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An extremal problem for the non-overlapping domains

Andrey Targonskii

(*Presented by V. Ya. Gutlyanskii*)

Abstract. Sharp estimates of product of inner radii for pairwise disjoint domains are obtained. In particular, we solve an extremal problem in the case of arbitrary finite number of free poles on the system points on the rays.

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Introduction

This paper belongs to the theory of extremal problems on classes of non-overlapping domain, which is a separate direction in geometric theory of functions of a complex variable. The begin of these investigations associated with the paper of M. A. Lavrent'ev [1] in 1934. He found the maximum of some functional with respect to two simply connected domains with two fixed points. We note that this result was needed him for applying to some aerodynamics problems. In 1947, G. M. Goluzin solved a similar problem for three fixed points on the complex plane [2]. Then the topic began to evolve rapidly. In this connection we may recall papers of many authors, including Y. E. Alenitsina, M. A. Lebedev, J. Jenkins, P. M. Tamrazov, P. P. Kufareva and others. Using the idea of P. M. Tamrazov, in 1975 G. P. Bakhtin solved first the problem with so-called “free poles”, on the unit circle, see, e.g., [3].

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An important step for the development of this topic was papers of V. N. Dubinin. He developed a new method of research that is method of piecewise-separating transformation. He also first solved numerous of extremal problems for an arbitrary but fixed multi connected non-overlapping domains (see, e.g., [4–6]). Now this type of extremal problems is used for investigations in holomorphic dynamics.

In the last decade actively used Bakhtin's method of "managing functional". He managed to solve a series of extremal problems for so-called "radial systems of points" (see, e.g., [4, 7–12]). In the present paper we use the mentioned about Bakhtin's method.

1. Theory

Let $\mathbb{N}$, $\mathbb{R}$ — the sets natural and real numbers conformity, $\mathbb{C}$ — the plain complex numbers, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ — the Riemannian sphere.

For fix number $n \in \mathbb{N}$ system points

$$A_n = \{a_k\}_{k=1}^n$$

the relations are executed:

$$0 = \arg a_1 < \arg a_2 < ... < \arg a_n < 2\pi. \tag{1.1}$$

For such systems of points we will consider the following sizes:

$$\sigma_k = \frac{1}{\pi}\left(\arg a_{k+1} - \arg a_k\right),\ k = 1, 2, ..., n, \quad a_{n+1} := a_1.$$

Let's consider system of angular domains:

$$M_k := \{w :\ \arg a_k < \arg w < \arg a_{k+1}\}, \quad k = \overline{1, n}, \quad a_{n+1} := a_1.$$

Let's consider the following "operating", functionalities for arbitrary A_n-system points

$$\mathrm{T}\left(A_n\right) = \prod_{k=1}^{n} \chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\sigma_k}}\right)|a_k|,$$

where $\chi(t) = \frac{1}{2}(t + \frac{1}{t})$.

Let $\{B_k\}_{k=1}^n$ — arbitrary non-overlapping domains such that

$$a_k \in B_k, \quad B_k \subset \overline{\mathbb{C}}, \quad k = \overline{1, n}. \tag{1.2}$$

Let

$$g_B\left(z, a\right) = h_{B,a}(z) + \log\frac{1}{|z - a|}$$

generalized Green's function of domains B with respect to a point $a \in B$. If $a = \infty$, then

$$g_B\left(z, \infty\right) = h_{B,\infty}(z) + \log \frac{1}{|z|}.$$

The value of

$$r(B, a) := \exp\left(h_{B,a}(z)\right)$$

the define of inner radius domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ (see [4–6, 13–15]).

We use the concept of a quadratic differential. Recall that a quadratic differential on a Riemann surface S is a map

$$\varphi : TS \to \mathbb{C}$$

satisfying

$$\varphi(\lambda v) = \lambda^2 \varphi(v)$$

for all $v \in TS$ and all $\lambda \in \mathbb{C}$, TS — tangent space. If $z \in U \to \mathbb{C}$, is a chart defined on some open set $U \subset S$ then φ is equal on U to

$$\varphi_U(z)dz^2$$

for some function φ_U defined on $z(U)$.

Suppose that two charts $z : U \to \mathbb{C}$ and $w : V \to \mathbb{C}$ on S overlap, and let

$$h := w \circ z^{-1}$$

be the transition function. If φ is represented both as $\varphi_U(z)dz^2$ and $\varphi_V(w)dw^2$ on $U \cap V$, then we have

$$\varphi_V\left(h(z)\right)\left(h'(z)\right)^2 = \varphi_U(z).$$

One way to say this is that quadratic differentials transform under pullbacks by the square of the derivative. As the main results associated with it can be found in [16].

2. Results

Subject of studying of our work are the following problems.

Problem. Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \geq 0$. Maximum functional be found

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k|^{\alpha} \cdot r\left(B_k, a_k\right)\right),$$

where $A_n = \{a_k\}_{k=1}^{n}$ — arbitrary system points on the rays, the satisfied condition (1.1), $\{B_k\}_{k=1}^{n}$ — arbitrary set non-overlapping domains, the satisfied condition (1.2), $a_k \in B_k \subset \overline{\mathbb{C}}$, and all extremal the describe $(k = \overline{1, n})$.

Lemma 1. *The function*

$$P(\tau) = \ln \sin \frac{\pi\tau}{2}$$

is convex for $\tau \in (0,2)$.

Proof. Find the second-order derivative

$$P''(\tau) = \frac{\pi}{2} \cdot \left(\mathrm{ctg}\frac{\pi\tau}{2}\right)' = -\left(\frac{\pi}{2}\right)^2 \cdot \frac{1}{\sin^2 \frac{\pi\tau}{2}}.$$

Consequently,

$$P''(\tau) < 0, \text{ for } \quad 0 < \tau < 2.$$

□

Theorem 1. *Let* $n \in \mathbb{N}$, $n \geq 2$, $\alpha \geq 0$. *Then for all system points* $A_n = \{a_k\}_{k=1}^n$, *the satisfied condition* (1.1) *and*

$$(|a_k| - |a_{k+1}|)^2 = 4\sin^2 \frac{\pi\sigma_k}{2} \left(1 - |a_k||a_{k+1}|\right), \; k = \overline{1,n},$$

and arbitrary set non-overlapping domains $\{B_k\}_{k=1}^n$, *the satisfied condition* (1.2), *be satisfied inequality*

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k|^{\alpha} \cdot r\left(B_k, a_k\right)\right) \leq \left(\frac{2^{\alpha+2}}{n} \cdot \sin^{\alpha} \frac{\pi}{n}\right)^n \cdot \mathrm{T}\left(A_n\right).$$

The equality obtain in this inequality, when points a_k *and domains* B_k *are, conformity, the poles and the circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{w^{n-2}}{\left(w^n - 1\right)^2} dw^2. \tag{2.3}$$

Proof. The theorem of the proof leans on a method of the piece-dividing transformation developed by Dubinin (see [4–6]).

Function

$$\zeta_k\left(w\right) = -i\left(e^{-i \arg a_k} w\right)^{\frac{1}{\sigma_k}}, \quad k = 1, 2, \ldots, n \tag{2.4}$$

realizes univalent and conformal transformations of domain M_k to the right half-plane $\mathrm{Re}\zeta > 0$, for all $k = \overline{1,n}$.

From a formula (2.4) we receive the following asymptotic expressions

$$\left|\zeta_k\left(w\right) - \zeta_k\left(a_m\right)\right| \sim \frac{1}{\sigma_k} \left|a_m\right|^{\frac{1}{\sigma_k} - 1} \left|w - a_m\right|,$$

$$w \to a_m, \qquad k = 1, 2, ..., n,\ m = k, k+1. \tag{2.5}$$

It's obvious that

$$\zeta_k\left(a_k\right) = -i|a_k|^{\frac{1}{\sigma_k}},\ \zeta_k\left(a_{k+1}\right) = i|a_{k+1}|^{\frac{1}{\sigma_k}},\ k = 1, 2, ..., n. \tag{2.6}$$

Family of functions $\{\zeta_k(w)\}_{k=1}^{n}$, set by equality (2.4), it is possible for by piece-dividing transformation (see [4–6]) domains $\left\{B_k : k = \overline{1,n}\right\}$ in relation to the system of corners $\{M_k\}_{k=1}^{n}$. For any domain $\Delta \in \mathbb{C}$ the define $(\Delta)^* := \left\{w \in \overline{\mathbb{C}} :\ \overline{w} \in \Delta\right\}$. Let $G_k^{(1)}$ the define connected component $\zeta_k\left(B_k \bigcap \overline{M}_k\right) \bigcup \left(\zeta_k\left(B_k \bigcap \overline{M}_k\right)\right)^*$, containing a point $(-i)$, $G_{k-1}^{(2)}$ – the define connected component $\zeta_{k-1}\left(B_k \bigcap \overline{M}_{k-1}\right) \bigcup \left(\zeta_{k-1}\left(B_k \bigcap \overline{M}_{k-1}\right)\right)^*$, containing a point i, $k = \overline{1,n}$, $\overline{M}_0 := \overline{M}_n$, $\zeta_0 := \zeta_n$, $G_0^{(2)} := G_n^{(2)}$. It is clear, that, $G_k^{(s)}$ generally speaking, domains are multiconnected domains, $k = \overline{1,n}$, $s = 1, 2$. Pair of domains $G_{k-1}^{(2)}$ and $G_k^{(1)}$ grows out of piece-dividing transformation domains B_k concerning families $\{M_{k-1}, M_k\}$, $\{\zeta_{k-1}, \zeta_k\}$ in point a_k, $k = \overline{1,n}$.

From the Theorem 1.9 [13] (see also [5,6]) and the formulae (2.5), we have the inequalities

$$r\left(B_k, a_k\right) \leq \left[|a_k|^{1-\frac{1}{\sigma_k}} \cdot \sigma_k \cdot r\left(G_k^{(1)}, \zeta_k\left(a_k\right)\right) \cdot \sigma_{k-1}\right.$$

$$\left.\times\, |a_k|^{1-\frac{1}{\sigma_{k-1}}} \cdot r\left(G_{k-1}^{(2)}, \zeta_{k-1}\left(a_k\right)\right)\right]^{\frac{1}{2}}, \quad k = 1, 2, ..., n. \tag{2.7}$$

From the condition that the points a_k, $k = 1, 2, .., n$, we get that

$$|a_{k+1} - a_k| = 2 \sin \frac{\pi \sigma_k}{2},\ k = 1, 2, ..., n. \tag{2.8}$$

Using formulas (2.7), (2.8) it is received the following ratio:

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k|^{\alpha} \cdot r\left(B_k, a_k\right)\right) \leq 2^{n\alpha} \cdot \prod_{k=1}^{n} \frac{\sigma_k\,|a_k|}{|a_k|^{\frac{1}{2\sigma_k}} \cdot |a_k|^{\frac{1}{2\sigma_{k-1}}}}$$

$$\times \prod_{k=1}^{n} \sin^{\alpha} \frac{\pi \sigma_k}{2} \cdot \prod_{k=1}^{n} \left(r\left(G_k^{(1)}, \zeta_k\left(a_k\right)\right) \cdot r\left(G_k^{(2)}, \zeta_k\left(a_{k+1}\right)\right)\right)^{\frac{1}{2}}. \tag{2.9}$$

Inequalities Lavrent'ev using [1] and (2.6), we get:

$$r\left(G_k^{(1)}, \zeta_k\left(a_k\right)\right) \cdot r\left(G_k^{(2)}, \zeta_k\left(a_{k+1}\right)\right)$$

$$\leq \left(|a_k|^{\frac{1}{\sigma_k}} + |a_{k+1}|^{\frac{1}{\sigma_k}}\right)^2, \quad k = 1, 2, ..., n.$$

Taking into account the last inequality, the expression (2.9) can be written as follows:

$$\prod_{k=1}^{n}\left(|a_{k+1}-a_k|\cdot r\left(B_k,a_k\right)\right)\leq 2^{n\alpha}\cdot\prod_{k=1}^{n}\sigma_k\sin^{\alpha}\frac{\pi\sigma_k}{2}$$

$$\times\prod_{k=1}^{n}\frac{|a_k|^{\frac{1}{\sigma_k}}+|a_{k+1}|^{\frac{1}{\sigma_k}}}{|a_k|^{\frac{1}{2\sigma_k}}\cdot|a_k|^{\frac{1}{2\sigma_{k-1}}}}\,|a_k|\,.$$

It's obvious that

$$\prod_{k=1}^{n}\frac{|a_k|^{\frac{1}{\sigma_k}}+|a_{k+1}|^{\frac{1}{\sigma_k}}}{|a_k|^{\frac{1}{2\sigma_k}}\cdot|a_k|^{\frac{1}{2\sigma_{k-1}}}}\,|a_k|=2^n\cdot\mathrm{T}\left(A_n\right).$$

Also,

$$\prod_{k=1}^{n}\sigma_k\leq\left(\frac{2}{n}\right)^n.$$

The equality obtain in this inequality, if and only if

$$\sigma_1=\sigma_2=...=\sigma_n=\frac{2}{n}.$$

Then, we have:

$$\prod_{k=1}^{n}\left(|a_{k+1}-a_k|\cdot r\left(B_k,a_k\right)\right)\leq\left(\frac{2^{\alpha+2}}{n}\right)^n\cdot\mathrm{T}\left(A_n\right)\cdot\prod_{k=1}^{n}\sin^{\alpha}\frac{\pi\sigma_k}{2}. \quad (2.10)$$

The equality obtain in this inequality, when points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential

$$Q(\zeta)d\zeta^2=\frac{d\zeta^2}{\left(\zeta^2+1\right)^2}. \quad (2.11)$$

Using the Lemma that the function $\alpha\ln\sin\frac{\pi\sigma_k}{2}$, is convex for $\sigma_k\in(0;\,2)$, $\alpha\geq 0$. Hence, when $\sigma_k\in(0;\,2)$, then

$$\frac{\alpha}{n}\cdot\sum_{k=1}^{n}\ln\sin\frac{\pi\sigma_k}{2}\leq\alpha\ln\sin\left(\frac{\pi}{2}\cdot\frac{1}{n}\sum_{k=1}^{n}\sigma_k\right).$$

Given that

$$\sum_{k=1}^{n}\sigma_k=2,$$

we obtain

$$\prod_{k=1}^{n} \sin^{\alpha} \frac{\pi\sigma_k}{2} \leq \sin^{n\alpha} \frac{\pi}{n}. \qquad (2.12)$$

The equality obtain in this inequality, if and only if

$$\sigma_1 = \sigma_2 = ... = \sigma_n = \frac{2}{n}.$$

Then from (2.10) using formulas (2.12) it is received the following ratio

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k| \cdot r\left(B_k, a_k\right)\right) \leq \left(\frac{2^{\alpha+2}}{n}\right)^{n} \cdot \mathrm{T}\left(A_n\right) \cdot \sin^{n\alpha} \frac{\pi}{n}.$$

The equality obtain in this inequality, when points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential (2.3). It is derived from the square of the quadratic differential (2.11) conversion using

$$\zeta = -iw^{\frac{n}{2}}.$$

□

Provided that, $|a_k| = 1$, $k = 1, 2, ..., n$ we obtain the well known result.

Corollary 1. *[4–6]. Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha > 0$. Then for all system points $A_n = \{a_k\}_{k=1}^{n}$, the satisfied condition* (1.1) *and $|a_k| = 1$, $k = 1, 2, ..., n$, and arbitrary set non-overlapping domains $\{B_k\}_{k=1}^{n}$, the satisfied condition* (1.2)*, be satisfied inequality*

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k|^{\alpha} \cdot r\left(B_k, a_k\right)\right) \leq \left(\frac{2^{\alpha+2}}{n} \cdot \sin^{\alpha} \frac{\pi}{n}\right)^{n}.$$

The equality obtain in this inequality, when points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential (2.3)*.*

As a consequence, at $\alpha = 0$, $|a_k| = 1$, $k = 1, 2, ..., n$ we obtain the well known result.

Corollary 2. *[4–6]. Let $n \in \mathbb{N}$, $n \geq 2$. Then for all system points $A_n = \{a_k\}_{k=1}^{n}$, the satisfied condition* (1.1) *and $|a_k| = 1$, $k = 1, 2, ..., n$, and* *arbitrary set non-overlapping domains* $\{B_k\}_{k=1}^{n}$, *the satisfied condition* (1.2), *be satisfied inequality*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leq \left(\frac{4}{n}\right)^n .$$

The equality obtain in this inequality, when points a_k *and domains* B_k *are, conformity, the poles and the circular domains of the quadratic differential* (2.3).

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Contact information

Andrey Targonskii Zhitomir State University,
Department of Mathematics,
Zhitomir, Ukraine
E-Mail: targonsk@mail.ru,
targonsk@zu.edu.ua