3.1, (3) and [1, Lemma 3.4] respectively. Moreover, in the case $N_+ = N_-$ (and only in this case) one has $\widetilde{\mathcal{H}}_b = \mathcal{H}_b$ and the identity (4.3) takes the form

$$[y, z]_b = (\Gamma_{0b}y, \Gamma_{1b}z) - (\Gamma_{1b}y, \Gamma_{0b}z) + i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z), \quad y, z \in \operatorname{dom} \mathcal{T}_{\max}.$$

Observe also that $\Gamma_b y$ is a singular boundary value of a function $y \in \operatorname{dom} \mathcal{T}_{\max}$ at the endpoint b (for more details see [1, Remark 3.5]).

The following lemma directly follows from definition of the operator Γ_a and its block representation (4.1).

Lemma 4.6. *Let $Y(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{K}, \mathbb{H}]$ be an operator solution of (3.3). Then*

$$\widetilde{U}^{-1}Y(a, \lambda) = \Gamma_a Y(\lambda) = (P_{\mathbb{H}, \mathbb{H}_0}\Gamma_a Y(\lambda), \Gamma_{1a}^1 Y(\lambda))^\top : \mathcal{K} \to \mathbb{H}_0 \oplus H_1^\perp, \tag{4.4}$$

where

$$P_{\mathbb{H}, \mathbb{H}_0}\Gamma_a = (\Gamma_{0a}^1, \Gamma_{0a}^2, \widehat{\Gamma}_a, \Gamma_{1a}^2)^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1. \tag{4.5}$$

Proposition 4.7. *Let $\mathcal{H}_0$ and $\mathcal{H}_1 \subset \mathcal{H}_0$ be finite dimensional Hilbert spaces and let $\Gamma'_j : \operatorname{dom} \mathcal{T}_{\max} \to \mathcal{H}_j$, $j \in \{0, 1\}$, be linear operators given by*

$$\mathcal{H}_0 = H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b, \quad \mathcal{H}_1 = H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \mathcal{H}_b \tag{4.6}$$

$$\Gamma'_0 = (-\Gamma_{1a}^1, -\Gamma_{1a}^2, i(\widehat{\Gamma}_a - \widehat{\Gamma}_b), \Gamma_{0b})^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \tag{4.7}$$

$$\Gamma'_1 = (\Gamma_{0a}^1, \Gamma_{0a}^2, \tfrac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b), -\Gamma_{1b})^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \mathcal{H}_b. \tag{4.8}$$

Then $\dim \mathcal{H}_0 = N_+$, $\dim \mathcal{H}_1 = N_-$ and a pair $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ with a linear relation $\Gamma : \mathfrak{H} \oplus \mathfrak{H} \to \mathcal{H}_0 \oplus \mathcal{H}_1$ defined by

$$\Gamma = \{\{\widetilde{\pi}_\Delta\{y, f\}, \Gamma'_0 y \oplus \Gamma'_1 y\} : \{y, f\} \in \mathcal{T}_{\max}\} \tag{4.9}$$

is a boundary pair for $T_{\max}$ such that $\mathcal{K}_\Gamma = \{0\}$ (for $\mathcal{K}_\Gamma$ see (2.15)).

Proof. The fact that $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ is a boundary pair for $T_{\max}$ as well as the equalities $\dim \mathcal{H}_0 = N_+$, $\dim \mathcal{H}_1 = N_-$ directly follow from [22, Theorem 5.3]. Next, according to [22] $\operatorname{mul} \Gamma = \{\{\Gamma'_0 y, \Gamma'_1 y\} : y \in \mathcal{N}\}$

and hence $\mathcal{K}_\Gamma = \{\Gamma'_1 y : y \in \mathcal{N}$ and $\Gamma'_0 y = 0\}$. Moreover, the equalities $\widetilde{U}^{-1}\theta = \mathbb{H}_0$ and (3.11), (4.1) yield the equivalence

$$y(a) \in \theta \iff \Gamma^1_{1a} y = 0, \quad y \in \operatorname{dom} \mathcal{T}_{\max}. \tag{4.10}$$

Since the system is θ-definite, this implies the equality $\mathcal{K}_\Gamma = \{0\}$. $\qquad \square$

Definition 4.8. The boundary pair $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ constructed in Proposition 4.7 is called a decomposing boundary pair for $T_{\max}$.

Let $\dot{\mathcal{H}}_0$ and $\dot{\mathcal{H}}_1 \subset \dot{\mathcal{H}}_0$ be finite dimensional Hilbert spaces and $\dot{\Gamma}'_j :$ $\operatorname{dom} \mathcal{T}_{\max} \to \dot{\mathcal{H}}_j$, $j \in \{0, 1\}$, be linear operators given by

$$\dot{\mathcal{H}}_0 = H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b, \quad \dot{\mathcal{H}}_1 = H_1 \oplus \widehat{H} \oplus \mathcal{H}_b \tag{4.11}$$

$$\dot{\Gamma}'_0 = (-\Gamma^2_{1a}, i(\widehat{\Gamma}_a - \widehat{\Gamma}_b), \Gamma_{0b})^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \tag{4.12}$$

$$\dot{\Gamma}'_1 = (\Gamma^2_{0a}, \tfrac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b), -\Gamma_{1b})^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1 \oplus \widehat{H} \oplus \mathcal{H}_b. \tag{4.13}$$

It follows from (4.6)–(4.8) that

$$\mathcal{H}_0 = H_1^\perp \oplus \dot{\mathcal{H}}_0, \qquad \mathcal{H}_1 = H_1^\perp \oplus \dot{\mathcal{H}}_1 \tag{4.14}$$

$$\Gamma'_0 = (-\Gamma^1_{1a}, \dot{\Gamma}'_0)^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1^\perp \oplus \dot{\mathcal{H}}_0 \tag{4.15}$$

$$\Gamma'_1 = (\Gamma^1_{0a}, \dot{\Gamma}'_1)^\top : \operatorname{dom} \mathcal{T}_{\max} \to H_1^\perp \oplus \dot{\mathcal{H}}_1. \tag{4.16}$$

Proposition 4.9. *Let $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be given by (3.12). Then:*
(1) T is a symmetric extension of $T_{\min}$ and the following equalities hold:

$$T = \{\widetilde{\pi}_\Delta \{y, f\} : \{y, f\} \in \mathcal{T}_{\max}, \ y \in \mathcal{D}_b, \ \Gamma^1_{1a} y = 0,$$
$$\Gamma^2_{0a} y = \Gamma^2_{1a} y = 0, \ \widehat{\Gamma}_a y = 0\} \tag{4.17}$$
$$T^* = \{\widetilde{\pi}_\Delta \{y, f\} : \{y, f\} \in \mathcal{T}_{\max}, \ \Gamma^1_{1a} y = 0\} \tag{4.18}$$

(2) For every $\{\widetilde{y}, \widetilde{f}\} \in T^*$ there exists a unique $y \in \operatorname{dom} \mathcal{T}_{\max}$ such that $\Gamma^1_{1a} y = 0$, $\pi_\Delta y = \widetilde{y}$ and $\{y, f\} \in \mathcal{T}_{\max}$ for any $f \in \widetilde{f}$.
(3) The collection $\dot{\Pi} = \{\dot{\mathcal{H}}_0 \oplus \dot{\mathcal{H}}_1, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ with operators $\dot{\Gamma}_j : T^* \to \dot{\mathcal{H}}_j$ of the form

$$\dot{\Gamma}_0\{\widetilde{y}, \widetilde{f}\} = \dot{\Gamma}'_0 y, \quad \dot{\Gamma}_1\{\widetilde{y}, \widetilde{f}\} = \dot{\Gamma}'_1 y, \quad \{\widetilde{y}, \widetilde{f}\} \in T^* \tag{4.19}$$

is a boundary triplet for T^*. In (4.19) $y \in \operatorname{dom} \mathcal{T}_{\max}$ is uniquely defined by $\{\widetilde{y}, \widetilde{f}\}$ in accordance with statement (2).

Proof. (1) Since $\widetilde{U}^{-1}\theta = \mathbb{H}_0$ and $\widetilde{U}^{-1}\theta^\times = \mathbb{H}_0^\times = H_1^\perp \oplus \{0\} \oplus \{0\} \oplus \{0\}$, the equivalences (4.10) and

$$y(a) \in \theta^\times \iff (\Gamma_{1a}^1 y = 0,\ \Gamma_{0a}^2 y = \Gamma_{1a}^2 y = 0,\ \widehat{\Gamma}_a y = 0), \quad y \in \operatorname{dom} \mathcal{T}_{\max}$$

are valid. This and (3.12), (3.13) yield (4.17) and (4.18).

By using θ-definiteness of the system one proves statement (2) similarly to [27, Proposition 4.5, (2)].

(3) Equalities (4.15), (4.16) and identity (2.14) for the decomposing boundary pair yield the Green's identity (2.9) for operators $\dot{\Gamma}_0$ and $\dot{\Gamma}_1$. To prove surjectivity of the operator $(\dot{\Gamma}_0, \dot{\Gamma}_1)^\top$ it is sufficient to show that

$$\ker \dot{\Gamma}_0 \cap \ker \dot{\Gamma}_1 = T, \qquad \dim \dot{\mathcal{H}}_0 = n_+(T), \qquad \dim \dot{\mathcal{H}}_1 = n_-(T). \quad (4.20)$$

Clearly, $\{\widetilde{y}, \widetilde{f}\} \in \ker \dot{\Gamma}_0 \cap \ker \dot{\Gamma}_1$ if and only if there is $\{y, f\} \in \mathcal{T}_{\max}$ such that $\widetilde{\pi}_\Delta \{y, f\} = \{\widetilde{y}, \widetilde{f}\}$ and $\Gamma_{1a}^1 y = 0,\ \Gamma_{0a}^2 y = \Gamma_{1a}^2 y = 0,\ \widehat{\Gamma}_a y = 0$, $\Gamma_b y = 0$. Moreover, in view of (4.3) and surjectivity of the operator Γ_b the equivalence $\Gamma_b y = 0 \iff y \in \mathcal{D}_b$ is valid. This yields the first equality in (4.20). Next assume that

$$\mathcal{T} = \{\{y, f\} \in \mathcal{T}_{\max} : y \in \mathcal{D}_b,\ y(a) \in \theta^\times \cap \theta_\mathcal{N}^\times\}.$$

It follows from (3.8) and (3.6) that $\dim(\operatorname{dom} \mathcal{T}/\operatorname{dom} \mathcal{T}_a) = \dim(\theta^\times \cap \theta_\mathcal{N}^\times)$ and $T = \widetilde{\pi}_\Delta \mathcal{T}$. If $\{y, f\} \in \mathcal{T}$ and $\widetilde{\pi}_\Delta \{y, f\} = 0$, then $y \in \mathcal{N}$ and $y(a) \in \theta^\times \subset \theta$. Therefore in view of θ-definiteness $y = 0$ and consequently $\ker \widetilde{\pi}_\Delta \restriction \mathcal{T} = \{0\}$. This and the obvious equality $\dim(\mathcal{T}/\mathcal{T}_a) = \dim(\operatorname{dom} \mathcal{T}/\operatorname{dom} \mathcal{T}_a)$ imply that

$$\dim(T/T_{\min}) = \dim(\theta^\times \cap \theta_\mathcal{N}^\times). \quad (4.21)$$

In view of θ-definiteness one has $\theta \cap \theta_\mathcal{N} = \{0\}$. Since obviously $\theta^\times \cap \theta_\mathcal{N}^\times = (\theta \dotplus \theta_N)^\times$, it follows that

$$\dim(\theta^\times \cap \theta_\mathcal{N}^\times) = n - \dim \theta - \dim \theta_\mathcal{N} = \operatorname{codim} \theta - \dim \mathcal{N}. \quad (4.22)$$

Combining (4.21) and (4.22) with the well known equality $n_\pm(T) = n_\pm(T_{\min}) - \dim(T/T_{\min})$ and taking (3.7) into account on gets $n_\pm(T) = N_\pm - \operatorname{codim} \theta$. Moreover, the equality $\widetilde{U}^{-1}\theta = \mathbb{H}_0$ yields $\operatorname{codim} \theta = \dim H_1^\perp$ and according to Proposition 4.7 $\dim \mathcal{H}_0 = N_+,\ \dim \mathcal{H}_1 = N_-$. This implies that

$$n_+(T) = \dim \mathcal{H}_0 - \dim H_1^\perp, \quad n_-(T) = \dim \mathcal{H}_1 - \dim H_1^\perp \quad (4.23)$$

Now combining (4.23) with (4.14) one obtains the second and third equalities in (4.20). $\qquad\square$

4.2. $\mathcal{L}^2_\Delta$-solutions of boundary problems

Definition 4.10. Let $\dot{\mathcal{H}}_0$ and $\dot{\mathcal{H}}_1$ be given by (4.11). A boundary parameter is a pair

$$\tau = \tau(\lambda) = \{C_0(\lambda), C_1(\lambda)\} \in \widetilde{R}(\dot{\mathcal{H}}_0, \dot{\mathcal{H}}_1), \quad \lambda \in \mathbb{C}_+, \qquad (4.24)$$

where $C_j(\lambda)(\in [\dot{\mathcal{H}}_j, \dot{\mathcal{H}}_0])$, $j \in \{0, 1\}$, are holomorphic operator functions satisfying (2.7).

In the case $N_+ = N_-$ (and only in this case) $\widetilde{\mathcal{H}}_b = \mathcal{H}_b$, $\dot{\mathcal{H}}_0 = \dot{\mathcal{H}}_1 =: \dot{\mathcal{H}}$ and $\tau \in \widetilde{R}(\dot{\mathcal{H}})$. If in addition $\tau = \tau(\lambda) \in \widetilde{R}^0(\dot{\mathcal{H}})$ is an operator pair (2.8), then a boundary parameter τ will be called self-adjoint.

Let τ be a boundary parameter (4.24). For a given $f \in \mathcal{L}^2_\Delta(\mathcal{I})$ consider the boundary value problem

$$Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I} \qquad (4.25)$$

$$\Gamma^1_{1a}y = 0, \quad C_0(\lambda)\dot{\Gamma}'_0 y - C_1(\lambda)\dot{\Gamma}'_1 y = 0, \quad \lambda \in \mathbb{C}_+ \qquad (4.26)$$

A function $y(\cdot, \cdot) : \mathcal{I} \times \mathbb{C}_+ \to \mathbb{H}$ is called a solution of this problem if for each $\lambda \in \mathbb{C}_+$ the function $y(\cdot, \lambda)$ belongs to $AC(\mathcal{I}; \mathbb{H}) \cap \mathcal{L}^2_\Delta(\mathcal{I})$ and satisfies the equation (4.25) a.e. on $\mathcal{I}$ (so that $y \in \operatorname{dom}\mathcal{T}_{\max}$) and the boundary conditions (4.26).

The following theorem is a consequence of Theorem 3.11 in [24] applied to the boundary triplet $\dot{\Pi}$ for T^*.

Theorem 4.11. *Let under the assumptions* (A1)–(A3) *T be a symmetric relation* (3.12) *(or equivalently* (4.17)*). If τ is a boundary parameter* (4.24)*, then for every $f \in \mathcal{L}^2_\Delta(\mathcal{I})$ the problem* (4.25)*,* (4.26) *has a unique solution $y(t, \lambda) = y_f(t, \lambda)$ and the equality*

$$R(\lambda)\widetilde{f} = \pi_\Delta(y_f(\cdot, \lambda)), \quad \widetilde{f} \in \mathfrak{H}, \quad f \in \widetilde{f}, \quad \lambda \in \mathbb{C}_+$$

defines a generalized resolvent $R(\lambda) =: R_\tau(\lambda)$ of T. Conversely, for each generalized resolvent $R(\lambda)$ of T there exists a unique boundary parameter τ such that $R(\lambda) = R_\tau(\lambda)$. Moreover, if $N_+ = N_-$, then $R_\tau(\lambda)$ is a canonical resolvent if and only if τ is a self-adjoint boundary parameter (2.8)*. In this case $R_\tau(\lambda) = (\widetilde{T}_\tau - \lambda)^{-1}$, where $\widetilde{T}_\tau \in \operatorname{Self}(T)$ is given by*

$$\widetilde{T}_\tau = \{\widetilde{\pi}_\Delta\{y, f\} : \{y, f\} \in \mathcal{T}_{\max}, \Gamma^1_{1a}y = 0, C_0\dot{\Gamma}'_0 y - C_1\dot{\Gamma}'_1 y = 0\}. \quad (4.27)$$

Proposition 4.12. *For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique collection of operator solutions $\xi_1(\cdot, \lambda) \in \mathcal{L}^2_\Delta[H^\perp_1, \mathbb{H}]$, $\xi_2(\cdot, \lambda) \in \mathcal{L}^2_\Delta[H_1, \mathbb{H}]$ and*

$\xi_3(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\widehat{H}, \mathbb{H}]$ *of the system* (3.3) *satisfying the boundary conditions*

$$\Gamma^1_{1a}\xi_1(\lambda) = -I_{H^\perp_1}, \quad \Gamma^2_{1a}\xi_1(\lambda) = 0, \quad \widehat{\Gamma}_a\xi_1(\lambda) = \widehat{\Gamma}_b\xi_1(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{4.28}$$

$$\Gamma_{0b}\xi_1(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \qquad P_{\widetilde{\mathcal{H}}_b, \mathcal{H}_b}\Gamma_{0b}\xi_1(\lambda) = 0, \quad \lambda \in \mathbb{C}_-. \tag{4.29}$$

$$\Gamma^1_{1a}\xi_2(\lambda) = 0, \quad \Gamma^2_{1a}\xi_2(\lambda) = -I_{H_1}, \quad \widehat{\Gamma}_a\xi_2(\lambda) = \widehat{\Gamma}_b\xi_2(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{4.30}$$

$$\Gamma_{0b}\xi_2(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \qquad P_{\widetilde{\mathcal{H}}_b, \mathcal{H}_b}\Gamma_{0b}\xi_2(\lambda) = 0, \quad \lambda \in \mathbb{C}_-. \tag{4.31}$$

$$\Gamma^1_{1a}\xi_3(\lambda) = 0, \quad \Gamma^2_{1a}\xi_3(\lambda) = 0, \quad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)\xi_3(\lambda) = I_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{4.32}$$

$$\Gamma_{0b}\xi_3(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \qquad P_{\widetilde{\mathcal{H}}_b, \mathcal{H}_b}\Gamma_{0b}\xi_3(\lambda) = 0, \quad \lambda \in \mathbb{C}_-. \tag{4.33}$$

Moreover, for any $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$) *there exists a unique operator solution* $u_+(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\widetilde{\mathcal{H}}_b, \mathbb{H}]$ *(resp.* $u_-(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_b, \mathbb{H}]$*) satisfying the boundary conditions*

$$\Gamma^1_{1a}u_\pm(\lambda) = 0, \quad \Gamma^2_{1a}u_\pm(\lambda) = 0, \quad \widehat{\Gamma}_a u_\pm(\lambda) = \widehat{\Gamma}_b u_\pm(\lambda), \quad \lambda \in \mathbb{C}_\pm \tag{4.34}$$

$$\Gamma_{0b}u_+(\lambda) = I_{\widetilde{\mathcal{H}}_b}, \quad \lambda \in \mathbb{C}_+; \qquad P_{\widetilde{\mathcal{H}}_b, \mathcal{H}_b}\Gamma_{0b}u_-(\lambda) = I_{\mathcal{H}_b}, \quad \lambda \in \mathbb{C}_-. \tag{4.35}$$

Proof. Let $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ be the decomposing boundary pair (4.9) for $T_{\max}$. Then the linear relation $\Gamma_0 = P_{\mathcal{H}_0 \oplus \{0\}}\Gamma : \mathfrak{H}^2 \to \mathcal{H}_0$ for this triplet is

$$\Gamma_0 = \{\{\widetilde{\pi}_\Delta\{y, f\}, \Gamma'_0 y\} : \{y, f\} \in \mathcal{T}_{\max}\}. \tag{4.36}$$

By using (4.36) one proves in the same way as in [27, Proposition 4.8] that

$$\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T_{\min}) = \{\{\widetilde{\pi}_\Delta\{y, \lambda y\}, \Gamma'_0 y\} : y \in \mathcal{N}_\lambda\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.37}$$

Since by Proposition 4.7 $\mathcal{K}_\Gamma = \{0\}$, it follows from Proposition 2.11 that $\operatorname{ran}\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T_{\min}) = \mathcal{H}_0$ and (4.37) yields $\Gamma'_0 \mathcal{N}_\lambda = \mathcal{H}_0$, $\lambda \in \mathbb{C}_+$. Moreover, by Proposition 4.6 $\dim \mathcal{N}_\lambda = \dim \mathcal{H}_0$ and hence for each $\lambda \in \mathbb{C}_+$ the operator $\Gamma'_0 \upharpoonright \mathcal{N}_\lambda$ isomorphically maps $\mathcal{N}_\lambda$ onto $\mathcal{H}_0$. Similarly by using (4.37) one proves that for each $\lambda \in \mathbb{C}_-$ the operator $P_1\Gamma'_0 \upharpoonright \mathcal{N}_\lambda$ isomorphically maps $\mathcal{N}_\lambda$ onto $\mathcal{H}_1$. Therefore the equalities $Z_+(\lambda) = (\Gamma'_0 \upharpoonright \mathcal{N}_\lambda)^{-1}$, $\lambda \in \mathbb{C}_+$, and $Z_-(\lambda) = (P_1\Gamma'_0 \upharpoonright \mathcal{N}_\lambda)^{-1}$, $\lambda \in \mathbb{C}_-$, define the isomorphisms $Z_+(\lambda) : \mathcal{H}_0 \to \mathcal{N}_\lambda$ and $Z_-(\lambda) : \mathcal{H}_1 \to \mathcal{N}_\lambda$ such that

$$\Gamma'_0 Z_+(\lambda) = I_{\mathcal{H}_0}, \quad \lambda \in \mathbb{C}_+; \qquad P_1\Gamma'_0 Z_-(\lambda) = I_{\mathcal{H}_1}, \quad \lambda \in \mathbb{C}_-. \tag{4.38}$$

Assume that the block representations of $Z_\pm(\lambda)$ are

$$Z_+(\lambda) = (\xi_1(\lambda),\, \xi_2(\lambda),\, \xi_3(\lambda),\, u_+(\lambda)) : H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \to \mathcal{N}_\lambda,$$
$$\lambda \in \mathbb{C}_+ \tag{4.39}$$

$$Z_-(\lambda) = (\xi_1(\lambda),\, \xi_2(\lambda),\, \xi_3(\lambda),\, u_-(\lambda)) : H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \mathcal{H}_b \to \mathcal{N}_\lambda,$$
$$\lambda \in \mathbb{C}_- \tag{4.40}$$

and let $\xi_1(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H_1^\perp,\mathbb{H}]$, $\xi_2(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H_1,\mathbb{H}]$, $\xi_3(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\widehat{H},\mathbb{H}]$, $u_+(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\widetilde{\mathcal{H}}_b,\mathbb{H}]$ and $u_-(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_b,\mathbb{H}]$ be the respective operator solutions of (3.3). It follows from (4.7) that

$$P_1\Gamma'_0 = (-\Gamma^1_{1a},\, -\Gamma^2_{1a},\, i(\widehat{\Gamma}_a - \widehat{\Gamma}_b),\, P_{\widetilde{\mathcal{H}}_b,\mathcal{H}_b}\Gamma_{0b})^\top : \operatorname{dom} \mathcal{T}_{\max}$$
$$\to H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \mathcal{H}_b \quad (4.41)$$

Now combining (4.38) with (4.7), (4.39), (4.41), (4.40) and taking the block representations of $I_{\mathcal{H}_0}$ and $I_{\mathcal{H}_1}$ into account one gets the equalities (4.28)–(4.35). Finally, uniqueness of specified operator solutions is implied by the equalities $\ker \Gamma'_0 \upharpoonright \mathcal{N}_\lambda = \{0\}$, $\lambda \in \mathbb{C}_+$, and $\ker P_1\Gamma'_0 \upharpoonright \mathcal{N}_\lambda = \{0\}$, $\lambda \in \mathbb{C}_-$. $\qquad\square$

Proposition 4.13. *The Weyl function $M_+ = M_+(\lambda)$, $\lambda \in \mathbb{C}_+$, of the decomposing boundary pair $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma\}$ for $T_{\max}$ admits the block representation*

$$M_+ = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} : \underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b}_{\mathcal{H}_0}$$
$$\to \underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \mathcal{H}_b}_{\mathcal{H}_1}, \quad (4.42)$$

with entries $M_{jk} = M_{jk}(\lambda)$, $\lambda \in \mathbb{C}_+$, defined by

$$M_{jk}(\lambda) = \Gamma^j_{0a}\xi_k(\lambda), \ \ j \in \{1,2\}, \ k \in \{1,2,3\};$$
$$M_{j4}(\lambda) = \Gamma^j_{0a}u_+(\lambda), \ \ j \in \{1,2\} \tag{4.43}$$
$$M_{3k}(\lambda) = \widehat{\Gamma}_a\xi_k(\lambda), \ k \in \{1,2\}; \ \ M_{33}(\lambda) = \widehat{\Gamma}_a\xi_3(\lambda) + \tfrac{i}{2}I_{\widehat{H}},$$
$$M_{34}(\lambda) = \widehat{\Gamma}_a u_+(\lambda) \tag{4.44}$$
$$M_{4k}(\lambda) = -\Gamma_{1b}\xi_k(\lambda), \ \ k \in \{1,2,3\}; \ \ \ M_{44}(\lambda) = -\Gamma_{1b}u_+(\lambda). \tag{4.45}$$

Moreover, for every $\lambda \in \mathbb{C}_-$ *one has*

$$M_{jk}^*(\overline{\lambda}) = \Gamma_{0a}^k \xi_j(\lambda), \;\; k \in \{1,2\}, \; j \in \{1,2,3\};$$
$$M_{4k}^*(\overline{\lambda}) = \Gamma_{0a}^k u_-(\lambda), \;\; k \in \{1,2\}, \tag{4.46}$$
$$M_{j3}^*(\overline{\lambda}) = \widehat{\Gamma}_a \xi_j(\lambda), \; j \in \{1,2\}; \;\;\; M_{33}^*(\overline{\lambda}) = \widehat{\Gamma}_a \xi_3(\lambda) + \tfrac{i}{2} I_{\widehat{H}},$$
$$M_{43}^*(\overline{\lambda}) = \widehat{\Gamma}_a u_-(\lambda). \tag{4.47}$$

Proof. Let $Z_\pm(\lambda)$ be the same as in the proof of Proposition 4.12. Then by (4.9)

$$\{\pi_\Delta Z_+(\lambda)h_0 \oplus \lambda \pi_\Delta Z_+(\lambda)h_0, \Gamma_0' Z_+(\lambda)h_0 \oplus \Gamma_1' Z_+(\lambda)h_0\} \in \Gamma,$$
$$h_0 \in \mathcal{H}_0, \lambda \in \mathbb{C}_+,$$
$$\{\pi_\Delta Z_-(\lambda)h_1 \oplus \lambda \pi_\Delta Z_-(\lambda)h_1, \Gamma_0' Z_-(\lambda)h_1 \oplus \Gamma_1' Z_-(\lambda)h_1\} \in \Gamma,$$
$$h_1 \in \mathcal{H}_1, \lambda \in \mathbb{C}_-$$

and in view of (2.16) and (2.17) one has

$$\Gamma_1' Z_+(\lambda) = M_+(\lambda)\Gamma_0' Z_+(\lambda); \;\; (\Gamma_1' + iP_2\Gamma_0')Z_-(\lambda) = M_+^*(\overline{\lambda})P_1\Gamma_0' Z_-(\lambda).$$

(the first equality holds for $\lambda \in \mathbb{C}_+$, while the second one for $\lambda \in \mathbb{C}_-$). This and (4.38) imply that

$$\Gamma_1' Z_+(\lambda) = M_+(\lambda), \;\; \lambda \in \mathbb{C}_+; \qquad (\Gamma_1' + iP_2\Gamma_0')Z_-(\lambda) = M_+^*(\overline{\lambda}), \;\; \lambda \in \mathbb{C}_-. \tag{4.48}$$

It follows from (4.7) and (4.8) that

$$\Gamma_1' + iP_2\Gamma_0' = (\Gamma_{0a}^1, \, \Gamma_{0a}^2, \, \tfrac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b), \, *)^\top : \operatorname{dom} \mathcal{T}_{\max}$$
$$\to H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \tag{4.49}$$

(the entry $*$ does not matter). Assume that (4.42) is the block representation of $M_+(\lambda)$. Combining the first equality in (4.48) with (4.8), (4.39) and taking the last equalities in (4.28), (4.30), (4.32) and (4.34) into account one gets (4.43)–(4.45). Similarly combining the second equality in (4.48) with (4.49) and (4.40) one obtains (4.46) and (4.47). $\square$

Using the entries $M_{ij} = M_{ij}(\lambda)$ from the block representation (4.42) of $M_+(\lambda)$ introduce the holomorphic operator-functions $m_0 = m_0(\lambda)(\in [\mathbb{H}_0])$, $S_1 = S_1(\lambda)(\in [\dot{\mathcal{H}}_0, \mathbb{H}_0])$, $S_2 = S_2(\lambda)(\in [\mathbb{H}_0, \dot{\mathcal{H}}_1])$ and $\dot{M}_+ =$

$M_+(\lambda)(\in [\dot{\mathcal{H}}_0, \dot{\mathcal{H}}_1]), \ \lambda \in \mathbb{C}_+$, by setting

$$
m_0 = \begin{pmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & -\frac{1}{2}I_{H_1} \\ M_{31} & M_{32} & M_{33} & 0 \\ 0 & -\frac{1}{2}I_{H_1} & 0 & 0 \end{pmatrix} : \underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1}_{\mathbb{H}_0}
$$

$$
\to \underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1}_{\mathbb{H}_0} \quad (4.50)
$$

$$
S_1 = \begin{pmatrix} M_{12} & M_{13} & M_{14} \\ M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} - \frac{i}{2}I_{\widehat{H}} & M_{34} \\ -I_{H_1} & 0 & 0 \end{pmatrix} : \underbrace{H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b}_{\dot{\mathcal{H}}_0}
$$

$$
\to \underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1}_{\mathbb{H}_0} \quad (4.51)
$$

$$
S_2 = \begin{pmatrix} M_{21} & M_{22} & M_{23} & -I_{H_1} \\ M_{31} & M_{32} & M_{33} + \frac{i}{2}I_{\widehat{H}} & 0 \\ M_{41} & M_{42} & M_{43} & 0 \end{pmatrix} : \underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1}_{\mathbb{H}_0}
$$

$$
\to \underbrace{H_1 \oplus \widehat{H} \oplus \mathcal{H}_b}_{\dot{\mathcal{H}}_1} \quad (4.52)
$$

$$
\dot{M}_+ = \begin{pmatrix} M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} & M_{34} \\ M_{42} & M_{43} & M_{44} \end{pmatrix} : \underbrace{H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b}_{\dot{\mathcal{H}}_0}
$$

$$
\to \underbrace{H_1 \oplus \widehat{H} \oplus \mathcal{H}_b}_{\dot{\mathcal{H}}_1} \quad (4.53)
$$

Lemma 4.14. *Let $\dot{\Pi} = \{\dot{\mathcal{H}}_0 \oplus \dot{\mathcal{H}}_1, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ be the boundary triplet* (4.19) *for T^*. Moreover, let $\dot{Z}_+(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\dot{\mathcal{H}}_0, \mathbb{H}]$ and $\dot{Z}_-(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\dot{\mathcal{H}}_1, \mathbb{H}]$ be operator solutions of* (3.3) *given by*

$$
\dot{Z}_+(t, \lambda) = (\xi_2(t, \lambda), \xi_3(t, \lambda), u_+(t, \lambda)) : H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \to \mathbb{H}, \ \ \lambda \in \mathbb{C}_+
$$
$$
(4.54)
$$

$$
\dot{Z}_-(t, \lambda) = (\xi_2(t, \lambda), \xi_3(t, \lambda), u_-(t, \lambda)) : H_1 \oplus \widehat{H} \oplus \mathcal{H}_b \to \mathbb{H}, \ \ \lambda \in \mathbb{C}_-
$$
$$
(4.55)
$$

and let $\dot{M}_+(\cdot)$ be the operator-function (4.53). Then:

(1) *The following equalities hold*

$$\widetilde{U}^{-1}\dot{Z}_+(a,\lambda) = \begin{pmatrix} P_{\mathbb{H},\mathbb{H}_0}\Gamma_a\dot{Z}_+(\lambda) \\ \Gamma_{1a}^1\dot{Z}_+(\lambda) \end{pmatrix} = \begin{pmatrix} S_1(\lambda) \\ 0 \end{pmatrix} : \dot{\mathcal{H}}_0 \to \mathbb{H}_0 \oplus H_1^\perp, \quad \lambda \in \mathbb{C}_+$$

(4.56)

$$\widetilde{U}^{-1}\dot{Z}_-(a,\lambda) = \begin{pmatrix} P_{\mathbb{H},\mathbb{H}_0}\Gamma_a\dot{Z}_-(\lambda) \\ \Gamma_{1a}^1\dot{Z}_-(\lambda) \end{pmatrix} = \begin{pmatrix} S_2^*(\overline{\lambda}) \\ 0 \end{pmatrix} : \dot{\mathcal{H}}_1 \to \mathbb{H}_0 \oplus H_1^\perp, \quad \lambda \in \mathbb{C}_-.$$

(4.57)

(2) γ-*fields* $\dot{\gamma}_\pm(\cdot)$ *of the triplet* $\dot{\Pi}$ *are*

$$\dot{\gamma}_+(\lambda) = \pi_\Delta\dot{Z}_+(\lambda), \ \lambda \in \mathbb{C}_+; \qquad \dot{\gamma}_-(\lambda) = \pi_\Delta\dot{Z}_-(\lambda), \ \lambda \in \mathbb{C}_- \quad (4.58)$$

and the Weyl function of $\dot{\Pi}$ *coincides with* $\dot{M}_+(\lambda)$.

(3) *If τ is a boundary parameter (4.24), then*

$$(C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1} \in [\dot{\mathcal{H}}_0]$$

and

$$-(\tau(\lambda) + \dot{M}_+(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1}C_1(\lambda), \quad \lambda \in \mathbb{C}_+.$$

(4.59)

Proof. (1) It follows from (4.5) and Propositions 4.12, 4.13 that

$$P_{\mathbb{H},\mathbb{H}_0}\Gamma_a\dot{Z}_+(\lambda) = S_1(\lambda), \qquad \Gamma_{1a}^1\dot{Z}_+(\lambda) = 0, \quad \lambda \in \mathbb{C}_+, \qquad (4.60)$$

and $P_{\mathbb{H},\mathbb{H}_0}\Gamma_a\dot{Z}_-(\lambda) = S_2^*(\overline{\lambda}), \Gamma_{1a}^1\dot{Z}_-(\lambda) = 0, \lambda \in \mathbb{C}_-$. This and Lemma 4.6 yield (4.56) and (4.57).

(2) Let $\dot{\gamma}_\pm(\lambda)$ be given by (4.58) and let $Z_\pm(\lambda)$ be the same as in the proof of Proposition 4.12. Comparing (4.54) and (4.55) with (4.39) and (4.40) one gets $\dot{Z}_+(\lambda) =$
$Z_+(\lambda) \upharpoonright \dot{\mathcal{H}}_0, \ \lambda \in \mathbb{C}_+$, and $\dot{Z}_-(\lambda) = Z_-(\lambda) \upharpoonright \dot{\mathcal{H}}_1, \ \lambda \in \mathbb{C}_-$. Therefore by (4.38)

$$\Gamma_0'\dot{Z}_+(\lambda)h_0 = h_0, \ h_0 \in \dot{\mathcal{H}}_0, \ \lambda \in \mathbb{C}_+; \ P_1\Gamma_0'\dot{Z}_-(\lambda)h_1 = h_1, \ h_1 \in \dot{\mathcal{H}}_1, \ \lambda \in \mathbb{C}_-$$

(4.61)

and in view of (4.15) $P_1\Gamma_0' = (-\Gamma_{1a}^1, \dot{P}_1\dot{\Gamma}_0')^\top$ with $\dot{P}_1 := P_{\dot{\mathcal{H}}_0,\dot{\mathcal{H}}_1}$. This and (4.14), (4.15) imply that

$$\Gamma_{1a}^1\dot{Z}_+(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \qquad \Gamma_{1a}^1\dot{Z}_-(\lambda) = 0, \quad \lambda \in \mathbb{C}_- \qquad (4.62)$$

$$\dot{\Gamma}_0'\dot{Z}_+(\lambda) = I_{\dot{\mathcal{H}}_0}, \ \lambda \in \mathbb{C}_+; \qquad \dot{P}_1\dot{\Gamma}_0'\dot{Z}_-(\lambda) = I_{\dot{\mathcal{H}}_1}, \ \lambda \in \mathbb{C}_-. \qquad (4.63)$$

It follows from (4.62) that $\dot\gamma_+(\lambda)\mathcal{H}_0 \subset \mathfrak{N}_\lambda(T)$, $\dot\gamma_-(\lambda)\mathcal{H}_1 \subset \mathfrak{N}_\lambda(T)$ and (4.63) yields

$$\dot\Gamma_0\{\dot\gamma_+(\lambda)h_0, \lambda\dot\gamma_+(\lambda)h_0\} = \dot\Gamma_0' \dot Z_+(\lambda)h_0 = h_0, \quad h_0 \in \dot{\mathcal{H}}_0, \quad \lambda \in \mathbb{C}_+$$
$$\dot P_1\dot\Gamma_0\{\dot\gamma_-(\lambda)h_1, \lambda\dot\gamma_-(\lambda)h_1\} = \dot P_1\dot\Gamma_0'\dot Z_-(\lambda)h_1 = h_1, \quad h_1 \in \dot{\mathcal{H}}_1, \quad \lambda \in \mathbb{C}_-.$$

Therefore according to definitions (2.12) and (2.13) $\dot\gamma_\pm(\cdot)$ are γ-fields of $\dot\Pi$.

Next assume that $\dot M_+(\cdot)$ is given by (4.53). Then in view of (4.42) and (4.14) $\dot M_+(\lambda) = P_{\mathcal{H}_1,\dot{\mathcal{H}}_1} M_+(\lambda) \upharpoonright \dot{\mathcal{H}}_0$ and by using (4.48) one obtains

$$\dot\Gamma_1'\dot Z_+(\lambda) = P_{\mathcal{H}_1,\dot{\mathcal{H}}_1}\Gamma_1' Z_+(\lambda) \upharpoonright \dot{\mathcal{H}}_0 = P_{\mathcal{H}_1,\dot{\mathcal{H}}_1} M_+(\lambda) \upharpoonright \dot{\mathcal{H}}_0 = \dot M_+(\lambda) \tag{4.64}$$

Hence $\dot\Gamma_1\{\dot\gamma_+(\lambda)h_0, \lambda\dot\gamma_+(\lambda)h_0\} = \dot\Gamma_1'\dot Z_+(\lambda)h_0 = \dot M_+(\lambda)h_0$, $h_0 \in \dot{\mathcal{H}}_0$, $\lambda \in \mathbb{C}_+$, and according to definition (2.11) $\dot M_+(\cdot)$ is the Weyl function of $\dot\Pi$.

Statement (3) follows from [24, Theorem 3.11] and [20, Lemma 2.1]. $\qquad\square$

Theorem 4.15. *Let τ be a boundary parameter* (4.24), *let*

$$C_0(\lambda) = (C_{0a}(\lambda), \widehat{C}_0(\lambda), C_{0b}(\lambda)) : H_1 \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \to \dot{\mathcal{H}}_0 \tag{4.65}$$
$$C_1(\lambda) = (C_{1a}(\lambda), \widehat{C}_1(\lambda), C_{1b}(\lambda)) : H_1 \oplus \widehat{H} \oplus \mathcal{H}_b \to \dot{\mathcal{H}}_0 \tag{4.66}$$

be the block representations of $C_0(\lambda)$ and $C_1(\lambda)$ and let

$$\Phi(\lambda) := (0, C_{0a}(\lambda), \widehat{C}_0(\lambda) + \tfrac{i}{2}\widehat{C}_1(\lambda), -C_{1a}(\lambda))$$
$$: H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1 \to \dot{\mathcal{H}}_0. \tag{4.67}$$

Then for each $\lambda \in \mathbb{C}_+$ there exists a unique operator solution $v_\tau(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathbb{H}_0, \mathbb{H}]$ of the system (3.3) *satisfying the boundary conditions*

$$\Gamma_{1a}^1 v_\tau(\lambda) = -P_{\mathbb{H}_0, H_1^\perp}, \qquad C_0(\lambda)\dot\Gamma_0' v_\tau(\lambda) - C_1(\lambda)\dot\Gamma_1' v_\tau(\lambda) = \Phi(\lambda) \tag{4.68}$$

(here $P_{\mathbb{H}_0, H_1^\perp}$ is the orthoprojection in $\mathbb{H}_0$ onto $H_1^\perp$ in accordance with decomposition (3.11) *of $\mathbb{H}_0$).*

Proof. Let $Z_0(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathbb{H}_0, \mathbb{H}]$ and $\dot Z_+(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\dot{\mathcal{H}}_0, \mathbb{H}]$ be operator solutions of (3.3) given by

$$Z_0(t, \lambda) = (\xi_1(t, \lambda), \xi_2(t, \lambda), \xi_3(t, \lambda), 0)$$
$$: H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1 \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+ \tag{4.69}$$

and (4.54) respectively and let $S_2(\lambda)$ be defined by (4.52). Then in view of Lemma 4.14,(3) the equality

$$v_\tau(t,\lambda) = Z_0(t,\lambda) + \dot{Z}_+(t,\lambda)(C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda), \ \lambda \in \mathbb{C}_+ \tag{4.70}$$

correctly defines the solution $v_\tau(\cdot,\lambda) \in \mathcal{L}_\Delta^2[\mathbb{H}_0, \mathbb{H}]$ of (3.3). Let us show that this solution satisfies (4.68).

It follows from (4.5) and Propositions 4.12, 4.13 that

$$P_{\mathbb{H},\mathbb{H}_0}\Gamma_a Z_0(\lambda) = m_0(\lambda) - \tfrac{1}{2}J_0, \quad \Gamma_{1a}^1 Z_0(\lambda) = -P_{\mathbb{H}_0, H_1^\perp}, \quad \lambda \in \mathbb{C}_+, \tag{4.71}$$

where $J_0 \in [\mathbb{H}_0]$ is the operator given by

$$J_0 = P_{\mathbb{H},\mathbb{H}_0}J \restriction \mathbb{H}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{H_1} \\ 0 & 0 & iI_{\widehat{H}} & 0 \\ 0 & I_{H_1} & 0 & 0 \end{pmatrix} \in [\underbrace{H_1^\perp \oplus H_1 \oplus \widehat{H} \oplus H_1}_{\mathbb{H}_0}]. \tag{4.72}$$

Combining (4.70) with the second equalities in (4.71) and (4.60) one gets the first equality in (4.68). Next, by (4.63) and (4.64)

$$(C_0(\lambda)\dot{\Gamma}_0' - C_1(\lambda)\dot{\Gamma}_1')v_\tau(\lambda) = (C_0(\lambda)\dot{\Gamma}_0' - C_1(\lambda)\dot{\Gamma}_1')Z_0(\lambda)$$
$$+ (C_0(\lambda)\dot{\Gamma}_0' - C_1(\lambda)\dot{\Gamma}_1')\dot{Z}_+(\lambda)(C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda)$$
$$= C_0(\lambda)\dot{\Gamma}_0'Z_0(\lambda) + C_1(\lambda)(S_2(\lambda) - \dot{\Gamma}_1'Z_0(\lambda)).$$

Moreover, by (4.28)–(4.33) and (4.43)–(4.45) one has

$$\dot{\Gamma}_0'Z_0(\lambda) = \begin{pmatrix} 0 & I_{H_1} & 0 & 0 \\ 0 & 0 & I_{\widehat{H}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\dot{\Gamma}_1'Z_0(\lambda) = \begin{pmatrix} M_{21}(\lambda) & M_{21}(\lambda) & M_{23}(\lambda) & 0 \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) & 0 \\ M_{41}(\lambda) & M_{42}(\lambda) & M_{43}(\lambda) & 0 \end{pmatrix}$$

and hence $S_2(\lambda) - \dot{\Gamma}_1'Z_0(\lambda) = \begin{pmatrix} 0 & 0 & 0 & -I_{H_1} \\ 0 & 0 & \frac{i}{2}I_{\widehat{H}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This and (4.65),

(4.66) imply that

$$C_0(\lambda)\dot{\Gamma}_0 v_\tau(\lambda) - C_1(\lambda)\dot{\Gamma}_1 v_\tau(\lambda)$$

$$= (C_{0a}(\lambda), \widehat{C}_0(\lambda), C_{0b}(\lambda)) \begin{pmatrix} 0 & I_{H_1} & 0 & 0 \\ 0 & 0 & I_{\widehat{H}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ (C_{1a}(\lambda), \widehat{C}_1(\lambda), C_{1b}(\lambda)) \begin{pmatrix} 0 & 0 & 0 & -I_{H_1} \\ 0 & 0 & \frac{i}{2}I_{\widehat{H}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \Phi(\lambda).$$

Thus the second equality in (4.68) is valid. Finally uniqueness of $v_\tau(\cdot, \lambda)$ is implied by uniqueness of the solution of the problem (4.25), (4.26) (see Theorem 4.11). $\qquad\square$

4.3. m-functions

Let τ be a boundary parameter (4.24), let $v_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}_0, \mathbb{H}]$ be the operator solution of (3.3) defined in Theorem 4.15 and let J_0 be the operator (4.72).

Definition 4.16. The operator function $m_\tau(\cdot) : \mathbb{C}_+ \to [\mathbb{H}_0]$ defined by

$$m_\tau(\lambda) = P_{\mathbb{H},\mathbb{H}_0}\Gamma_a v_\tau(\lambda) + \tfrac{1}{2}J_0, \quad \lambda \in \mathbb{C}_+ \tag{4.73}$$

is called the m-function corresponding to the boundary parameter τ or, equivalently, to the boundary value problem (4.25), (4.26).

In the following theorem we provide a description of all m-functions immediately in terms of the boundary parameter τ.

Theorem 4.17. *Let the assumptions* (A1)–(A3) *after Proposition 4.4 be satisfied and let $\dot{\mathcal{H}}_0$ and $\dot{\mathcal{H}}_1$ be finite-dimensional Hilbert spaces* (4.11). *Assume also that $M_+(\cdot)$ is the operator-function defined by* (4.42)–(4.45) *and $m_0(\cdot)$, $S_1(\cdot)$, $S_2(\cdot)$ and $\dot{M}_+(\cdot)$ are the operator-functions* (4.50)– (4.53). *Then:*

(1) $m_0(\cdot)$ is the m-function corresponding to the boundary parameter $\tau_0 = \{I_{\dot{\mathcal{H}}_0}, 0_{\dot{\mathcal{H}}_1, \dot{\mathcal{H}}_0}\}$;

(2) for every boundary parameter τ of the form (4.24) *the corresponding m-function $m_\tau(\cdot)$ admits the representation*

$$m_\tau(\lambda) = m_0(\lambda) + S_1(\lambda)(C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.74}$$

Proof. Applying the operator $P_{\mathbb{H},\mathbb{H}_0}\Gamma_a$ to the equality (4.70) and taking the first equalities in (4.71) and (4.60) into account one gets (4.74). Statement (1) of the theorem is immediate from (4.74). $\qquad\square$

Proposition 4.18. *The m-function $m_\tau(\cdot)$ belongs to the class $R[\mathbb{H}_0]$ and satisfies*

$$\operatorname{Im} m_\tau(\lambda) \geq \int_{\mathcal{I}} v_\tau^*(t,\lambda)\Delta(t)v_\tau(t,\lambda)\,dt, \quad \lambda \in \mathbb{C}_+. \qquad (4.75)$$

If $N_+ = N_-$ and τ is a self-adjoint boundary parameter, then the inequality (4.75) turns into the equality.

Proof. It follows from (4.74) that $m_\tau(\cdot)$ is holomorphic in $\mathbb{C}_+$. Moreover, one can prove inequality (4.75) in the same way as similar inequalities (5.10) in [1] and (4.66) in [25]. Therefore $m_\tau(\cdot) \in R[\mathbb{H}_0]$. $\qquad\square$

4.4. Generalized resolvents and characteristic matrices

In the sequel we denote by $Y_{\widetilde{U}}(\cdot,\lambda)$ the $[\mathbb{H}]$-valued operator solution of (3.3) satisfying $Y_{\widetilde{U}}(a,\lambda) = \widetilde{U}$, $\lambda \in \mathbb{C}$.

The following theorem is well known (see e.g. [5, 9, 33]).

Theorem 4.19. *For each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists an operator-function $\Omega(\cdot) \in R[\mathbb{H}]$ (the characteristic matrix of $R(\lambda)$) such that for any $\widetilde{f} \in \mathfrak{H}$ and $\lambda \in \mathbb{C}_+$*

$$R(\lambda)\widetilde{f} = \pi_\Delta\left(\int_{\mathcal{I}} Y_{\widetilde{U}}(x,\lambda)(\Omega(\lambda) + \tfrac{1}{2}\operatorname{sgn}(t-x)J)Y_{\widetilde{U}}^*(t,\overline{\lambda})\Delta(t)f(t)\,dt\right),$$

$$f \in \widetilde{f}. \quad (4.76)$$

Proposition 4.20. *Let τ be a boundary parameter (4.24) and let $R_\tau(\lambda)$ be the corresponding generalized resolvent of T (and hence of $T_{\min}$) in accordance with Theorem 4.11. Moreover, let $P_{\mathbb{H}_0,H_1^\perp}$ and $I_{H_1^\perp,\mathbb{H}_0}$ be the orthoprojection in $\mathbb{H}_0$ onto $H_1^\perp$ and the embedding operator of $H_1^\perp$ into $\mathbb{H}_0$ respectively (see decomposition (3.11) of $\mathbb{H}_0$). Then the equality*

$$\Omega(\lambda) = \begin{pmatrix} m_\tau(\lambda) & -\tfrac{1}{2}I_{H_1^\perp,\mathbb{H}_0} \\ -\tfrac{1}{2}P_{\mathbb{H}_0,H_1^\perp} & 0 \end{pmatrix} : \underbrace{\mathbb{H}_0 \oplus H_1^\perp}_{\mathbb{H}} \to \underbrace{\mathbb{H}_0 \oplus H_1^\perp}_{\mathbb{H}}, \ \lambda \in \mathbb{C}_+$$

$$(4.77)$$

defines a characteristic matrix $\Omega(\cdot)$ of $R_\tau(\lambda)$.

Proof. Assume that $\dot{\gamma}_\pm(\cdot)$ are γ-fields and $\dot{M}_+(\cdot)$ is the Weyl function of the boundary triplet $\dot{\Pi} = \{\dot{\mathcal{H}}_0 \oplus \dot{\mathcal{H}}_1, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ for T^* defined in Proposition 4.9. Moreover let

$$B_\tau(\lambda) := -(\tau(\lambda) + \dot{M}_+(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1}C_1(\lambda), \ \lambda \in \mathbb{C}_+$$

$$(4.78)$$

(see (4.59)). Then according to [24, Theorem 3.11] the Krein type formula for generalized resolvents

$$R(\lambda) = R_\tau(\lambda) = (A_0 - \lambda)^{-1} + \dot{\gamma}_+(\lambda)B_\tau(\lambda)\dot{\gamma}_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+ \qquad (4.79)$$

holds with the maximal symmetric extension A_0 of T given by

$$A_0 := \ker \dot{\Gamma}_0$$
$$= \{\widetilde{\pi}_\Delta\{y, f\} : \{y, f\} \in \mathcal{T}_{\max}, \Gamma_{1a}^1 y = 0, \Gamma_{1a}^2 y = 0, \widehat{\Gamma}_a y = \widehat{\Gamma}_b y, \Gamma_{0b} y = 0\}.$$

According to [25, (4.36)] for each $\widetilde{f} \in \mathfrak{H}$ and $\lambda \in \mathbb{C}_+$

$$(A_0 - \lambda)^{-1}\widetilde{f}$$
$$= \pi_\Delta \left(\int_{\mathcal{I}} Y_{\widetilde{U}}(x, \lambda)(\Omega_0(\lambda) + \tfrac{1}{2}\,\mathrm{sgn}(t - x)J)Y_{\widetilde{U}}^*(t, \overline{\lambda})\Delta(t)f(t)dt \right), \quad (4.80)$$

where $f(\cdot) \in \widetilde{f}$ and $\Omega_0(\lambda)$ is the operator function defined in [25, (4.30)] (actually (4.80) is proved in [25] for definite systems but the proof is suitable for the case of a θ-definite system as well). One can easily verify that $\Omega_0(\lambda)$ admits the representation

$$\Omega_0(\lambda) = \begin{pmatrix} m_0(\lambda) & -\tfrac{1}{2}I_{H_1^\perp, \mathbb{H}_0} \\ -\tfrac{1}{2}P_{\mathbb{H}_0, H_1^\perp} & 0 \end{pmatrix} : \mathbb{H}_0 \oplus H_1^\perp \to \mathbb{H}_0 \oplus H_1^\perp, \quad \lambda \in \mathbb{C}_+$$

$$(4.81)$$

with $m_0(\lambda)$ given by (4.50). Next, $\dot{Z}_\pm(t, \lambda) = Y_{\widetilde{U}}(t, \lambda)\widetilde{U}^{-1}\dot{Z}_\pm(a, \lambda)$ and in view of the second equality in (4.58) and [1, Lemma 3.3] one has

$$\dot{\gamma}_-^*(\overline{\lambda})\widetilde{f} = \int_{\mathcal{I}} \dot{Z}_-^*(t, \overline{\lambda})\Delta(t)f(t)\,dt$$
$$= \int_{\mathcal{I}} (\widetilde{U}^{-1}\dot{Z}_-(a, \overline{\lambda}))^*Y_{\widetilde{U}}^*(t, \overline{\lambda})\Delta(t)f(t)\,dt, \quad f(\cdot) \in \widetilde{f}.$$

This and the first equality in (4.58) imply that for any $\widetilde{f} \in \mathfrak{H}$ and $\lambda \in \mathbb{C}_+$

$$\dot{\gamma}_+(\lambda)B_\tau(\lambda)\dot{\gamma}_-^*(\overline{\lambda})\widetilde{f}$$
$$= \pi_\Delta \int_{\mathcal{I}} Y_{\widetilde{U}}(\cdot, \lambda)(\widetilde{U}^{-1}\dot{Z}_+(a, \lambda))B_\tau(\lambda)(\widetilde{U}^{-1}\dot{Z}_-(a, \overline{\lambda}))^*Y_{\widetilde{U}}^*(t, \overline{\lambda})\Delta(t)f(t)\,dt$$
$$= \pi_\Delta \int_{\mathcal{I}} Y_{\widetilde{U}}(\cdot, \lambda)\overline{\Omega}(\lambda)Y_{\widetilde{U}}^*(t, \overline{\lambda})\Delta(t)f(t)\,dt, \quad f(\cdot) \in \widetilde{f},$$

where

$$\overline{\Omega}(\lambda) = (\widetilde{U}^{-1}\dot{Z}_+(a, \lambda))B_\tau(\lambda)(\widetilde{U}^{-1}\dot{Z}_-(a, \overline{\lambda}))^* = \begin{pmatrix} S_1(\lambda) \\ 0 \end{pmatrix} B_\tau(\lambda)\,(S_2(\lambda), 0)$$
$$= \begin{pmatrix} S_1(\lambda)B_\tau(\lambda)S_2(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$$

(here we made use of (4.56) and (4.57)). Combining these relations with (4.79) and (4.80) one obtains the equality (4.76) with

$$\Omega(\lambda) = \Omega_0(\lambda) + \overline{\Omega}(\lambda) = \begin{pmatrix} m_0(\lambda) + S_1(\lambda)B_\tau(\lambda)S_2(\lambda) & -\frac{1}{2}I_{H_1^\perp,\mathbb{H}_0} \\ -\frac{1}{2}P_{\mathbb{H}_0,H_1^\perp} & 0 \end{pmatrix}.$$

Hence $\Omega(\lambda)$ is a characteristic matrix of $R_\tau(\lambda)$ and in view of (4.74) and (4.78) the equality (4.77) is valid. $\qquad\square$

5. Parametrization of pseudospectral and spectral functions

As before we suppose in this section (unless otherwise stated) the assumptions (A1)–(A3) specified after Proposition 4.4.

Let T be a symmetric relation (3.12). Then according to Theorem 4.11 the boundary value problem (4.25), (4.26) induces parametrizations $R(\lambda) = R_\tau(\lambda)$, $\widetilde{T} = \widetilde{T}_\tau$ and $F(\cdot) = F_\tau(\cdot)$ of all generalized resolvents $R(\lambda)$, exit space extensions $\widetilde{T} \in \widetilde{\mathrm{Self}}(T)$ and spectral functions $F(\cdot)$ of T respectively by means of the boundary parameter τ. Here $\widetilde{T}_\tau(\in \widetilde{\mathrm{Self}}(T))$ is the extension of T generating $R_\tau(\lambda)$ and $F_\tau(\cdot)$ is the respective spectral function of T.

Definition 5.1. Let $\dot{M}_+ = \dot{M}_+(\lambda)$ be given by (4.53). A boundary parameter τ of the form (4.24) is called admissible if

$$\lim_{y\to+\infty} \tfrac{1}{iy} P_{\dot{\mathcal{H}}_0,\dot{\mathcal{H}}_1}(C_0(iy) - C_1(iy)\dot{M}_+(iy))^{-1}C_1(iy) = 0, \qquad (5.1)$$

$$\lim_{y\to+\infty} \tfrac{1}{iy}\dot{M}_+(iy)(C_0(iy) - C_1(iy)\dot{M}_+(iy))^{-1}C_0(iy) \upharpoonright \dot{\mathcal{H}}_1 = 0. \qquad (5.2)$$

Proposition 5.2. *An extension $\widetilde{T} = \widetilde{T}^\tau$ belongs to $\mathrm{Self}_0(T)$ if and only if the boundary parameter τ is admissible. Therefore the set of admissible boundary parameters is not empty.*

Proof. According to Lemma 4.14, (2) $\dot{M}_+(\cdot)$ is the Weyl function of the boundary triplet $\dot{\Pi}$ for T^*. Therefore the required result follows from [26, Theorem 2.15]. $\qquad\square$

In the following with the operator $\widetilde{U}$ from assumption (A2) we associate the operator $U = U_\theta \in [\mathbb{H}_0, \mathbb{H}]$ given by $U = \widetilde{U} \upharpoonright \mathbb{H}_0$. Moreover, we denote by $\varphi_U(\cdot, \lambda)$ the $[\mathbb{H}_0, \mathbb{H}]$-valued operator solution of (3.3) with $\varphi_U(a, \lambda) = U$. Clearly $\ker U = \{0\}$ and $U\mathbb{H}_0 = \theta$.

Theorem 5.3. *Let τ be an admissible boundary parameter, let $F(\cdot) = F_\tau(\cdot)$ be the corresponding spectral function of T and let $m_\tau(\cdot)$ be the m-function (4.73). Then there exists a unique pseudospectral function $\sigma(\cdot) = \sigma_\tau(\cdot)$ of the system (3.3) (with respect to $U \in [\mathbb{H}_0, \mathbb{H}]$) satisfying (3.19). This pseudospectral function is defined by the Stieltjes inversion formula*

$$\sigma_\tau(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} m_\tau(u + i\varepsilon)\, du. \qquad (5.3)$$

Proof. Assume that $\Omega(\cdot) \in R[\mathbb{H}]$ is the characteristic matrix (4.77) of $R_\tau(\lambda)$ and $\Sigma(\cdot) : \mathbb{R} \to [\mathbb{H}]$ is the distribution function defined by

$$\Sigma(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} \Omega(u + i\varepsilon)\, du.$$

Using (4.76) and the Stieltjes–Livšic formula one proves as in [9, 33] the equality

$$((F(\beta) - F(\alpha))\widetilde{f}, \widetilde{f}) = \int_{[\alpha,\beta)} (d\Sigma(s)\widehat{f_0}(s), \widehat{f_0}(s)), \ \widetilde{f} \in \mathfrak{H}_b,$$
$$- \infty < \alpha < \beta < \infty \quad (5.4)$$

with the function $\widehat{f_0} : \mathbb{R} \to \mathbb{H}$ defined for each $\widetilde{f} \in \mathfrak{H}_b$ by $\widehat{f_0}(s) = \int_{\mathcal{I}} Y_{\widetilde{u}}^*(t, s)\Delta(t)f(t)\, dt$, $f(\cdot) \in \widetilde{f}$. Let $\widetilde{f} \in \mathfrak{H}_b$, let

$$\widehat{f}(s) = \int_{\mathcal{I}} \varphi_u^*(t, s)\Delta(t)f(t)\, dt, \quad f(\cdot) \in \widetilde{f} \qquad (5.5)$$

and let $\sigma(\cdot) = \sigma_\tau(\cdot)$ be the distribution function (5.3). Since $\varphi_u(t, \lambda) = Y_{\widetilde{u}}(t, \lambda) \upharpoonright \mathbb{H}_0$, it follows that $\widehat{f}(s) = P_{\mathbb{H},\mathbb{H}_0} \widehat{f_0}(s)$. Moreover, by (4.77) one has

$$\Sigma(s) = \begin{pmatrix} \sigma(s) & 0 \\ 0 & 0 \end{pmatrix} : \mathbb{H}_0 \oplus H_1^\perp \to \mathbb{H}_0 \oplus H_1^\perp.$$

This and (5.4) yield the equality (3.19). Next by using (3.19) and Proposition 5.2 one proves that $\sigma(\cdot)$ is a pseudospectral function (with respect to U) in the same way as in [26, Theorem 3.20] and [27, Theorem 5.4].

Let us prove that $\sigma(\cdot) = \sigma_\tau(\cdot)$ is a unique pseudospectral function satisfying (3.19) (we give only the sketch of the proof because it is similar to that of the alike result in [27, Theorem 5.4]). Let $\widetilde{\sigma}(\cdot)$ be a pseudospectral function (with respect to U) such that (3.19) holds with $\widetilde{\sigma}(\cdot)$ instead

of $\sigma(\cdot)$. Then according to [10] there exists a scalar measure μ on Borel sets in $\mathbb{R}$ and functions $\Psi_j(\cdot) : \mathbb{R} \to [\mathbb{H}_0]$, $j \in \{1, 2\}$, such that

$$\sigma(\beta) - \sigma(\alpha) = \int_\delta \Psi_1(s)\, d\mu(s),\ \widetilde{\sigma}(\beta) - \widetilde{\sigma}(\alpha) = \int_\delta \Psi_2(s)\, d\mu(s),\ \ \delta = [\alpha, \beta].$$
(5.6)

Let $\Psi(s) := \Psi_1(s) - \Psi_2(s)$ and let μ_0 be the Lebesgue measure on Borel sets in $\mathcal{I}$. Denote also by $\mathcal{G}$ the set of all functions $\widehat{f}(\cdot) : \mathbb{R} \to \mathbb{H}_0$ admitting the representation (5.5) with some $\widetilde{f} \in \mathfrak{H}_b$. As in [27, Theorem 5.4] one proves that for each $\widehat{f} \in \mathcal{G}$ there is a Borel set $C_{\widehat{f}} \subset \mathbb{R}$ such that

$$\mu(\mathbb{R} \setminus C_{\widehat{f}}) = 0 \text{ and } \mu_0(\{t \in \mathcal{I} : \Delta(t)\varphi_U(t, s)\Psi(s)\widehat{f}(s) \neq 0\}) = 0,\ s \in C_{\widehat{f}}.$$
(5.7)

Let $s \in C_{\widehat{f}}$ and let $y = y(t) = \varphi_U(t, s)\Psi(s)\widehat{f}(s)$. Then y is a solution of the system (3.3) with $\lambda = s$ and by (5.7) $\Delta(t)y(t) = 0$ (μ_0 a.e. on $\mathcal{I}$). Hence $y \in \mathcal{N}$. Moreover, $y(a) = U\Psi(s)\widehat{f}(s) \in \theta$. Since system is θ-definite, this implies that $y = 0$ and, consequently, $\Psi(s)\widehat{f}(s) = 0$. Thus for any $\widehat{f} \in \mathcal{G}$ there exists a Borel set $C_{\widehat{f}} \subset \mathbb{R}$ such that

$$\mu(\mathbb{R} \setminus C_{\widehat{f}}) = 0 \ \text{ and } \ \Psi(s)\widehat{f}(s) = 0,\ \ s \in C_{\widehat{f}}.$$
(5.8)

Next we prove the following statement:

(S) for any $s \in \mathbb{R}$ and $h \in \mathbb{H}_0$ there is $\widehat{f}(\cdot) \in \mathcal{G}$ such that $\widehat{f}(s) = h$. Indeed, let $s \in \mathbb{R}$, $h' \in \mathbb{H}_0$ and $(\widehat{f}(s), h') = 0$ for any $\widehat{f}(\cdot) \in \mathcal{G}$. Put $y = y(t) = \varphi_U(t, s)h'$. Then for any $\beta \in \mathcal{I}$ one has $\widehat{f}_\beta(\cdot) := \int_{[a,\beta]} \varphi_U^*(t, \cdot)\Delta(t)y(t)\, dt \in \mathcal{G}$ and, consequently,

$$0 = (\widehat{f}_\beta(s), h') = \int_{[a,\beta]} (\varphi_U^*(t, s)\Delta(t)y(t), h')\, dt$$

$$= \int_{[a,\beta]} (\Delta(t)y(t), y(t))\, dt, \quad \beta \in \mathcal{I}.$$

Hence $y \in \mathcal{N}$. Moreover, $y(a) = Uh' \in \theta$ and θ-definiteness of the system implies that $y = 0$. Therefore $h' = 0$, which proves statement (S).

Next by using (5.8) and statement (S) one proves the equality $\Psi(s) = 0$ (μ-a.e. on $\mathbb{R}$) in the same way as in [27, Theorem 5.4]. Thus $\Psi_1(s) = \Psi_2(s)$ (μ-a.e. on $\mathbb{R}$) and by (5.6) $\widetilde{\sigma}(s) = \sigma(s)$. $\square$

Corollary 5.4. (1) *Let the assumption* (A1) *from Section 4.1 be satisfied. Then the set of pseudospectral functions (with respect to $K_\theta \in [\mathbb{H}_0', \mathbb{H}]$) is not empty.*

(2) *Let system* (3.3) *be definite, let* $N_- \leq N_+$ *and let* θ *be a subspace in* $\mathbb{H}$. *Then the set of pseudospectral functions (with respect to* $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$) *is not empty if and only if* $\theta^\times \in Sym(\mathbb{H})$.

Proof. Statement (1) is immediate from Proposition 5.2 and Theorem 5.3. Statement (2) follows from statement (1), Remark 4.2 and Proposition 3.16. $\qquad\square$

A parametrization of all pseudospectral functions $\sigma(\cdot)$ (with respect to $U \in [\mathbb{H}_0, \mathbb{H}]$) immediately in terms of a boundary parameter τ is given by the following theorem.

Theorem 5.5. *Let the assumptions be the same as in Theorem 4.17. Then the equality*

$$m_\tau(\lambda) = m_0(\lambda) + S_1(\lambda)(C_0(\lambda) - C_1(\lambda)\dot{M}_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{5.9}$$

together with formula (5.3) *establishes a bijective correspondence* $\sigma(s) = \sigma_\tau(s)$ *between all admissible boundary parameters* τ *defined by* (4.24) *and all pseudospectral functions* $\sigma(\cdot)$ *of the system* (3.3) *(with respect to* $U \in [\mathbb{H}_0, \mathbb{H}]$).

The proof of Theorem 5.5 is based on Theorems 5.3, 4.17 and Propositions 3.14, 5.2. We omit this proof because it is similar to that of Theorem 5.7 in [27].

The following theorem directly follows from Theorem 5.3 and Propositions 3.14, 4.4.

Theorem 5.6. *Let the assumptions* (A1) *and* (A2) *from Section 4.1 be satisfied. Then there is a one to one correspondence* $\sigma(\cdot) = \sigma_{\widetilde{T}}(\cdot)$ *between all extensions* $\widetilde{T} \in \widetilde{\mathrm{Self}}_0(T)$ *and all pseudospectral functions* $\sigma(\cdot)$ *of the system* (3.3) *(with respect to* $U \in [\mathbb{H}_0, \mathbb{H}]$). *This correspondence is given by the equality* (3.19), *where* $F(\cdot)$ *is a spectral function of* T *generated by* $\widetilde{T}$. *Moreover, the operators* $\widetilde{T}_0$ *(the operator part of* $\widetilde{T}$) *and* Λ_σ *are unitarily equivalent and hence they have the same spectral properties. In particular this implies that the spectral multiplicity of* $\widetilde{T}_0$ *does not exceed* $\dim \mathbb{H}_0$.

Corollary 5.7. *Let under the assumptions* (A1)–(A3) τ *be an admissible boundary parameter, let* $\sigma(\cdot) = \sigma_\tau(\cdot)$ *be a pseudospectral function (with respect to* U) *and let* $V_{0,\sigma}(= V_\sigma \upharpoonright \mathfrak{H}_0)$ *be the corresponding isometry from* $\mathfrak{H}_0$ *to* $L^2(\sigma; \mathbb{H}_0)$. *Then* $V_{0,\sigma}$ *is a unitary operator if and only if the parameter* τ *is self-adjoint. If this condition is satisfied, then the boundary conditions* (4.27) *defines an extension* $\widetilde{T}^\tau \in \mathrm{Self}_0(T)$ *and the*

operators $\widetilde{T}_{0,\tau}$ (the operator part of $\widetilde{T}^\tau$) and Λ_σ are unitarily equivalent by means of $V_{0,\sigma}$

Proof. The first statement is a consequence of Proposition 3.14 and Theorem 4.11. The second statement is implied by Theorems 4.11 and 5.6. □

The criterion which enables one to describe all pseudospectral functions in terms of an arbitrary (not necessarily admissible) boundary parameter is given in the following theorem.

Theorem 5.8. *The following statements are equivalent:*
(1) *each boundary parameter τ is admissible;*
(2) $\lim\limits_{y\to+\infty} \dot{M}_+(iy) \restriction \dot{\mathcal{H}}_1 = 0$ *and*

$$\lim_{y\to+\infty} y\left(\mathrm{Im}(\dot{M}_+(iy)h,h)_{\dot{\mathcal{H}}_0} + \tfrac{1}{2}||\dot{P}_2 h||^2\right) = +\infty,$$

where $h \in \dot{\mathcal{H}}_0$, $h \neq 0$ and $\dot{P}_2$ is the orthoprojection in $\dot{\mathcal{H}}_0$ onto $\dot{\mathcal{H}}_2 :=$ $\dot{\mathcal{H}}_0 \ominus \dot{\mathcal{H}}_1$;
(3) $\mathrm{mul}\, T = \mathrm{mul}\, T^*$, *i.e., the condition (C2) in Assertion 4.3 is fulfilled;*
(4) *statement of Theorem 5.5 holds for arbitrary boundary parameters τ.*

Proof. Proposition 5.2 and (2.5) yield the equivalence (1)⇔(3). Since by Lemma 4.14, (2) $\dot{M}_+(\cdot)$ is the Weyl function of the boundary triplet $\dot{\Pi}$, the equivalence (2)⇔(3) is implied by [24, Theorem 4.6]. The equivalence (1)⇔(4) follows from Theorem 5.5. □

Combining the results of this section with Proposition 3.11 we get the following theorem.

Theorem 5.9. *Let the assumptions* (A1) *and* (A2) *be satisfied. Then the set of spectral functions of the system* (3.3) *(with respect to $U \in [\mathbb{H}_0, \mathbb{H}]$) is not empty if and only if $\mathrm{mul}\, T = \{0\}$ or equivalently if and only if the condition (C1) in Assertion 4.3 is fulfilled. If this condition is satisfied, then the sets of spectral and pseudospectral functions of the system* (3.3) *coincide and hence Theorems 5.5, 5.6, 5.8 and Corollary 5.7 are valid for spectral functions (instead of pseudospectral ones). In this case $\widetilde{T}_0$, $\widetilde{T}_{0,\tau}$ and $V_{0,\sigma}$ in Theorem 5.6 and Corollary 5.7 should be replaced with $\widetilde{T}$, $\widetilde{T}_\tau$ and V_σ respectively. Moreover, in this case statement* (3) *in Theorem 5.8 takes the following form:*
(3') $\mathrm{mul}\, T^* = \{0\}$, *i.e., the condition (C3) in Assertion 4.3 is fulfilled.*

Remark 5.10. Assume that $N_- \leq N_+$ and θ is a subspace in $\mathbb{H}$ such that $\theta^\times \in Sym(\mathbb{H})$ and system (3.3) is θ-definite. Moreover, let $\mathbb{H}'_0$ be a subspace in $\mathbb{H}$ and let $K_\theta \in [\mathbb{H}'_0, \mathbb{H}]$ be an operator with $\ker K_\theta = \{0\}$ and $K_\theta \mathbb{H}'_0 = \theta$. It follows from Proposition 3.12 and Remark 3.13 that Theorems 5.5, 5.6, 5.8, 5.9 and Corollary 5.7 are valid, with some corrections, for pseudospectral and spectral functions $\sigma(\cdot)$ with respect to K_θ in place of U. We leave to the reader the precise formulation of the specified results.

6. The case of the minimally possible $\dim \theta$. Spectral functions of the minimal dimension

It follows from Lemma 3.1, (1) that the minimally possible dimension of the subspace $\theta \subset \mathbb{H}$ satisfying the assumption (A1) in Section 4.1 is

$$\dim \theta = \nu + \widehat{\nu}. \tag{6.1}$$

If θ satisfies (A1) and (6.1) then the previous results become essentially simpler. Namely, in this case the subspace $\mathbb{H}_0$ from assumption (A2) satisfies $\dim \mathbb{H}_0 = \dim(H \oplus \widehat{H})$ and hence $H_1 = \{0\}$, $H_1^\perp = H$ and

$$\mathbb{H}_0 = H \oplus \widehat{H}. \tag{6.2}$$

Therefore the assumption (A2) in Section 4.1 takes the following form:

(A2$'$) $\mathbb{H}_0$ is the subspace (6.2), $\widetilde{U}$ and Γ_a are the same as in the assumption (A2) and

$$\Gamma_a = (\Gamma_{0a}, \widehat{\Gamma}_a, \Gamma_{1a})^\top : \operatorname{dom} \mathcal{T}_{\max} \to H \oplus \widehat{H} \oplus H \tag{6.3}$$

is the block representation of Γ_a.

Below we suppose (unless otherwise is stated) the following assumption (A$_{\min}$), which is equivalent to the assumptions (A1) - (A3) and the equality (6.1):

(A$_{\min}$) In addition to (A1) the equality (6.1) holds and the assumptions (A2$'$) and (A3) are satisfied.

Under this assumption the equalities (4.11) take the form

$$\dot{\mathcal{H}}_0 = \widehat{H} \oplus \widetilde{\mathcal{H}}_b, \qquad \dot{\mathcal{H}}_1 = \widehat{H} \oplus \mathcal{H}_b \tag{6.4}$$

and a boundary parameter is the same as in definition 4.10.

Theorem 6.1. *Let τ be a boundary parameter* (4.24) *and let*

$$C_0(\lambda) = (\widehat{C}_0(\lambda), C_{0b}(\lambda)) : \widehat{H} \oplus \widetilde{\mathcal{H}}_b \to \dot{\mathcal{H}}_0,$$
$$C_1(\lambda) = (\widehat{C}_1(\lambda), C_{1b}(\lambda)) : \widehat{H} \oplus \mathcal{H}_b \to \dot{\mathcal{H}}_0$$

be the block representations of $C_0(\lambda)$ and $C_1(\lambda)$. Then for each $\lambda \in \mathbb{C}_+$ there exists a unique pair of operator solutions $\xi_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[H, \mathbb{H}]$ and $\widehat{\xi}_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\widehat{H}, \mathbb{H}]$ of the system (3.3) satisfying the boundary conditions

$$\Gamma_{1a}\xi_\tau(\lambda) = -I_H, \tag{6.5}$$

$$[(i\widehat{C}_0(\lambda) - \tfrac{1}{2}\widehat{C}_1(\lambda))\widehat{\Gamma}_a + C_{0b}(\lambda)\Gamma_{0b} - (i\widehat{C}_0(\lambda) + \tfrac{1}{2}\widehat{C}_1(\lambda))\widehat{\Gamma}_b$$
$$+ C_{1b}(\lambda)\Gamma_{1b}]\xi_\tau(\lambda) = 0, \tag{6.6}$$

$$\Gamma_{1a}\widehat{\xi}_\tau(\lambda) = 0, \tag{6.7}$$

$$[(i\widehat{C}_0(\lambda) - \tfrac{1}{2}\widehat{C}_1(\lambda))\widehat{\Gamma}_a + C_{0b}(\lambda)\Gamma_{0b}$$
$$- (i\widehat{C}_0(\lambda) + \tfrac{1}{2}\widehat{C}_1(\lambda))\widehat{\Gamma}_b + C_{1b}(\lambda)\Gamma_{1b}]\widehat{\xi}_\tau(\lambda) = \widehat{C}_0(\lambda) + \tfrac{i}{2}\widehat{C}_1(\lambda). \tag{6.8}$$

Proof. Let $v_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}_0, \mathbb{H}]$ be the solution of (3.3) defined in theorem 4.15 and let

$$v_\tau(t, \lambda) = (\xi_\tau(t, \lambda), \widehat{\xi}_\tau(t, \lambda)) : H \oplus \widehat{H} \to \mathbb{H} \tag{6.9}$$

be the block representation of $v_\tau(t, \lambda)$. Then the first condition in (4.68) takes the form $\Gamma_{1a}(\xi_\tau(\lambda), \widehat{\xi}_\tau(\lambda)) = (-I_H, 0)$, which is equivalent to (6.5) and (6.7). Moreover, (4.12), (4.13) and (4.67) take the form

$$\dot{\Gamma}'_0 = (i(\widehat{\Gamma}_a - \widehat{\Gamma}_b), \Gamma_{0b})^\top, \quad \dot{\Gamma}'_1 = (\tfrac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b), -\Gamma_{1b})^\top,$$
$$\Phi(\lambda) = (0, \widehat{C}_0(\lambda) + \tfrac{i}{2}\widehat{C}_1(\lambda)).$$

Therefore the second condition in (4.68) is equivalent to (6.6) and (6.8). Now the required statement is implied by Theorem 4.15. $\qquad\square$

It follows from (6.2), (6.3) and (4.72) that $P_{\mathbb{H},\mathbb{H}_0}\Gamma_a = (\Gamma_{0a}, \widehat{\Gamma}_a)^\top$ and $J_0 = \begin{pmatrix} 0 & 0 \\ 0 & iI_{\widehat{H}} \end{pmatrix}$. This and (4.73) imply that in the case (6.1) (i.e., under the assumption $(A_{\min})$) the m-function $m_\tau(\cdot)$ can be defined as

$$m_\tau(\lambda) = \begin{pmatrix} \Gamma_{0a}\xi_\tau(\lambda) & \Gamma_{0a}\widehat{\xi}_\tau(\lambda) \\ \widehat{\Gamma}_a\xi_\tau(\lambda) & \widehat{\Gamma}_a\widehat{\xi}_\tau(\lambda) + \tfrac{i}{2}I_{\widehat{H}} \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{\mathbb{H}_0} \to \underbrace{H \oplus \widehat{H}}_{\mathbb{H}_0}, \quad \lambda \in \mathbb{C}_+.$$

The following proposition is implied by Proposition 4.12.

Proposition 6.2. *For any $\lambda \in \mathbb{C}_+$ there exists a unique collection of operator solutions $\xi_0(\cdot, \lambda) \in \mathcal{L}^2_\Delta[H, \mathbb{H}]$, $\widehat{\xi}_0(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\widehat{H}, \mathbb{H}]$ and $u_+(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\widetilde{\mathcal{H}}_b, \mathbb{H}]$ of the system (3.3) satisfying the boundary conditions*

$$\Gamma_{1a}\xi_0(\lambda) = -I_H, \qquad \widehat{\Gamma}_a\xi_0(\lambda) = \widehat{\Gamma}_b\xi_0(\lambda), \qquad \Gamma_{0b}\xi_0(\lambda) = 0,$$

$$\Gamma_{1a}\widehat{\xi}_0(\lambda) = 0, \qquad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)\widehat{\xi}_0(\lambda) = I_{\widehat{H}}, \qquad \Gamma_{0b}\widehat{\xi}_0(\lambda) = 0,$$

$$\Gamma_{1a}u_+(\lambda) = 0, \qquad \widehat{\Gamma}_a u_+(\lambda) = \widehat{\Gamma}_b u_+(\lambda), \qquad \Gamma_{0b}u_+(\lambda) = I_{\widetilde{\mathcal{H}}_b}.$$

If the assumption $(A_{\min})$ is satisfied, then the operator function $M_+(\cdot)$ from Proposition 4.13 takes the form

$$M_+(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} : H \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b \to H \oplus \widehat{H} \oplus \mathcal{H}_b,$$

where $\lambda \in \mathbb{C}_+$ and

$$M_{11}(\lambda) = \Gamma_{0a}\xi_0(\lambda), \quad M_{12}(\lambda) = \Gamma_{0a}\widehat{\xi}_0(\lambda), \quad M_{13}(\lambda) = \Gamma_{0a}u_+(\lambda),$$

$$M_{21}(\lambda) = \widehat{\Gamma}_a\xi_0(\lambda), \quad M_{22}(\lambda) = \widehat{\Gamma}_a\widehat{\xi}_0(\lambda) + \tfrac{i}{2}I_{\widehat{H}}, \quad M_{23}(\lambda) = \widehat{\Gamma}_a u_+(\lambda),$$

$$M_{31}(\lambda) = -\Gamma_{1b}\xi_0(\lambda), \quad M_{32}(\lambda) = -\Gamma_{1b}\widehat{\xi}_0(\lambda), \quad M_{33}(\lambda) = -\Gamma_{1b}u_+(\lambda).$$

Moreover, the operator functions $m_0(\cdot)$, $S_1(\cdot)$, $S_2(\cdot)$ and $\dot{M}_+(\cdot)$ in Theorem 5.5 take the following simpler form (cf. (4.50)-(4.53)):

$$m_0(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{\mathbb{H}_0} \to \underbrace{H \oplus \widehat{H}}_{\mathbb{H}_0}, \quad \lambda \in \mathbb{C}_+$$

$$S_1(\lambda) = \begin{pmatrix} M_{12}(\lambda) & M_{13}(\lambda) \\ M_{22}(\lambda) - \tfrac{i}{2}I_{\widehat{H}} & M_{23}(\lambda) \end{pmatrix} : \underbrace{\widehat{H} \oplus \widetilde{\mathcal{H}}_b}_{\dot{\mathcal{H}}_0} \to \underbrace{H \oplus \widehat{H}}_{\mathbb{H}_0}, \quad \lambda \in \mathbb{C}_+$$

$$S_2(\lambda) = \begin{pmatrix} M_{21}(\lambda) & M_{22}(\lambda) + \tfrac{i}{2}I_{\widehat{H}} \\ M_{31}(\lambda) & M_{32}(\lambda) \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{\mathbb{H}_0} \to \underbrace{\widehat{H} \oplus \mathcal{H}_b}_{\dot{\mathcal{H}}_1}, \quad \lambda \in \mathbb{C}_+$$

$$\dot{M}_+(\lambda) = \begin{pmatrix} M_{22}(\lambda) & M_{23}(\lambda) \\ M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} : \underbrace{\widehat{H} \oplus \widetilde{\mathcal{H}}_b}_{\dot{\mathcal{H}}_0} \to \underbrace{\widehat{H} \oplus \mathcal{H}_b}_{\dot{\mathcal{H}}_1}, \quad \lambda \in \mathbb{C}_+.$$

In the following theorem we characterize spectral functions of the minimal dimension.

Theorem 6.3. *Let system* (3.3) *be definite (see Definition 3.15) and let $N_- \leq N_+$. Then the following statements are equivalent:*

(i) $\operatorname{mul} T_{\min} = \{0\}$, *i.e., the condition* (C0) *in Assertion 4.3 is fulfilled;*

(ii) *The set of spectral functions of the system is not empty, i.e., there exist subspaces θ and $\mathbb{H}'_0$ in $\mathbb{H}$ and a spectral function $\sigma(\cdot)$ of the system (with respect to $K_\theta \in\in [\mathbb{H}'_0, \mathbb{H}]$).*

If the statement (i) *holds, then the dimension n_σ of each spectral function $\sigma(\cdot)$ (see Definition 3.10) satisfies*

$$\nu + \widehat{\nu} \leq n_\sigma \leq n \tag{6.10}$$

and there exists a spectral function $\sigma(\cdot)$ with the minimally possible dimension $n_\sigma = \nu + \widehat{\nu}$.

Proof. Assume statement (i). Then by Lemma 3.4 there exists a subspace $\theta \subset \mathbb{H}$ such that $\theta^\times \in Sym(\mathbb{H})$, $\dim \theta = \nu + \widehat{\nu}$ and the relation T of the form (3.12) satisfies $\operatorname{mul} T = \{0\}$. Therefore by Corollary 5.4, (2) and Proposition 3.11 there exists a spectral function $\sigma(\cdot)$ (with respect to K_θ). Moreover, $n_\sigma (= \dim \theta) = \nu + \widehat{\nu}$.

Next assume that θ is a subspace in $\mathbb{H}$ and $\sigma(\cdot)$ is a spectral function (with respect to K_θ). Since the system is definite, it follows from Proposition 3.16 that $\theta^\times \in Sym(\mathbb{H})$. Therefore by Lemma 3.1, (1) $n_\sigma (= \dim \theta) \geq \nu + \widehat{\nu}$, which yields (6.10).

Conversely, let statement (ii) holds. If $\sigma(\cdot)$ is a spectral function (with respect to K_θ), then according to Proposition 3.11 $\operatorname{mul} T = \{0\}$. This and the obvious inclusion $\operatorname{mul} T_{\min} \subset \operatorname{mul} T$ yield statement (i).

$\square$

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