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Generalized integral theorems for the quaternionic G-monogenic mappings

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Abstract. For G-monogenic mappings taking values in the algebra of complex quaternions we generalize some analogues of classical integral theorems of the holomorphic function theory of a complex variable (the surface and the curvilinear Cauchy integral theorems).

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1. Introduction

The Cauchy integral theorems for analytic functions of the complex variable are fundamental results of the classical complex analysis. Analogues of these results are also important tools in the quaternionic analysis.

In the papers [1–3], some analogues of classical integral theorems for G-monogenic mappings taking values in the algebra of complex quaternions were established. Namely, in the paper [1] the Stokes formula, a curvilinear analogue of the Cauchy integral theorem in the case where a curve of integration lies in a domain of G-monogeneity, the Cauchy integral formula, the Gauss–Ostrogradsky formula and the surface Cauchy integral theorem were proved. The analogues of the Cauchy integral theorems are of the form

$$\int\limits_{\Gamma} \widehat{\Phi}\,\sigma = 0, \qquad \int\limits_{\Gamma} \sigma\,\Phi = 0, \tag{1.1}$$

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where Γ is a closed surface (or a closed curve), σ is a special differential form, and $\widehat{\Phi}$, Φ are left-G-monogenic mapping and right-G-monogenic mapping, respectively.

In the paper [2], the formulae (1.1) was proved in the case where a curve of integration lies on the boundary of a domain of G-monogeneity. In the paper [3], the analogue of the Morera theorem was established.

In the present paper we generalize analogues of the surface and curvilinear Cauchy integral theorems for G-monogenic mappings to "two sides" integrals. Namely, the equality

$$\int\limits_{\Gamma} \widehat{\Phi}\, \sigma\, \Phi = 0 \tag{1.2}$$

will be proved under some assumptions. In the papers [4] and [5] the formula of the type (1.2) was proved for another class of quaternionic differentiable functions.

2. G-monogenic mappings in the algebra of complex quaternions

Let $\mathbb{H}(\mathbb{C})$ be the quaternion algebra over the field of complex numbers $\mathbb{C}$, whose basis consists of the unit 1 of the algebra and of the elements I, J, K satisfying the multiplication rules:

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

In the algebra $\mathbb{H}(\mathbb{C})$ there exists another basis $\{e_1, e_2, e_3, e_4\}$ such that multiplication table in this basis can be represented as

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	e_1
e_4	e_4	0	e_2	0

.

The unit of the algebra can be decomposed as $1 = e_1 + e_2$.

Let us consider the vectors

$$i_1 = e_1 + e_2, \quad i_2 = a_1 e_1 + a_2 e_2, \quad i_3 = b_1 e_1 + b_2 e_2, \tag{2.1}$$

where $a_k, b_k \in \mathbb{C}$, $k = 1, 2$, which are linearly independent over the field of real numbers $\mathbb{R}$. It means that the equality $\alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_3 = 0$ for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

In the algebra $\mathbb{H}(\mathbb{C})$ we consider the linear span

$$E_3 := \{\zeta = xi_1 + yi_2 + zi_3 : x, y, z \in \mathbb{R}\}$$

generated by the vectors i_1, i_2, i_3 over the field $\mathbb{R}$.

A set $S \subset \mathbb{R}^3$ is associated with the set $S_\zeta := \{\zeta = xi_1 + yi_2 + zi_3 : (x, y, z) \in S\}$ in E_3. We understand topological properties of the set S_ζ in E_3 as the same topological properties of the set S in $\mathbb{R}^3$.

In the paper [6] we introduced a new class of quaternionic mappings, so-called, G-monogenic mappings.

We say that a continuous mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})\big)$ is *right-G-monogenic* $\big($or *left-G-monogenic*$\big)$ in a domain $\Omega_\zeta \subset E_3$, if Φ $\big($or $\widehat{\Phi}\big)$ is differentiable in the sense of the Gâteaux at every point of Ω_ζ, i. e. for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi}'(\zeta) \in \mathbb{H}(\mathbb{C})\big)$ such that

$$\lim_{\varepsilon \to 0+0} \Big(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\Big)\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_3$$

$$\left(\text{or} \quad \lim_{\varepsilon \to 0+0} \Big(\widehat{\Phi}(\zeta + \varepsilon h) - \widehat{\Phi}(\zeta)\Big)\,\varepsilon^{-1} = \widehat{\Phi}'(\zeta)h \quad \forall\, h \in E_3\right).$$

Consider the decomposition of the mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\Phi(\zeta) = \sum_{k=1}^{4} U_k(x, y, z)e_k.$$

In the case where functions $U_k : \Omega \to \mathbb{C}$ are $\mathbb{R}$-differentiable in Ω, i. e. for every $(x, y, z) \in \Omega$

$$U_k(x + \Delta x, y + \Delta y, z + \Delta z) - U_k(x, y, z)$$
$$= \frac{\partial U_k}{\partial x}\Delta x + \frac{\partial U_k}{\partial y}\Delta y + \frac{\partial U_k}{\partial z}\Delta z + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\right),$$
$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \to 0,$$

the mapping Φ is right-G-monogenic and $\widehat{\Phi}$ is left-G-monogenic in the domain Ω_ζ if and only if the following analogues of the Cauchy – Riemann conditions are satisfied in Ω_ζ:

$$\frac{\partial \Phi}{\partial y} = i_2 \frac{\partial \Phi}{\partial x}, \qquad \frac{\partial \Phi}{\partial z} = i_3 \frac{\partial \Phi}{\partial x} \tag{2.2}$$

and

$$\frac{\partial \widehat{\Phi}}{\partial y} = \frac{\partial \widehat{\Phi}}{\partial x} i_2, \qquad \frac{\partial \widehat{\Phi}}{\partial z} = \frac{\partial \widehat{\Phi}}{\partial x} i_3. \tag{2.3}$$

3. Cauchy integral theorem for a surface integral

Let Ω_ζ be a bounded domain in E_3. For a continuous mapping $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ of the form

$$\varphi(\zeta) = \sum_{k=1}^{4} U_k(x,y,z)e_k + i\sum_{k=1}^{4} V_k(x,y,z)e_k,$$

where $(x,y,z) \in \Omega$ and $U_k : \Omega \to \mathbb{R}$, $V_k : \Omega \to \mathbb{R}$, we define *a volume integral* by the equality

$$\int\limits_{\Omega_\zeta} \varphi(\zeta)dxdydz := \sum_{k=1}^{4} e_k \int\limits_{\Omega} U_k(x,y,z)dxdydz$$

$$+i\sum_{k=1}^{4} e_k \int\limits_{\Omega} V_k(x,y,z)dxdydz.$$

Let Σ_ζ be a piece-smooth surface in E_3. For continuous mappings $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ of the forms

$$\varphi(\zeta) = \sum_{k=1}^{4} U_k(x,y,z)e_k + i\sum_{k=1}^{4} V_k(x,y,z)e_k, \tag{3.1}$$

$$\psi(\zeta) = \sum_{m=1}^{4} P_m(x,y,z)e_m + i\sum_{m=1}^{4} Q_m(x,y,z)e_m, \tag{3.2}$$

where $(x,y,z) \in \Sigma$, $U_k : \Sigma \to \mathbb{R}$, $V_k : \Sigma \to \mathbb{R}$ and $P_m : \Sigma \to \mathbb{R}$, $Q_m : \Sigma \to \mathbb{R}$, we define *a surface integral* on a piece-smooth surface Σ_ζ with the differential form

$$\sigma := dydz + dzdxi_2 + dxdyi_3$$

by the equality

$$\int_{\Sigma_\zeta} \varphi(\zeta)\,\sigma\,\psi(\zeta) := \sum_{k,m=1}^{4} e_k e_m \int_{\Sigma} \big(U_k\,P_m - V_k\,Q_m\big)dydz$$
$$+ \sum_{k,m=1}^{4} e_k i_2 e_m \int_{\Sigma} \big(U_k\,P_m - V_k\,Q_m\big)dzdx$$
$$+ \sum_{k,m=1}^{4} e_k i_3 e_m \int_{\Sigma} \big(U_k\,P_m - V_k\,Q_m\big)dxdy$$
$$+ i\sum_{k,m=1}^{4} e_k e_m \int_{\Sigma} \big(V_k\,P_m + U_k\,Q_m\big)dydz$$
$$+ i\sum_{k,m=1}^{4} e_k i_2 e_m \int_{\Sigma} \big(V_k\,P_m + U_k\,Q_m\big)dzdx$$
$$+ i\sum_{k,m=1}^{4} e_k i_3 e_m \int_{\Sigma} \big(V_k\,P_m + U_k\,Q_m\big)dxdy.$$

If a domain $\Omega \subset \mathbb{R}^3$ has a closed piece-smooth boundary $\partial\Omega$ and mappings $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ are continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$, then the following analogues of *the Gauss–Ostrogradsky formula* is true:

$$\int_{\partial\Omega_\zeta} \varphi(\zeta)\,\sigma\,\psi(\zeta)$$

$$= \int_{\Omega_\zeta} \left(\frac{\partial\varphi}{\partial x}\psi + \varphi\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\,i_2\,\psi + \varphi\,i_3\,\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\,i_3\,\psi + \varphi\,i_3\,\frac{\partial\psi}{\partial z}\right) dxdydz. \tag{3.3}$$

Using the equality (3.3) and the conditions (2.2), (2.3) we obtain the following theorem.

Theorem 3.1. *Suppose that a domain Ω_ζ has a closed piece-smooth boundary $\partial\Omega_\zeta$ and $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a right-G-monogenic mapping in Ω_ζ, and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a left-G-monogenic in Ω_ζ and continuous together with partial derivatives of the first order up to the boundary*

$\partial\Omega_\zeta$. *Then*

$$\int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\,\sigma\,\Phi(\zeta)$$
$$= \int\limits_{\Omega_\zeta} \Big[\widehat{\Phi}'(\zeta)(1+i_2^2+i_3^2)\Phi(\zeta) + \widehat{\Phi}(\zeta)(1+i_2^2+i_3^2)\Phi'(\zeta)\Big] dxdydz. \quad (3.4)$$

Proof. Using the conditions (2.2), (2.3) we have

$$\int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\,\sigma\,\Phi(\zeta)$$

$$= \int\limits_{\Omega_\zeta} \Big(\widehat{\Phi}'\,\Phi + \widehat{\Phi}\,\Phi' + \widehat{\Phi}'\,i_2^2\,\Phi + \widehat{\Phi}\,i_2^2\,\Phi' + \widehat{\Phi}'\,i_3^2\,\Phi + \widehat{\Phi}\,i_3^2\,\Phi'\Big) dxdydz$$

$$= \int\limits_{\Omega_\zeta} \Big[(\widehat{\Phi}' + \widehat{\Phi}'\,i_2^2 + \widehat{\Phi}'\,i_3^2)\,\Phi + \widehat{\Phi}(\Phi' + i_2^2\,\Phi' + i_3^2\,\Phi')\Big] dxdydz$$

$$= \int\limits_{\Omega_\zeta} \Big[\widehat{\Phi}'\,(1+i_2^2+i_3^2)\,\Phi + \widehat{\Phi}(1+i_2^2+i_3^2)\,\Phi'\Big] dxdydz.$$

□

The following statement is a consequence of Theorem 3.1.

Theorem 3.2. *Under conditions of Theorem* 3.1 *with the additional assumption* $1+i_2^2+i_3^2=0$*, i. e. mappings* Φ *and* $\widehat{\Phi}$ *are solutions of the three-dimensional Laplace equation, the equality (3.4) can be rewritten in the form*

$$\int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\,\sigma\,\Phi(\zeta) = 0.$$

4. Cauchy integral theorem for a curvilinear integral

Let γ_ζ be a Jordan rectifiable curve in E_3. For continuous mappings $\varphi : \gamma_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \gamma_\zeta \to \mathbb{H}(\mathbb{C})$ of the forms (3.1) and (3.2), respectively, where $(x,y,z) \in \gamma$, $U_k : \gamma \to \mathbb{R}$, $V_k : \gamma \to \mathbb{R}$ and $P_m : \gamma \to \mathbb{R}$, $Q_m : \gamma \to \mathbb{R}$, we define *a curvilinear integral* along a Jordan rectifiable curve γ_ζ by the equality:

$$\int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) := \sum_{k,m=1}^{4} e_k e_m \int\limits_{\gamma} \big(U_k\, P_m - V_k\, Q_m\big) dx$$

$$+ \sum_{k,m=1}^{4} e_k i_2 e_m \int\limits_{\gamma} \big(U_k\, P_m - V_k\, Q_m\big) dy$$

$$+ \sum_{k,m=1}^{4} e_k i_3 e_m \int\limits_{\gamma} \big(U_k\, P_m - V_k\, Q_m\big) dz$$

$$+ i \sum_{k,m=1}^{4} e_k e_m \int\limits_{\gamma} \big(V_k\, P_m - U_k\, Q_m\big) dx$$

$$+ i \sum_{k,m=1}^{4} e_k i_2 e_m \int\limits_{\gamma} \big(V_k\, P_m - U_k\, Q_m\big) dy$$

$$+ i \sum_{k,m=1}^{4} e_k i_3 e_m \int\limits_{\gamma} \big(V_k\, P_m - U_k\, Q_m\big) dz,$$

where $d\zeta := dx + i_2 dy + i_3 dz$.

If mappings $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ are continuous together with partial derivatives of the first order in a domain Ω_ζ and Σ_ζ is an arbitrary piece-smooth surface in Ω_ζ with a rectifiable Jordan edge γ_ζ, then the following analogue of *the Stokes formula* is true:

$$\int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) = \int\limits_{\Sigma_\zeta} \left(\frac{\partial \varphi}{\partial x}\, i_2\, \psi + \varphi\, i_2\, \frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y}\, \psi - \varphi\, \frac{\partial \psi}{\partial y} \right) dxdy$$

$$+ \left(\frac{\partial \varphi}{\partial y}\, i_3\, \psi + \varphi\, i_3\, \frac{\partial \psi}{\partial y} - \frac{\partial \varphi}{\partial z}\, i_2\, \psi - \varphi\, i_2\, \frac{\partial \psi}{\partial z} \right) dydz$$

$$+ \left(\frac{\partial \varphi}{\partial z}\, \psi + \varphi\, \frac{\partial \psi}{\partial z} - \frac{\partial \varphi}{\partial x}\, i_3\, \psi - \varphi\, i_3\, \frac{\partial \psi}{\partial x} \right) dzdx. \qquad (4.1)$$

In the next theorem we show that the right-hand side of the equality (4.1) equals zero for the right-G-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and the left-G-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$. Note that the following theorem is a generalization of Theorem 1 of [1].

Theorem 4.1. *Suppose that* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is a right-G-monogenic mapping and* $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is a left-G-monogenic mapping in a domain* Ω_ζ *, and* γ_ζ *is a rectifiable Jordan edge of some piece-smooth surface in* Ω_ζ*. Then*

$$\int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = 0. \tag{4.2}$$

Proof. Using the formula (4.1) and the conditions (2.2) and (2.3), we obtain

$$\int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\Sigma_\zeta} \left(\frac{\partial \widehat{\Phi}}{\partial x}\, i_2\, \Phi + \widehat{\Phi}\; i_2\, \frac{\partial \Phi}{\partial x} - \frac{\partial \widehat{\Phi}}{\partial y}\, \Phi - \widehat{\Phi}\, \frac{\partial \Phi}{\partial y} \right) dxdy$$

$$+ \left(\frac{\partial \widehat{\Phi}}{\partial y}\, i_3\, \Phi + \widehat{\Phi}\, i_3\, \frac{\partial \Phi}{\partial y} - \frac{\partial \widehat{\Phi}}{\partial z}\, i_2\, \Phi - \widehat{\Phi}\; i_2\, \frac{\partial \Phi}{\partial z} \right) dydz$$

$$+ \left(\frac{\partial \widehat{\Phi}}{\partial z}\, \Phi + \widehat{\Phi}\, \frac{\partial \Phi}{\partial z} - \frac{\partial \widehat{\Phi}}{\partial x}\, i_3\, \Phi - \widehat{\Phi}\; i_3\, \frac{\partial \Phi}{\partial x} \right) dzdx$$

$$= \int\limits_{\Sigma_\zeta} \Big(\widehat{\Phi}'(\zeta)\, i_2\, \Phi(\zeta) + \widehat{\Phi}(\zeta)\, i_2\, \Phi'(\zeta) - \widehat{\Phi}'(\zeta)\, i_2\, \Phi(\zeta) - \widehat{\Phi}(\zeta)\, i_2\, \Phi'(\zeta) \Big) dxdy$$

$$+ \Big(\widehat{\Phi}'(\zeta)\, i_2 i_3\, \Phi(\zeta) + \widehat{\Phi}(\zeta)\, i_3 i_2\, \Phi'(\zeta) - \widehat{\Phi}'(\zeta)\, i_3 i_2\, \Phi(\zeta) - \widehat{\Phi}(\zeta)\, i_2 i_3\, \Phi'(\zeta) \Big) dydz$$

$$+ \Big(\widehat{\Phi}'(\zeta)\, i_3\, \Phi(\zeta) + \widehat{\Phi}(\zeta)\, i_3\, \Phi'(\zeta) - \widehat{\Phi}'(\zeta)\, i_3\, \Phi(\zeta) - \widehat{\Phi}(\zeta)\, i_3\, \Phi'(\zeta) \Big) dzdx = 0.$$

□

We understand a triangle $\triangle_\zeta$ as a plane figure bounded by three line segments connecting three its vertices. Denote by $\partial\triangle_\zeta$ the boundary of the triangle $\triangle_\zeta$ in the relative topology of its plane. Also we assume that the triangle $\triangle_\zeta$ includes the boundary $\partial\triangle_\zeta$.

Since every triangle $\triangle_\zeta \subset \Omega_\zeta$ can be included into a convex subset of a domain Ω_ζ, the following statement is a consequence of Theorem 4.1.

Corollary 4.1. *If* $\Omega_\zeta \subset E_3$ *is a convex domain, a mapping* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is right-G-monogenic and a mapping* $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is left-G-monogenic, then for an arbitrary triangle* $\triangle_\zeta$ *such that* $\overline{\triangle_\zeta} \subset \Omega_\zeta$*, the following equality is true:*

$$\int\limits_{\partial\triangle_\zeta} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = 0. \tag{4.3}$$

Let us consider the algebra $\widetilde{\mathbb{H}}(\mathbb{R})$ with the basis $\{e_k, ie_k\}_{k=1}^4$ over the field $\mathbb{R}$ which is isomorphic to the algebra $\mathbb{H}(\mathbb{C})$ over the field $\mathbb{C}$. In the algebra $\widetilde{\mathbb{H}}(\mathbb{R})$ there exist another basis $\{i_k\}_{k=1}^8$, where the vectors i_1, i_2, i_3 are the same as in the equalities (2.1).

For the element $a := \sum\limits_{k=1}^{8} a_k i_k,\ a_k \in \mathbb{R}$, we define the Euclidian norm

$$\|a\| := \sqrt{\sum_{k=1}^{8} a_k^2}\,.$$

Accordingly, $\|\zeta\| = \sqrt{x^2+y^2+z^2}$ and $\|i_1\| = \|i_2\| = \|i_3\| = 1$.

Now we apply a scheme of the proof of the corresponding lemma for a function given in the complex plane (see, e. g., [7]) to the proof of the following statement.

Lemma 4.1. *Suppose that $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ are continuous mappings in a simply connected domain Ω_ζ, and γ_ζ is a rectifiable curve in Ω_ζ. Then for an arbitrary $\varepsilon > 0$ there exists a broken line $\Lambda_\zeta \subset \Omega_\zeta$, vertexes of which lie on the curve γ_ζ, and such that*

$$\left\| \int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) - \int\limits_{\Lambda_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) \right\| < \varepsilon. \tag{4.4}$$

Proof. Let us consider a closed domain $\overline{D}_\zeta \subset \Omega_\zeta$, containing inside the curve γ_ζ. Since φ and ψ are continuous at every point of the domain $\overline{D}_\zeta$, then it is uniformly continuous in this domain. It means that the product of these mappings is uniformly continuous too. Thus, for an arbitrary $\varepsilon_1 > 0$ there exists a number $\delta(\varepsilon) > 0$ such that

$$\|\varphi(\zeta')\,\psi(\zeta') - \varphi(\zeta'')\,\psi(\zeta'')\| < \varepsilon_1, \tag{4.5}$$

if $|\zeta' - \zeta''| < \delta(\varepsilon)$, where ζ', ζ'' are any points of the domain $\overline{D}_\zeta$. In addition, under the same assumptions, the following inequalities are true:

$$\|\varphi(\zeta')\, i_2\, \psi(\zeta') - \varphi(\zeta'')\, i_2\, \psi(\zeta'')\| < \varepsilon_2, \tag{4.6}$$

$$\|\varphi(\zeta')\, i_3\, \psi(\zeta') - \varphi(\zeta'')\, i_3\, \psi(\zeta'')\| < \varepsilon_3. \tag{4.7}$$

Let us divide the curve γ_ζ into the n arcs $Q_\zeta^0,\ Q_\zeta^1, \ldots, Q_\zeta^{n-1}$ so that the length of each of them was less than δ and enter the broken curve Λ_ζ so that their broken links $L_\zeta^0, L_\zeta^1, \ldots, L_\zeta^{n-1}$ tied these arcs. By ζ_0, $\zeta_1, \ldots, \zeta_{n-1}, \zeta_n$ denote the vertexes of the broken curve Λ_ζ. Since the length of every arc Q_ζ^k is less than δ, the distance between any two points on the same arc especially less than δ. The same is true for links L_ζ^k.

We compare the value of integral along the curve γ_ζ with the value of the same integral along the broken curve Λ_ζ. For this goal we consider a sum, which is an approximate value of the integral $\int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta)$:

$$S := \varphi(\zeta_0)\, \Delta\zeta_0\, \psi(\zeta_0) + \varphi(\zeta_1)\, \Delta\zeta_1\, \psi(\zeta_1) + \cdots + \varphi(\zeta_{n-1})\, \Delta\, \zeta_{n-1}\psi(\zeta_{n-1})\,. \tag{4.8}$$

Since $\Delta\zeta_k = \int\limits_{Q_\zeta^k} d\zeta$, the equality (4.8) can be represented in the form

$$S := \int\limits_{Q_\zeta^0} \varphi(\zeta_0)\, d\zeta\, \psi(\zeta_0) + \int\limits_{Q_\zeta^1} \varphi(\zeta_1)\, d\zeta\, \psi(\zeta_1) + \cdots + \int\limits_{Q_\zeta^{n-1}} \varphi(\zeta_{n-1})\, d\zeta\, \psi(\zeta_{n-1}). \tag{4.9}$$

On the other hand, the integral $\int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \Psi(\zeta)$ can be represented in the form of the sum of the integrals along the arcs Q_ζ^k:

$$\int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) = \int\limits_{Q_\zeta^0} \varphi(\zeta)\, d\zeta\, \psi(\zeta)$$
$$+ \int\limits_{Q_\zeta^1} \varphi(\zeta)\, d\zeta\, \psi(\zeta) + \cdots + \int\limits_{Q_\zeta^{n-1}} \varphi(\zeta)\, d\zeta\, \psi(\zeta). \tag{4.10}$$

Consider the difference of the equations (4.10) and (4.9):

$$\int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) - S = \int\limits_{Q_\zeta^0} \Big(\varphi(\zeta)\, d\zeta\, \psi(\zeta) - \varphi(\zeta_0)\, d\zeta\, \psi(\zeta_0)\Big)$$
$$+ \int\limits_{Q_\zeta^1} \Big(\varphi(\zeta)\, d\zeta\, \psi(\zeta) - \varphi(\zeta_1)\, d\zeta\, \psi(\zeta_1)\Big)$$
$$+ \cdots + \int\limits_{Q_\zeta^{n-1}} \Big(\varphi(\zeta)\, d\zeta\, \psi(\zeta) - \varphi(\zeta_{n-1})\, d\zeta\, \psi(\zeta_{n-1})\Big)$$
$$= \int\limits_{Q^0} \Big(\varphi(\zeta)\, \psi(\zeta) - \varphi(\zeta_0)\, \psi(\zeta_0)\Big) dx + \int\limits_{Q^0} \Big(\varphi(\zeta)\, i_2\, \psi(\zeta) - \varphi(\zeta_0)\, i_2\, \psi(\zeta_0)\Big) dy$$

$$+\int\limits_{Q^0}\Big(\varphi(\zeta)\,i_3\,\psi(\zeta)-\varphi(\zeta_0)\,i_3\,\psi(\zeta_0)\Big)dz$$

$$+\cdots+\int\limits_{Q^{n-1}}\Big(\varphi(\zeta)\,\psi(\zeta)-\varphi(\zeta_0)\,\psi(\zeta_0)\Big)dx$$

$$+\int\limits_{Q^{n-1}}\Big(\varphi(\zeta)\,i_2\,\psi(\zeta)-\varphi(\zeta_0)\,i_2\,\psi(\zeta_0)\Big)dy$$

$$+\int\limits_{Q^{n-1}}\Big(\varphi(\zeta)\,i_3\,\psi(\zeta)-\varphi(\zeta_0)\,i_3\,\psi(\zeta_0)\Big)dz.$$

Since on the every arc Q^k_ζ the inequalities (4.5) – (4.7) are true, we obtain

$$\Bigg\|\int\limits_{\gamma_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)-S\Bigg\|<\big(\varepsilon_1\,Q^0_x+\varepsilon_2\,Q^0_y+\varepsilon_3\,Q^0_z\big)+\ldots$$

$$\ldots+\big(\varepsilon_1\cdot Q^{n-1}_x+\varepsilon_2\,Q^{n-1}_y+\varepsilon_3\,Q^{n-1}_z\big)<\varepsilon\,Q^0+\cdots+\varepsilon\,Q^{n-1}<\varepsilon\,L, \quad (4.11)$$

where Q^j_x, Q^j_y, Q^j_z are lengths of the projections of the arc Q^j into the axes Ox, Oy, Oz, respectively, $\varepsilon:=\max\{\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3\}$ and L is the length of the curve γ_ζ.

In the same way we estimate the difference $\int\limits_{\Lambda_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)-S$ and obtain

$$\Bigg\|\int\limits_{\gamma_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)-S\Bigg\|<\varepsilon\,L. \quad (4.12)$$

Taking into account the inequalities (4.11) and (4.12), we have

$$\Bigg\|\int\limits_{\gamma_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)-\int\limits_{\Lambda_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)\Bigg\|\le\Bigg\|\int\limits_{\gamma_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)-S\Bigg\|$$

$$+\Bigg\|S-\int\limits_{\Lambda_\zeta}\varphi(\zeta)\,d\zeta\,\psi(\zeta)\Bigg\|<2\varepsilon L.$$

□

Now, using Corollary 4.1 and Lemma 4.1, we prove the following analogue of the Cauchy theorem for an arbitrary rectifiable curve in a convex domain.

Theorem 4.2. *Suppose that $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a right-G-monogenic mapping and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a left-G-monogenic mapping in a convex domain Ω_ζ. Then for any closed rectifiable Jordan curve $\gamma_\zeta \subset \Omega_\zeta$ the equality* (4.2) *is true.*

Proof. Basing on Lemma 4.1, we inscribe the broken curve Λ_ζ into the curve γ_ζ so that the inequality (4.4) holds. Then we divide Λ_ζ into triangles by the diagonals starting from a fixed vertex of Λ_ζ. Since the domain Ω_ζ is convex, all obtained triangles contained in Ω_ζ. By Corollary 4.1, the integral along the every triangle equals to zero. Then the integral along the broken curve equals to zero too:

$$\int\limits_{\Lambda_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) = 0. \tag{4.13}$$

Now, the equality (4.2) is a consequence of the relations (4.4) and (4.13). □

In the case where Ω_ζ is an arbitrary domain, similarly to the proof of Theorem 3.2 [8], we can prove the following statement.

Theorem 4.3. *Let $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ be a right-G-monogenic mapping and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ be a left-G-monogenic mapping in a domain Ω_ζ. Then for every closed Jordan rectifiable curve γ_ζ homotopic to a point in Ω_ζ, the equality* (4.2) *is true.*

Proof. Let a curve γ_ζ be defined by the equality $\zeta = \phi(t)$, $0 \le t \le 1$, where $\phi(0) = \phi(1) = \zeta_0$, and let γ_ζ be homotopic to the point ζ_0. Then there exists a continuous on the square $Q := [0,1]\times[0,1]$ mapping $H(s,t)$ of two real variables s and t, which takes values in the domain Ω_ζ and such that

$$H(0,t) = \phi(t), \qquad H(1,t) \equiv \zeta_0 \qquad \forall\, t \in [0,1],$$

$$H(s,0) = H(s,1) = \zeta_0 \qquad \forall\, s \in [0,1].$$

Since the mapping H is continuous on a compact set Q, its image $K := \{H(s,t) : (s,t) \in Q\}$ is a compact set in Ω_ζ.

Denote by $\rho := \min\limits_{\zeta' \in K,\, \zeta'' \in \partial\Omega_\zeta} \|\zeta' - \zeta''\|$.

The mapping H is also uniformly continuous on the set Q. It means that there exists $\delta > 0$ such that

$$\forall\,(s,t),(s',t') : |s'-s| < \delta,\ |t'-t| < \delta \Rightarrow \|H(s',t')-H(s,t)\| < \frac{\rho}{2}. \tag{4.14}$$

Let us choose a set of numbers $0 = t_0 < t_1 < \ldots < t_n = 1$, which are satisfying the inequalities $t_j - t_{j-1} < \delta$, $j = 1, 2, \ldots, n$, and put $s_1 = t_1$. Let $\zeta_{0,j} := H(0, t_j)$, $\zeta_{1,j} := H(s_1, t_j)$ for $j = 1, 2, \ldots, n-1$ and denote by L_ζ^j a segment, beginning at the point $\zeta_{0,j}$ and ending at the point $\zeta_{1,j}$. Also consider a curve $\gamma_\zeta^{[1]} := \{H(s_1, t) : 0 \le t \le 1\}$.

For a Jordan oriented curve γ_ζ, by $\gamma_\zeta[\zeta_1, \zeta_2]$ denote the arc beginning at the point ζ_1 and ending at the point ζ_2.

Since of the inequality (4.14), the arcs $\gamma_\zeta[\zeta_0, \zeta_{01}]$, $\gamma_\zeta^{[1]}[\zeta_0, \zeta_{11}]$ and the segment L_ζ^1 are contained in the ball $S(\zeta_0) := \{\zeta \in E_3 : ||\zeta - \zeta_0|| < \rho\}$. Since $S(\zeta_0)$ is a convex set and is contained in the domain Ω_ζ, the following equality is a consequence of Theorem 4.2

$$\int\limits_{\gamma_\zeta[\zeta_0,\zeta_{01}]} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) + \int\limits_{L_\zeta^1} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta^{[1]}[\zeta_0,\zeta_{11}]} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta). \quad (4.15)$$

The next inequalities follows from the inequalities (4.14):

$$||\zeta - \zeta_{0,j}|| < \frac{\rho}{2} \qquad \forall\, \zeta \in \gamma_\zeta[\zeta_{0,j}, \zeta_{0,j+1}],$$

$$||\zeta - \zeta_{1,j}|| < \frac{\rho}{2} \qquad \forall\, \zeta \in \gamma_\zeta^{[1]}[\zeta_{1,j}, \zeta_{1,j+1}], \qquad ||\zeta_{1,j} - \zeta_{0,j}|| < \frac{\rho}{2}$$

for $j = 1, 2, \ldots, n-2$. Then the arcs $\gamma_\zeta[\zeta_{0,j}, \zeta_{0,j+1}]$, $\gamma_\zeta^{[1]}[\zeta_{1,j}, \zeta_{1,j+1}]$ and the segments L_ζ^1, L_ζ^{j+1} are contained in the ball $S(\zeta_{0,j}) := \{\zeta \in E_3 : ||\zeta - \zeta_{0,j}|| < \rho\}$ for $j = 1, 2, \ldots, n-2$. Since $S(\zeta_{0,j})$ is a convex set and is contained in Ω_ζ, the next equalities follows from the Theorem 4.2

$$-\int\limits_{L_\zeta^j} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) + \int\limits_{\gamma_\zeta[\zeta_{0,j},\zeta_{0,j+1}]} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) +$$

$$+ \int\limits_{L_\zeta^{j+1}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta^{[1]}[\zeta_{1,j},\zeta_{1,j+1}]} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) \quad (4.16)$$

for $j = 1, 2, \ldots, n-2$.

Finally, similarly to the equality (4.15) we obtain the equality

$$-\int\limits_{L_\zeta^{n-1}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) + \int\limits_{\gamma_\zeta[\zeta_{0,n-1},\zeta_0]} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta^{[1]}[\zeta_{1,n-1},\zeta_0]} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta). \quad (4.17)$$

Adding all the equalities (4.15)–(4.17), we obtain the equality

$$\int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta^{[1]}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) \tag{4.18}$$

Then we put $s_j = t_j$ and consider the curve $\gamma_\zeta^{[j]} := \{H(s_j, t) : 0 \le t \le 1\}$ for $j = 1, 2, \ldots, n$. Similarly to the equality (4.18), we obtain the equalities

$$\int\limits_{\gamma_\zeta^{[1]}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta^{[2]}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \ldots = \int\limits_{\gamma_\zeta^{[n]}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta).$$

Hence, we have

$$\int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta^{[n]}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta),$$

where the curve $\gamma_\zeta^{[n]}$ degenerates to the point, because $H(1, t) \equiv \zeta_0$. Now, taking into account the equality

$$\int\limits_{\gamma_\zeta^{[n]}} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = 0,$$

we complete the proof of the theorem. □

Now, let us consider a curvilinear Cauchy integral theorem for G-monogenic mappings in the case where a curve of integration lies on the boundary of a domain of G-monogeneity.

Let a closed Jordan rectifiable curve $\gamma_\zeta \equiv \gamma_\zeta(t)$, where $0 \le t \le 1$, which is homotopic to an interior point $\zeta_0 \in \Omega_\zeta$, be given on the boundary $\partial\Omega_\zeta$ of the domain Ω_ζ. It means that there exists a mapping $H(s, t)$, which is continuous on the square $[0, 1] \times [0, 1]$, and such that $H(0, t) = \gamma_\zeta(t)$, $H(1, t) \equiv \zeta_0$, and all curves $\gamma_\zeta^s \equiv \gamma_\zeta^s(t) := \{\zeta = H(s, t) : 0 \le t \le 1\}$ for $0 < s < 1$ are contained in the domain Ω_ζ.

Consider also the curves $\Gamma_\zeta^t \equiv \Gamma_\zeta^t(s) := \{\zeta = H(s, t) : 0 \le s \le 1\}$. By mes denote the linear Lebesque measure of a rectifiable curve.

The following theorem can be proved similarly to the proof of Theorem 2 in [2] and Theorem 4 in [9].

Theorem 4.4. *Suppose that* $\Phi : \overline{\Omega}_\zeta \to \mathbb{H}(\mathbb{C})$ *and* $\widehat{\Phi} : \overline{\Omega}_\zeta \to \mathbb{H}(\mathbb{C})$ *are continuous mapping in the closure* $\overline{\Omega}_\zeta$ *of a domain* Ω_ζ, Φ *is right-G-monogenic and* $\widehat{\Phi}$ *is left-G-monogenic mapping in* Ω_ζ. *Suppose also that* $\gamma_\zeta \subset \partial\Omega_\zeta$ *is any closed Jordan rectifiable curve homotopic to a point* $\zeta_0 \in \Omega_\zeta$ *such that the curves of the family* $\{\Gamma^t_\zeta : 0 \le t \le 1\}$ *are rectifiable and the set* $\{mes\,\gamma^s_\zeta : 0 \le s \le 1\}$ *is bounded. Then the equality* (4.2), *is true.*

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