

Convolution equations and mean value theorems for solutions of linear elliptic equations with constant coefficients in the complex plane

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Abstract. In terms of the Bessel functions we characterize smooth solutions of some convolution equations in the complex plane and prove a two-radius theorem for solutions of homogeneous linear elliptic equations with constant coefficients whose left hand side is representable in the form of the product of some non-negative integer powers of the complex differentiation operators ∂ and $\bar{\partial}$.

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Introduction

Convolution equations generated by distributions with compact supports and the corresponding mean value theorems was investigated by many authors (see, e.g., [1, 2]). In particular, Volchkov [2, Part 3, Chapter 2] described a wide class of radial distributions with compact supports such that solutions of the corresponding convolution equations in open Euclidean balls can be efficiently characterized in terms of the Bessel functions and proved general uniqueness and two-radius theorems for solutions of these equations that go back to classical results of John [3, Chapter 6] and Delsarte [4, Part 3] about spherical means, respectively.

Let $0 < r < R \leq +\infty$, $m \in \mathbb{N} := \{1, 2, \dots\}$, and $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $s \leq m$. In the present paper, we study smooth functions $f(z)$ defined in

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the disk $B_R := \{z \in \mathbb{C} : |z| < R\}$ that satisfy the convolution equation

$$\frac{1}{2\pi} \iint_{|\zeta-z|\leq r} f(\zeta)(\zeta - z)^s d\xi d\eta = \sum_{p=s}^{m-1} \frac{r^{2p+2}}{(2p+2)(p-s)!p!} \partial^{p-s} \bar{\partial}^p f(z) \quad (1)$$

for all $z \in B_{R-r}$, where i is the imaginary unit, $z = x + iy$, $\zeta = \xi + i\eta$ ($x, y, \xi, \eta \in \mathbb{R}$),

$$\partial f = \frac{\partial f}{\partial z} := \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),$$

and the right hand side of (1) is zero for $s = m$, i.e., in this case Eq. (1) has the form

$$\iint_{|\zeta-z|\leq r} f(\zeta)(\zeta - z)^s d\xi d\eta = 0.$$

One of the main results of the paper is Theorem 1 that describes functions $f \in C^\infty(B_R)$ satisfying Eq. (1) in terms of the representation of the Fourier coefficients of the function $\partial^{m-s} \bar{\partial}^m f$ by series of special functions. Similar but more complicated description of such functions f was obtained by the author [5] under the condition $s < m$ in terms of the representation of the Fourier coefficients of the function f . Theorems 2 and 3 are a sharp uniqueness theorem for solutions of the convolution equation (1) and a two-radius theorem characterizing solutions of the elliptic equation

$$\partial^{m-s} \bar{\partial}^m f = 0, \quad (2)$$

respectively.

A remarkable feature of the convolution equation (1) is that for $s > 0$ this equation is generated by non-radial distributions. The use of the function $\partial^{m-s} \bar{\partial}^m f$ in Theorem 1 instead of the function f in [5] reduces the general case of this theorem to the investigation of some concrete radial distributions with compact supports and their spherical transformations that justifies the validity of the application of Volchkov’s results [2, Part 3, Chapter 2] on the representation of solutions of convolution equations generated by radial distributions with compact supports.

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1. Main results

To each function $f \in C(B_R)$ we assign its Fourier series

$$f(z) \sim \sum_{k=-\infty}^{+\infty} f_k(\rho) e^{ik\varphi}, \quad (3)$$

where $z = \rho e^{i\varphi}$ is the trigonometric form of z ,

$$f_k(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{it}) e^{-itk} dt \quad (z \neq 0, \quad k \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}).$$

For $z = 0$ we define the Fourier coefficients by continuity, i.e., $f_0(0) = f(0)$, $f_k(0) = 0$ for all integer $k \neq 0$. In the sequel we shall use the following well-known property of the Fourier coefficients [2, Part 1, §5.1]: if $f \in C^\infty(B_R)$, then for any $k \in \mathbb{Z}$ the function $f_k(\rho) e^{ik\varphi}$ is infinitely differentiable with respect to x and y ($z = x + iy = \rho e^{i\varphi}$) and the Fourier series (3) converges to the function f in the space $\mathcal{E}(B_R)$, i.e., converges uniformly together with its all partial derivatives of arbitrary order on each compact subset of the ball B_R .

Let

$$J_{s+1}(z) := \left(\frac{z}{2}\right)^{s+1} \sum_{p=0}^{\infty} \frac{(-1)^p}{p! \Gamma(s+p+2)} \left(\frac{z}{2}\right)^{2p} \quad (z \in \mathbb{C})$$

be the Bessel function of order $s + 1$. Denote by $Z(g_r)$ the set of all zeroes of the entire function

$$g_r(z) := \frac{J_{s+1}(zr)}{(zr)^{s+1}} - \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)} (-1)^{p-s}}{(p+1)! (p-s)! 2^{2p-s+1}} \quad \text{for } s < m,$$

$$g_r(z) := \frac{J_{s+1}(zr)}{(zr)^{s+1}} \quad \text{for } s = m.$$

To each $k \in \mathbb{Z}$, $\lambda \in Z_r := Z(g_r) \setminus \{0\}$, and $\eta \in \{0, \dots, n_\lambda - 1\}$, where n_λ is the multiplicity of the zero λ , we assign the function

$$\Phi_{\lambda,\eta,k}(\rho) = \left(\frac{d}{dz}\right)^\eta (J_k(z\rho))|_{z=\lambda} \quad (\rho > 0).$$

Using the introduced notation and the above assumptions we can formulate our main results as follows.

Theorem 1. *A function $f \in C^\infty(B_R)$ satisfies Eq. (1) for all $z \in B_{R-r}$ if and only if the Fourier coefficients $f_k^*(\rho)$ of the function $f^*(z) := \partial^{m-s} \bar{\partial}^m f(z)$ is represented in the form*

$$f_k^*(\rho) = \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda-1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) \quad (0 < \rho < R, \quad k \in \mathbb{Z}), \quad (4)$$

where for any $\alpha > 0$ and $k \in \mathbb{Z}$ we have

$$\max_{\eta=0, \dots, n_\lambda-1} |c_{\lambda, \eta, k}| = O(|\lambda|^{-\alpha}) \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in Z_r. \quad (5)$$

Under the condition (5) the series (4) converges in $\mathcal{E}(B_R)$ and all the coefficients $c_{\lambda, \eta, k}$ are defined uniquely with respect to f^* .

It follows from Theorem 1 that solutions of the elliptic equation (2) in B_R satisfy the convolution equation (1) for all $r \in (0, R)$ and $z \in B_{R-r}$. Notice that $f \in C^\infty(B_R)$ implies $f^* \in C^\infty(B_R)$. Therefore all the terms $f_k^*(\rho)e^{ik\varphi}$ of the Fourier series of the function f^* are infinitely differentiable functions with respect to x and y and this series converges to the function f^* in the space $\mathcal{E}(B_R)$. On the other hand, we shall justify in Lemmas 1 and 2 that all zeroes λ of the function $g_r(z)$ with sufficiently large $|\lambda|$ are simple. Hence we have from (5) that for each $k \in \mathbb{Z}$ all the terms of the series

$$\sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda-1} c_{\lambda, \eta, k} \Phi_{\lambda, \eta, k}(\rho) e^{ik\varphi}$$

are infinitely differentiable functions with respect x and y and this series converges to the k -th term of the Fourier series of the function f^* in $\mathcal{E}(B_R)$.

Theorem 2. *The following assertions are true:*

(a) if $0 < r < R$, $f \in C^\infty(B_R)$, Eq. (1) holds for any $z \in B_{R-r}$, and $f(z) = 0$ for all $z \in B_r$, then $f(z) \equiv 0$ in B_R ;

(b) for any $r > 0$ and $\varepsilon \in (0, r)$ there exists a function $f \in C^\infty(\mathbb{C})$ such that Eq. (1) is satisfied everywhere in \mathbb{C} , $f(z) = 0$ for all $z \in B_{r-\varepsilon}$, and $f(z) \not\equiv 0$.

Theorem 3. *Let r_1 and r_2 be real positive numbers. Then the following assertions are valid:*

(a) if $R \geq r_1 + r_2$, $Z_{r_1} \cap Z_{r_2} = \emptyset$, $f \in C^\infty(B_R)$, and Eq. (1) holds for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, then f is a real analytic function that satisfies the elliptic equation (2) in B_R ;

(b) if $\max\{r_1, r_2\} < R < r_1 + r_2$ or $Z_{r_1} \cap Z_{r_2} \neq \emptyset$, then there exists a function $f \in C^\infty(B_R)$ satisfying Eq. (1) for all $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$ and such that $\partial^{m-s} \bar{\partial}^m f \not\equiv 0$ in B_R .

Corollary 1. *If $s = m$, r_1 and r_2 are real positive numbers such that $R > r_1 + r_2$ and r_1/r_2 is not a ratio of two distinct zeroes of the function J_{s+1} , $f \in C(B_R)$, and Eq. (1) holds for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, then f is an m -analytic function in B_R .*

Proof. Suppose that the conditions of Corollary 1 are satisfied. Since all zeros of the Bessel function J_{s+1} are real, then the condition that r_1/r_2 is not a ratio of two distinct zeroes of the function J_{s+1} is equivalent to the condition $Z_{r_1} \cap Z_{r_2} = \emptyset$. Fix a function $\varphi \in C_0^\infty(\mathbb{C})$ supported in the closed unit disk \bar{B}_1 such that $\iint_{\mathbb{C}} \varphi(z) dx dy = 1$ ($z = x + iy$). Then for any $\varepsilon \in (0, R - r_1 - r_2)$ the function

$$f_\varepsilon(z) := \varepsilon^{-n} \iint_{\bar{B}_\varepsilon} f(z - \zeta) \varphi(\zeta/\varepsilon) d\xi d\eta \quad (\zeta = \xi + i\eta)$$

is defined in the disk $B_{R-\varepsilon}$, belongs to the class $C^\infty(B_{R-\varepsilon})$, and satisfies the convolution equation (1) for all $r = \{r_1, r_2\}$ and $z \in B_{R-r-\varepsilon}$. It follows from Theorem 3 that for each $z \in B_{R-\varepsilon}$ we have the equality $\bar{\partial}^m f_\varepsilon(z) = 0$. Since $f_\varepsilon(z)$ converges to $f(z)$ as $\varepsilon \rightarrow +0$ uniformly on compact subsets of B_R , then f is a distributional solution of the elliptic equation $\bar{\partial}^m f = 0$ in B_R . Hence f is an m -analytic in B_R . \square

The following corollary of Theorem 3 is proved similarly.

Corollary 2. *If $s < m$, r_1 and r_2 are real positive numbers such that $r_1 + r_2 < R$ and $Z_{r_1} \cap Z_{r_2} = \emptyset$, $f \in C^{2m-s-2}(B_R)$, and Eq. (1) holds for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, then f is a real analytic function that satisfies the elliptic equation (2) in B_R .*

For $s < m$ Theorem 2 was proved in [6]. Corollary 1 for $R = +\infty$ was obtained by Zalcman [7]. Volchkov [8] has proved Theorem 3 for $s = m$. On the other hand, the case $m = 1$ and $s = 0$ of Theorem 3 is the case $n = 2$ of the local version of the classical Delsarte’s two-radius theorem characterizing harmonic functions in \mathbb{R}^n [2, Part 5, §5.1]. Corollary 2 was obtained in [5].

2. Auxiliary results and constructions

Throughout this paper we consider linear spaces of functions and distributions over the field of complex numbers \mathbb{C} . Recall that a distribution $T \in \mathcal{D}'(\mathbb{C})$ is said to be radial if for any function $\varphi(w) \in C_0^\infty(\mathbb{C})$ and any $\alpha \in \mathbb{R}$ we have $T(\varphi(w)) = T(\varphi(e^{i\alpha}w))$, where T acts on the function $\varphi(e^{i\alpha}w)$ in w .

Let T be a radial distribution with compact support in \mathbb{C} . Then the function $\tilde{T}(z) := T(J_0(z|w|))$ ($z \in \mathbb{C}$) is called the spherical transformation of the distribution T . The function $\tilde{T}(z)$ is an even entire function of exponential type and characterized by the following Paley–Wiener–Schwartz theorem for the spherical transformation [2, Part 1, §6.2, Theorem 6.5]: an even entire function $f(z)$ is the spherical transformation

of some radial distribution supported in the closed disk $\overline{B}_r := \{z \in \mathbb{C} : |z| \leq r\}$ if and only if there are $c > 0$ and $N > 0$ such that for any $z \in \mathbb{C}$ we have $|f(z)| \leq c(1 + |z|^N)e^{r|\operatorname{Im}z|}$.

As usual, Δ and δ denote the Laplace operator and the Dirac delta-function. The following properties of the spherical transformation are easily deduced from the definition:

$$\widetilde{\Delta T}(z) \equiv -z^2 \widetilde{T}(z), \quad \widetilde{\delta}(z) \equiv 1, \quad \widetilde{\Delta^k \delta}(z) \equiv (-z^2)^k \quad (k \in \mathbb{N}_0). \quad (6)$$

Consider the following radial distributions supported in \overline{B}_r :

$$U_r(\varphi) := \frac{1}{2\pi} \iint_{|w| \leq r} \partial^s \varphi(w) w^s du dv \quad (\varphi \in C_0^\infty(\mathbb{C}), \quad w = u + iv),$$

$$S_r := \sum_{p=s}^{m-1} \frac{r^{2p+2}}{2(p+1)(p-s)!p!} \left(\frac{\Delta}{4}\right)^p \delta \quad (S_r := 0 \text{ for } s = m),$$

$$T_r := U_r - S_r.$$

Then we obtain consequently from (6),

$$\begin{aligned} \widetilde{S}_r(z) &= \sum_{p=s}^{m-1} \frac{r^{2p+2}(-1)^p}{2(p+1)(p-s)!p!4^p} z^{2p} \\ &= (-1)^s 2^{-2s} z^{2s} \sum_{p=s}^{m-1} \frac{r^{2p+2}(-1)^{p-s}}{2(p+1)(p-s)!p!2^{2(p-s)}} z^{2(p-s)} \\ &= (-1)^s 2^{-2s} z^{2s} r^{2s+2} \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-2s+1}} \\ &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} \sum_{p=s}^{m-1} \frac{(zr)^{2p-s}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-2s+1}} \\ &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \sum_{p=s}^{m-1} \frac{(zr)^{2p-s}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \quad (\widetilde{S}_r(z) \equiv 0 \text{ for } s = m), \end{aligned}$$

$$\begin{aligned}
 \widetilde{U}_r(z) &= \frac{1}{2\pi} \iint_{|w| \leq r} w^s \partial^s \left(\sum_{p=0}^{\infty} \frac{(-1)^p}{p!p!} \left(\frac{z}{2}\right)^{2p} w^p \bar{w}^p \right) du dv \\
 &= \frac{1}{2\pi} \iint_{|w| \leq r} \left(\sum_{p=s}^{\infty} \frac{(-1)^p}{p!(p-s)!} \left(\frac{z}{2}\right)^{2p} |w|^{2p} \right) du dv \\
 &= \sum_{q=0}^{\infty} \frac{(-1)^{q+s} r^{2(q+s+1)}}{2(q+s+1)!q!} \left(\frac{z}{2}\right)^{2(q+s)} \\
 &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma(q+s+2)q!} \left(\frac{zr}{2}\right)^{2q} \\
 &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \frac{J_{s+1}(zr)}{(zr)^{s+1}},
 \end{aligned}$$

$$\begin{aligned}
 \widetilde{U}_r(z) - \widetilde{S}_r(z) &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \frac{J_{s+1}(zr)}{(zr)^{s+1}} \\
 &\quad - (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)} (-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \\
 &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \left(\frac{J_{s+1}(zr)}{(zr)^{s+1}} - \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)} (-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \right) \\
 &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} g_r(z) = r^{2s+2} \left(-\frac{z^2}{4}\right)^s g_r(z),
 \end{aligned}$$

and

$$\widetilde{T}_r(z) \equiv r^{2s+2} \left(-\frac{z^2}{4}\right)^s g_r(z).$$

Applying the Paley–Wiener–Schwartz theorem for the spherical transformation we conclude that there exists a unique radial distribution V_r supported in \overline{B}_r such that $\widetilde{V}(z) \equiv 2^{-2s} r^{2s+2} g_r(z)$ and $T_r(\varphi) = \Delta^s V_r(\varphi)$ for all $\varphi \in C^\infty(\mathbb{C})$. Since the function $g_r(z)$ has zero of order $2(m-s)$ at the point $z = 0$, then the function $\widetilde{T}_r(z)$ has zero of order $2m$ at this point. Hence, by the mentioned Paley–Wiener–Schwartz theorem, there exists a unique radial distribution W_r supported in \overline{B}_r such that

$$\widetilde{T}_r(z) \equiv z^{2m} \widetilde{W}_r(z).$$

This implies that $\widetilde{W}_r(0) \neq 0$ and

$$\widetilde{W}_r(z) \equiv z^{-2m} r^{2s+2} \left(-\frac{z^2}{4}\right)^s g_r(z).$$

In the sequel we shall use the following properties of the Bessel functions.

Lemma 1. *Let $s \in \mathbb{N}_0$ and let $r > 0$. Then the following assertions are valid:*

(a) *for any $\varepsilon \in (0, \pi)$ the asymptotic expansion*

$$J_{s+1}(z) = (2/(\pi z))^{1/2} [\cos(z - (s+1)\pi/2 - \pi/4) + O(z^{-1}e^{|\operatorname{Im} z|})]$$

holds as $z \rightarrow \infty$, $|\arg z| \leq \pi - \varepsilon$;

(b) *all zeroes of the function $J_{s+1}(\lambda r)/(\lambda r)^{s+1}$ are real;*

(c) *there is a constant $c > 0$ such that*

$$\left| \frac{d}{d\lambda} \left(\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}} \right) \right| \geq \frac{c}{|\lambda|^{s+3/2}} \quad \text{as } \lambda \rightarrow \infty, \quad J_{s+1}(\lambda r) = 0. \quad (7)$$

Proof. Assertions (a) and (b) are well-known (see, e.g., [10, §12.2] and [9, §23], respectively). The proof of assertion (c) is based on the following differentiation formula for the Bessel functions [9, §6]:

$$\frac{d}{dz} \left(\frac{J_{s+1}(z)}{z^{s+1}} \right) = -\frac{J_{s+2}(z)}{z^{s+1}}. \quad (8)$$

Since the function $J_{s+1}(\lambda r)/(\lambda r)^{s+1}$ is even, we can assume without loss of generality that λ is a real positive number in (7) and, consequently, that $\operatorname{Im}(\lambda r) = 0$. Applying this formula we have

$$\begin{aligned} \frac{d}{d\lambda} \left(\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}} \right) &= -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r = -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r + i \frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}} r \\ &= -\frac{r}{(\lambda r)^{s+1}} (J_{s+2}(\lambda r) - i J_{s+1}(\lambda r)) \\ &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} [\cos(\lambda r - (s+2)\pi/2 - \pi/4) \\ &\quad - i \cos(\lambda r - (s+1)\pi/2 - \pi/4) + O((\lambda r)^{-1}e^{|\operatorname{Im} \lambda r|})] \\ &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} [\cos(\lambda r - (s+2)\pi/2 - \pi/4) \\ &\quad + i \sin(\lambda r - (s+2)\pi/2 - \pi/4) + O((\lambda r)^{-1})] \\ &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} [\exp i(\lambda r - (s+2)\pi/2 - \pi/4) + O((\lambda r)^{-1})] \end{aligned}$$

as $\lambda \rightarrow \infty$, $J_{s+1}(\lambda r) = 0$, $\lambda > 0$.

This implies the asymptotic equality

$$\left| \frac{d}{d\lambda} \left(\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}} \right) \right| = \sqrt{\frac{2}{\pi}} \cdot \frac{r}{|\lambda r|^{s+3/2}} (1 + o(1)) \text{ as } \lambda \rightarrow \infty, J_{s+1}(\lambda r) = 0,$$

whence we obtain (7) with $c = (2\pi)^{-1} r^{-s-1/2}$. □

The above arguments show that $g_r(z)$ is the spherical transformation of some radial distribution supported in the disk \overline{B}_r . This implies that the set Z_r is symmetric with respect to the origin and infinite. Indeed, the symmetry of Z_r follows from the evenness of the function $g_r(z)$. Suppose that the set Z_r is finite, i.e., $Z_r = \{z_1, -z_1, \dots, z_n, -z_n\}$ for some $n \in \mathbb{N}$. Then we have

$$g_r(z) = z^{2(m-s)} \prod_{j=1}^n \left(1 - \left(\frac{z}{z_j} \right)^2 \right) e^{F(z)},$$

where $F(z)$ is an entire function such that $e^{F(z)} \equiv e^{F(-z)}$. Therefore $e^{F(z)-F(-z)} \equiv 1$ whence we obtain $F(z) - F(-z) \equiv 2k\pi i$ for some integer k . Since $\lim_{z \rightarrow 0} (F(z) - F(-z)) = 0$, then $k = 0$. This means that the entire function $F(z)$ is even.

The Paley–Wiener–Schwartz theorem for the spherical transformation implies the existence of a constant $C > 0$ such that the inequality $|F(z)| \leq C \log(1 + |z|) |\operatorname{Im} z|$ holds everywhere in \mathbb{C} . Therefore $|F(z)| = o(|z|^2)$ as $z \rightarrow \infty$ whence passage to the limit as $R \rightarrow \infty$ in the Cauchy inequalities $|a_{2k}| \leq R^{-2k} \max_{\overline{B}_R} |F(z)|$ for the Taylor coefficients of the function $F(z) = \sum_{k=0}^\infty a_{2k} z^{2k}$ shows that $F(z) \equiv a_0$. Hence the function $g_r(z)$ must be a polynomial. This contradiction justifies that the set Z_r is infinite.

The following lemma plays the key role in this paper.

Lemma 2. *Let $s < m$. Then there are positive constants c_1, c_2 and c_3 such that for any $\lambda \in Z_r \cap \{z \in \mathbb{C} : |z| \geq c_3\}$ we have the estimations*

$$|\operatorname{Im} \lambda| \leq c_1 \log(2 + |\lambda|) \tag{9}$$

and

$$|g'_r(\lambda)| \geq c_2 |\lambda|^{2m-s-1}. \tag{10}$$

In particular, any $\lambda \in Z_r \cap \{z \in \mathbb{C} : |z| \geq c_3\}$ is a simple zero of the function $g_r(z)$.

Proof. Define the polynomial

$$P(z) := z^{s+1} \sum_{p=s}^{m-1} \frac{z^{2p-s+1} (-1)^{p-s}}{(p-s)!(p+1)! 2^{2p-s+1}} \quad (z \in \mathbb{C}).$$

Since the set Z_r is symmetric with respect to the origin, then it is sufficient to prove Lemma 2 under the assumption $|\arg \lambda r| < 3\pi/4$. In this case, applying the first assertion of Lemma 1, rewriting the condition $g_r(\lambda) = 0$ in the form

$$J_{s+1}(\lambda r) = P(\lambda r), \quad (11)$$

using the formula $\cos z = (e^{iz} + e^{-iz})/2$, and taking into account that $P(z)$ is a polynomial of degree $2m$, we have

$$J_{s+1}(\lambda r) = (2/(\pi \lambda r))^{1/2} [\cos(\lambda r - (s+1)\pi/2 - \pi/4) + O((\lambda r)^{-1} e^{|\operatorname{Im}(\lambda r)|})]$$

as $\lambda \rightarrow \infty$, $\lambda \in Z_r$, (12)

$$\begin{aligned} & \cos(\lambda r - (s+1)\pi/2 - \pi/4) \\ &= \frac{1}{2} [\exp(i\lambda r - i(s+1)\pi/2 - i\pi/4) + \exp(-i\lambda r + i(s+1)\pi/2 + i\pi/4)] \\ &= \frac{1}{2} [\exp(-\operatorname{Im}(\lambda r) + i\operatorname{Re}(\lambda r) - i(s+1)\pi/2 - i\pi/4) \\ & \quad + \exp(\operatorname{Im}(\lambda r) - i\operatorname{Re}(\lambda r) + i(s+1)\pi/2 + i\pi/4)]. \end{aligned} \quad (13)$$

This implies that $|\operatorname{Im}(\lambda r)| \rightarrow \infty$ as $\lambda \rightarrow \infty$, $\lambda \in Z_r$. Otherwise, we have from (12) and (13) the asymptotic $J_{s+1}(\lambda r) \rightarrow 0$ as $\lambda \rightarrow \infty$, $\lambda \in Z_r$, which contradicts (11). Therefore (11), (12) and (13) imply the asymptotic formulas

$$|J_{s+1}(\lambda r)| = (2/(\pi|\lambda|r))^{1/2} \frac{e^{|\operatorname{Im}(\lambda r)|}}{2} (1 + o(1)) \text{ as } \lambda \rightarrow \infty, \lambda \in Z_r$$

and

$$\frac{e^{|\operatorname{Im}(\lambda r)|}}{2} = (\pi|\lambda|r/2)^{1/2} |P(\lambda r)| (1 + o(1)) \text{ as } \lambda \rightarrow \infty, \lambda \in Z_r, \quad (14)$$

whence we obtain (9).

In order to prove (10) we shall use the differentiation formula (8) and the first assertion of Lemma 1. Then the following chain of relations is

valid as $\lambda \rightarrow \infty$, $\lambda \in Z_r$, $0 \leq \arg \lambda \leq 3\pi/4$:

$$\begin{aligned}
 g'_r(\lambda) &= \frac{d}{d\lambda} \left(\frac{J_{s+1}(\lambda r) - P(\lambda r)}{(\lambda r)^{s+1}} \right) = -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) \\
 &= -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) + ir g_r(\lambda) \\
 &\quad - \frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) + ir \frac{J_{s+1}(\lambda r) - P(\lambda r)}{(\lambda r)^{s+1}} \\
 &= -\frac{J_{s+2}(\lambda r) - iJ_{s+1}(\lambda r)}{(\lambda r)^{s+1}} r - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
 &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} [\cos(\lambda r - (s+2)\pi/2 - \pi/4) \\
 &\quad - i \cos(\lambda r - (s+1)\pi/2 - \pi/4) + O(|\lambda r|^{-1} e^{|\operatorname{Im} \lambda r|})] \\
 &\quad - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
 &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} [\cos(\lambda r - (s+2)\pi/2 - \pi/4) \\
 &\quad + i \sin(\lambda r - (s+2)\pi/2 - \pi/4) + O(|\lambda r|^{-1} e^{|\operatorname{Im} \lambda r|})] \\
 &\quad - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
 &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} \\
 &\quad \times [\exp(i(\lambda r - (s+2)\pi/2 - \pi/4)) + O(|\lambda r|^{-1} e^{|\operatorname{Im} \lambda r|})] \\
 &\quad - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}}.
 \end{aligned}$$

Using (14) and the estimate

$$|\exp(i(\lambda r - (s+2)\pi/2 - \pi/4))| = \exp(-\operatorname{Im}(\lambda r)) \leq 1,$$

which holds for $\operatorname{Im}(\lambda r) \geq 0$, we can continue this chain of relations as follows:

$$\begin{aligned}
 g'_r(\lambda) &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} \exp(i(\lambda r - (s+2)\pi/2 - \pi/4)) \\
 &\quad + O(|\lambda r|^{-s-2} |P(\lambda r)|) \\
 &\quad - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} = -\frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
 &\quad + O(|\lambda r|^{-s-3/2} |\exp(i(\lambda r - (s+2)\pi/2 - \pi/4))|) + O(|\lambda r|^{-s-2} |P(\lambda r)|)
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} + O(|\lambda r|^{-s-3/2}) + O(|\lambda r|^{-s-2}|P(\lambda r)|) \\
 &= -ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} (1 + o(1)) = -icr(\lambda r)^{2m-s-1} (1 + o(1)) \\
 &= -icr^{2m-s} \lambda^{2m-s-1} (1 + o(1)) \text{ as } \lambda \rightarrow \infty, \lambda \in Z_r, 0 \leq \arg \lambda \leq 3\pi/4,
 \end{aligned} \tag{15}$$

where c is the leading coefficient of the polynomial $P(z)$. Similarly, we have

$$\begin{aligned}
 g'_r(\lambda) &= -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) - ir g_r(\lambda) \\
 &= -\frac{J_{s+2}(\lambda r) + iJ_{s+1}(\lambda r)}{(\lambda r)^{s+1}} r - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) + ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
 &= -\frac{r}{(\lambda r)^{s+1}} (2/(\pi \lambda r))^{1/2} \\
 &\quad \times [\exp(-i(\lambda r - (s+2)\pi/2 - \pi/4)) + O(|\lambda r|^{-1} e^{|\operatorname{Im} \lambda r|})] \\
 &\quad - \frac{d}{d\lambda} \left(\frac{P(\lambda r)}{(\lambda r)^{s+1}} \right) + ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
 &= ir \frac{P(\lambda r)}{(\lambda r)^{s+1}} (1 + o(1)) = icr^{2m-s} \lambda^{2m-s-1} (1 + o(1))
 \end{aligned}$$

$$\text{as } \lambda \rightarrow \infty, \lambda \in Z_r, -3\pi/4 \leq \arg \lambda < 0. \tag{16}$$

Since $c \neq 0$ and the function $g_r(z)$ is even, then (15) and (16) justify the validity of the estimation (10) with $c_2 = 2^{-1}|c|r^{2m-s}$. \square

Let φ be a radial distribution with compact support in \mathbb{C} , let $r(\varphi) := \inf\{r > 0 : \operatorname{supp} \varphi \subset B_r\}$, and let $R \in (r(\varphi), +\infty]$. As usual, we say that a function $f \in C^\infty(B_R)$ satisfies the convolution equation $f * \varphi = 0$ in B_R if the equality $\varphi(f(x-y)) = 0$ holds for all $x \in B_{R-r(\varphi)}$, where the distribution φ acts on the function $f(x-y)$ in y . Denote by $Z(\tilde{\varphi})$ the set of all zeroes of the spherical transformation $\tilde{\varphi}(z)$ of the distribution φ . Let \mathcal{R} be the set of all radial distributions φ with compact supports in \mathbb{C} satisfying the following conditions: (a) $r(\varphi) > 0$; (b) there are constants $c_1 \geq 0$ and $c_2 \geq 0$ such that for any $\lambda \in Z(\tilde{\varphi})$ we have the estimates $|\operatorname{Im} \lambda| \leq c_1 \log(2 + |\lambda|)$ and $|\partial^{n_\lambda} \tilde{\varphi}(\lambda)| \geq (2 + |\lambda|)^{n_\lambda - c_2}$, where n_λ denotes the multiplicity of the zero λ .

Combining assertion (c) of Lemma 1, Lemma 2, and the equality

$$\widetilde{W}'_r(\lambda) = \lambda^{-2m} r^{2s+2} \left(-\frac{\lambda^2}{4} \right)^s g'_r(\lambda),$$

which is valid for all $\lambda \in Z_r$, we obtain the following lemma.

Lemma 3. *For any $r > 0$ the radial distribution W_r belongs to the class \mathcal{R} .*

The following four lemmas are special cases of the Volchkov results [2, Part 3, §2.4, Theorem 2.3], [2, Part 3, §2.3, Theorem 2.1], [2, Part 3, §4.2, Theorem 4.8], and [2, Part 3, §4.2, Theorem 4.9], respectively.

Lemma 4. *Let $\varphi \in \mathcal{R}$, $\tilde{\varphi}(0) \neq 0$, and let $R \in (r(\varphi), +\infty]$. Then a function $f \in C^\infty(B_R)$ is a solution of the convolution equation $f * \varphi = 0$ in B_R if and only if the Fourier coefficients $f_k(\rho)$ of the function f is represented in the form*

$$f_k(\rho) = \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda-1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) \quad (0 < \rho < R, \quad k \in \mathbb{Z}), \quad (17)$$

where for any $\alpha > 0$ and $k \in \mathbb{Z}$ we have

$$\max_{\eta=0,\dots,n_\lambda-1} |c_{\lambda,\eta,k}| = O(|\lambda|^{-\alpha}) \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in Z_r. \quad (18)$$

Under the condition (18) the series (17) converges in $\mathcal{E}(B_R)$ and all the coefficients $c_{\lambda,\eta,k}$ are defined uniquely with respect to f .

Lemma 5. *Let φ be a radial distribution with compact support in \mathbb{C} such that $r(\varphi) > 0$ and let $R \in (r(\varphi), +\infty]$. Then the following statements hold:*

(a) *if f is a function from $C^\infty(B_R)$ satisfying the convolution equation $f * \varphi = 0$ in B_R and $f(z) = 0$ everywhere in $B_{r(\varphi)}$, then $f(z) \equiv 0$ in B_R ;*

(b) *if $\varphi \in \mathcal{R}$, then for any $\varepsilon \in (0, r(\varphi))$ there exists is a nonzero radial function $f \in C^\infty(\mathbb{R}^n)$ that satisfies the convolution equation $f * \varphi = 0$ in \mathbb{R}^n and equals zero in $B_{r(\varphi)-\varepsilon}$.*

Lemma 6. *Let $\varphi_1, \varphi_2 \in \mathcal{R}$, $Z(\tilde{\varphi}_1) \cap Z(\tilde{\varphi}_2) = \emptyset$, $R \in [r(\varphi_1) + r(\varphi_2), +\infty]$, and let f be a function from $C^\infty(B_R)$ that satisfies the convolution equations $f * \varphi_1 = 0$ and $f * \varphi_2 = 0$ in B_R . Then $f \equiv 0$ in B_R .*

Lemma 7. *Let $\varphi_1, \varphi_2 \in \mathcal{R}$ and let $R \in (\max\{r(\varphi_1), r(\varphi_2)\}, r(\varphi_1) + r(\varphi_2))$. Then there exists a nonzero function $f \in C^\infty(B_R)$ that satisfies the convolution equations $f * \varphi_1 = 0$ and $f * \varphi_2 = 0$ in B_R .*

3. Proof of Theorem 1

Proof. Let a function $f \in C^\infty(B_R)$ satisfy Eq. (1) for all $z \in B_{R-r}$ and let

$$G(z, \zeta) := \log \left| \frac{R^2 - \zeta \bar{z}}{R(\zeta - z)} \right|, \quad z, \zeta \in B_R, \quad z \neq \zeta,$$

be the Green function for Laplace operator in the disk B_R . Then the function

$$Lf(z) := \int_{B_R} G(z, \zeta) f(\zeta) d\xi d\eta \quad (\zeta = \xi + i\eta) \tag{19}$$

belongs to the class $C^\infty(B_R)$, vanishes on the boundary of B_R , and satisfies the Poisson equation $\Delta(Lf) = f$ in the disk B_R [11, §1.6].

Define the function $d(z) := 4^s \bar{\partial}^s (L^s f)(z)$, $z \in B_R$. Then this function belongs to the class $C^\infty(B_R)$ and for any $z \in B_R$ we have

$$\begin{aligned} \partial^s d(z) &= 4^s \partial^s \bar{\partial}^s (L^s f)(z) = \Delta^s (L^s f)(z) \\ &= \Delta^{s-1} (L^{s-1} f)(z) = \dots = \Delta(Lf)(z) = f(z). \end{aligned}$$

It was shown in Section 2 that the function $f(z)$ satisfies the condition (1) for all $z \in B_{R-r}$ if and only if the function $d(z)$ is a solution of the convolution equation $d * T_r = 0$ in B_R . On the other hand, by the Paley–Wiener–Schwartz theorem for the spherical transformation, the condition (1) holds for all $z \in B_{R-r}$ if and only if the function $d(z)$ is a solution of the convolution equation $(\Delta^m d) * W_r = 0$ in the disk B_R , where the radial distribution W_r is defined in Section 2. By Lemma 3 and Lemma 4, this is equivalent to that the Fourier coefficients of the function $f^*(z) = 4^{-m} \Delta^m d(z)$ are represented in the form (4), where the coefficients $c_{\lambda, \eta, k}$ are defined uniquely with respect to f^* and satisfy the condition (5). Since $f^*(z) = \partial^{m-s} \bar{\partial}^m f(z)$ for all $z \in B_R$, then we obtain the assertion of Theorem 1. □

4. Proof of Theorem 2

Suppose that $0 < r < R$, $f \in C^\infty(B_R)$, Eq. (1) is valid for all $z \in B_{R-r}$, and the condition $f(z) = 0$ holds everywhere in B_r . Using the notation from the proof of Theorem 1, consider the function $f^*(z) := \partial^{m-s} \bar{\partial}^m f(z)$. This function belongs to the class $C^\infty(B_R)$ and satisfies the convolution equation $f^* * W_r = 0$ in B_R . Since $f(z) = 0$ holds everywhere in B_r , then we have $f^*(z) = 0$ for all $z \in B_r$. The radial distribution W_r satisfies the conditions of Lemma 5 (assertion (a)) whence we have $f^*(z) \equiv 0$ in B_R . This means that the function f satisfies the

equation $\partial^{m-s}\bar{\partial}^m f = 0$ in B_R . Since solutions of this equation are real analytic functions, then the first assertion of Theorem 2 follows from the uniqueness theorem for real analytic functions.

The second assertion of Theorem 2 is deduced from assertion (b) of Lemma 5 applied to the radial distribution T_r defined in Section 2, which implies that there is a function $g \in C^\infty(\mathbb{C})$ such that the convolution equation $g * T_r = 0$ is valid everywhere in \mathbb{C} , $g(z) \not\equiv 0$, and $g(z) = 0$ for all $B_{r-\varepsilon}$. Therefore the function $f(z) := \partial^s g(z)$ equals zero everywhere in $B_{r-\varepsilon}$ and, by the uniqueness theorem for real analytic functions, we have $f(z) \not\equiv 0$. On the other hand, it follows from the definition of the distribution T_r that the function $f(z)$ satisfies Eq. (1) in the whole of \mathbb{C} .

5. Proof of Theorem 3

Suppose that $R \geq r_1 + r_2$, $Z_{r_1} \cap Z_{r_2} = \emptyset$, $f \in C^\infty(B_R)$, and Eq. (1) holds for all $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$. It was shown in the proof of Theorem 1 that the function $f^*(z) := \partial^{m-s}\bar{\partial}^m f(z)$, $z \in B_{R-\varepsilon}$, satisfies the convolution equations $f^* * W_{r_1} = 0$ and $f^* * W_{r_2} = 0$ in $B_{R-\varepsilon}$. Applying Lemma 6 we obtain the first assertion of Theorem 3.

Prove the second assertion. If $Z_{r_1} \cap Z_{r_2} \neq \emptyset$, then there is a $\lambda \in Z_{r_1} \cap Z_{r_2}$, $\lambda \neq 0$. Theorem 1 implies that each solution of the equation

$$\partial^{m-s}\bar{\partial}^m f = J_0(\lambda|z|)$$

in the disk B_R belongs to the class $C^\infty(B_R)$ and satisfies Eq. (1) for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$.

Consider the case $Z_{r_1} \cap Z_{r_2} = \emptyset$ and $\max\{r_1, r_2\} < R < r_1 + r_2$. Then the function $\widetilde{W_{r_1}}(z)/\widetilde{W_{r_2}}(z)$ is entire and has no zeroes in \mathbb{C} . Therefore each of the functions $\widetilde{W_{r_1}}(z)$ and $\widetilde{W_{r_2}}(z)$ has no zeroes in \mathbb{C} . It follows from Lemma 7 that there is a nonzero function $g \in C^\infty(B_R)$ that satisfy the convolution equations $g * W_{r_1} = 0$ and $g * W_{r_2} = 0$ in B_R . Then the function $f(z) := \partial^s L^m g(z)$, where the operator $L : C^\infty(B_R) \rightarrow C^\infty(B_R)$ is defined by (19), belongs to the class $C^\infty(B_R)$ and satisfies Eq. (1) for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, but $\partial^{m-s}\bar{\partial}^m f(z) \not\equiv 0$ in B_R (otherwise we have $g(z) \equiv 0$ in B_R).

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