Український математичний вісник
Том 14 **(2017)**, № 3, 418 – 439

# Boundary triples for integral systems on finite intervals

Dmytro Strelnikov

(*Presented by M.M. Malamud*)

**Abstract.** Let $P$, $Q$ and $W$ be real functions of bounded variation on $[0,l]$ and let $W$ be nondecreasing. The following integral system

$$J\vec{f}(x) - J\vec{a} = \int_0^x \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f}(t), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{0.1}$$

on a finite compact interval $[0,l]$ has been studied in [6]. A maximal and a minimal linear relation $A_{max}$ and $A_{min}$ associated with the integral system (9) are studied in the Hilbert space $L^2(W)$. It is shown that the linear relation $A_{min}$ is symmetric with deficiency indices $n_\pm(A_{min}) = 2$ and $A_{max} = A_{min}^*$. Boundary triples for $A_{max}$ are constructed and the corresponding Weyl functions are calculated.

**2010 MSC.** Primary 34B20; Secondary 45A05; 47A06.

**Key words and phrases.** Integral system, boundary triple, symmetric linear relation, deficiency indices, Weyl function.

## 1. Introduction

This paper focuses on the following integral system

$$J\vec{f}(x) - J\vec{a} = \int_0^x dS(t) \cdot \vec{f}(t) \tag{1.1}$$

where $J$ and $dS$ are $2 \times 2$ matrices of the form:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad dS = \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix}, \tag{1.2}$$

*Received* 12.09.2017
*The author is grateful to professor V. Derkach for constant attention to this work.*

ISSN 1810 – 3200. 

$\lambda \in \mathbb{C}$, all functions $P$, $Q$ and $W$ are real of bounded variation on $[0,l]$ and $W$ is nondecreasing on $[0,l]$. Such systems were studied in [2,3,6]. System (1.1) contains Sturm-Liouville systems, Stieltjes string and Krein-Feller string [13, 18] as special cases.

We associate with system (1.1) minimal $A_{min}$ and maximal $A_{max}$ linear relations. In contrast to the Sturm–Liouville case $A_{min}$ and $A_{max}$ may be multivalued, therefore we use for them a term linear relation (see [1]). It turns out that the linear relation $A_{min}$ is symmetric with deficiency indices $(2,2)$.

The notions of the boundary triple and Weyl function introduced in [7,8,19] and [10], respectively, were proved to be useful in the study of spectral problems and extension theory problems for symmetric operators, see [11,12,14]. Boundary triples for various differential and difference operators were constructed in [4, 10, 11, 14, 19, 21, 22].

A boundary triple for the linear relation $A_{max}$ is constructed in the paper and the corresponding matrix Weyl function is calculated. In a similar way some intermediate extensions of the linear relation $A_{min}$ with deficiency indices $(1,1)$ are considered and their scalar Weyl functions are found.

## 2. Preliminaries

### 2.1. Linear relations

Let $\mathfrak{H}$ be a Hilbert space. Any linear supspace of $\mathfrak{H} \times \mathfrak{H}$ is called a *linear relation* in $\mathfrak{H}$, [1].

The *domain*, the *range*, the *kernel*, and the *multivalued part* of a linear relation $T$ are defined by the following equalities (see [1,5]):

$$\operatorname{dom} T := \left\{ f : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}, \qquad \operatorname{ran} T := \left\{ g : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}, \tag{2.1}$$

$$\ker T := \left\{ f : \begin{pmatrix} f \\ 0 \end{pmatrix} \in T \right\}, \qquad \operatorname{mul} T := \left\{ g : \begin{pmatrix} 0 \\ g \end{pmatrix} \in T \right\}. \tag{2.2}$$

The adjoint linear relation $T^*$ is defined by

$$T^* := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{H} \times \mathfrak{H} : \ (v, f)_{\mathfrak{H}} = (u, g)_{\mathfrak{H}} \text{ for some } \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}. \tag{2.3}$$

A linear relation $T$ in $\mathfrak{H}$ is called *closed* if $T$ is closed as a subspace of $\mathfrak{H} \times \mathfrak{H}$. The set of all closed linear operators (relations) is denoted by $\mathcal{C}(\mathfrak{H})$ $(\widetilde{\mathcal{C}}(\mathfrak{H}))$. Identifying a linear operator $T \in \mathcal{C}(\mathfrak{H})$ with its graph one can consider $\mathcal{C}(\mathfrak{H})$ as a part of $\widetilde{\mathcal{C}}(\mathfrak{H})$.

**Definition 2.1.** Suppose $T$ is a linear relation, $\lambda \in \mathbb{C}$ then

$$T - \lambda I := \left\{ \begin{pmatrix} f \\ g - \lambda f \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}. \tag{2.4}$$

A point $\lambda \in \mathbb{C}$ such that $\ker(T - \lambda I) = \{0\}$ and $\operatorname{ran}(T - \lambda I) = \mathfrak{H}$ is called a *regular point* of the linear relation $T$ and is written $\lambda \in \rho(T)$.

The *point spectrum* and the *continuous spectrum* of the linear relation $T$ are defined by

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}, \tag{2.5}$$

$$\sigma_c(T) := \{\lambda \in \mathbb{C} : \lambda \notin \sigma_p(T),\ \operatorname{ran}(T - \lambda I) \neq \overline{\operatorname{ran}(T - \lambda I)} = \mathfrak{H}\}. \tag{2.6}$$

For $\lambda \in \mathbb{C}_\pm$ let us set $\mathfrak{N}_\lambda(T) := \ker(T^* - \lambda I)$ and

$$\hat{\mathfrak{N}}_\lambda(T) := \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} : f_\lambda \in \mathfrak{N}_\lambda \right\}. \tag{2.7}$$

A linear relation $A$ is called *symmetric* if $A \subseteq A^*$. The *deficiency indices* of a symmetric linear relation $A$ are defined by

$$n_\pm(A) := \dim \ker(A^* \mp iI). \tag{2.8}$$

## 2.2. Boundary triples

In the case of densely defined operators a boundary triple notion was introduced in [7,8,14,19] (in different forms). Following the paper [21] we shall give a general definition of a boundary triple for the linear relation $T$.

**Definition 2.2.** The tuple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$where $\mathcal{H}$ is a Hilbert space, $\Gamma_0$ and $\Gamma_1$ are linear mappings from $T$ to $\mathcal{H}$ is called a *boundary triple* for linear relation $T$, if the following conditions hold:

(i) generalized Green's identity

$$(g, u)_\mathfrak{H} - (f, v)_\mathfrak{H} = \left( \Gamma_1 \begin{pmatrix} f \\ g \end{pmatrix}, \Gamma_0 \begin{pmatrix} u \\ v \end{pmatrix} \right)_\mathcal{H} - \left( \Gamma_0 \begin{pmatrix} f \\ g \end{pmatrix}, \Gamma_1 \begin{pmatrix} u \\ v \end{pmatrix} \right)_\mathcal{H} \tag{2.9}$$

holds for all $\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in T$;

(ii) the mapping $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : T \to \mathcal{H} \times \mathcal{H}$ is surjective.

If the linear relation $T$ is adjoint to some symmetric linear relation $A$ then there exists a boundary triple for $T$ if and only if the deficiency indices of $A$ coincide ($n_+(A) = n_-(A)$), see [11, 19, 21].

An extension $\widetilde{A}$ of a symmetric linear relation $A$ is called *proper* if $A \subsetneq \widetilde{A} \subsetneq A^*$. The class of all proper extensions of the linear relation $A$ completed with relations $A$ and $A^*$ is denoted by $\mathrm{Ext}(A)$. Denote also

$$A_\Theta := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in A^* : \ \Gamma \begin{pmatrix} f \\ g \end{pmatrix} \in \Theta \right\}. \tag{2.10}$$

**Proposition 2.3.** [11] *Let $A$ be a symmetric linear relation, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the adjoint linear relation $A^*$. Then the mapping $\Gamma : \widetilde{A} = A_\Theta \to \Theta = \Gamma\widetilde{A}$ is a one-to-one mapping from $\mathrm{Ext}(A)$ to $\widetilde{\mathcal{C}}(\mathfrak{H})$. Notice also that $A_\Theta$ is selfadjoint if and only if the linear relation $\Theta$ is selfadjoint.*

In particular, linear relations

$$A_0 := \ker \Gamma_0, \qquad A_1 := \ker \Gamma_1 \tag{2.11}$$

are disjoint, i.e. $A_0 \cap A_1 = A$, and they are selfadjoint extensions of the symmetric linear relation $A$ (see [11]).

Suppose $A$ is adjoint for the linear relation $T$ from Definition 2.2 The conditions ensuring the symmetry of $A$ are provided by the next theorem. In the case of single-valued linear operator $T$ the corresponding theorem was proved in [12].

**Theorem 2.4.** [12] *Let $T$ be a linear relation in the Hilbert space $\mathfrak{H}$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$be its boundary triple such that $n := \dim \mathcal{H} < \infty$ and $A = \ker \Gamma$. If the following conditions hold:*

*(i)* $\mathrm{ran}\, T = \mathfrak{H}$*;*

*(ii)* $\dim \ker T = n$ *and* $\ker A = \{0\}$,

*then linear relations $A$, $T$ are closed, $T = A^*$ and $n_+(A) = n_-(A) = n$.*

**Definition 2.5.** [10, 11] Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for linear relation $A^*$. Operator valued functions $M(\cdot)$, $\gamma(\cdot)$ defined by

$$M(\lambda)\Gamma_0 \hat{f}_\lambda = \Gamma_1 \hat{f}_\lambda, \quad \gamma(\lambda)\Gamma_0 \hat{f}_\lambda = f_\lambda, \quad \hat{f}_\lambda \in \hat{\mathfrak{N}}_\lambda, \quad \lambda \in \rho(A_0) \tag{2.12}$$

are called the Weyl function and the $\gamma$-field of the symmetric linear relation $A$ with respect to the boundary triple $\Pi$.

**Definition 2.6.** An operator valued function $F : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathcal{B}(\mathcal{H})$ is said to belong to the class $R[\mathcal{H}]$ if the following conditions hold:

(i) $F$ is holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$;

(ii) $\operatorname{Im} F(\lambda) \geq 0$ as $\lambda \in \mathbb{C}_+$;

(iii) $F(\overline{\lambda}) = F^*(\lambda), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

If $\mathcal{H} = \mathbb{C}$ then $R[\mathcal{H}]$ is denoted by $R$.

It is known that the Weyl function $M(\lambda)$ of a linear relation $A$ from Definition 2.5 belongs to the class $R[\mathcal{H}]$. The next proposition gives a description of the spectrum of a linear $\widetilde{A} \in \operatorname{Ext}(A)$.

**Proposition 2.7.** [11] *Let $A$ be a symmetric linear relation in $\mathfrak{H}$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$, $M(\lambda)$ be the corresponding Weyl function of $A$, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, and $\lambda \in \rho(A_0)$. Then:*

*(i)* $\lambda \in \rho(\widetilde{A}_\Theta) \Longleftrightarrow 0 \in \rho(\Theta - M(\lambda))$;

*(ii)* $\lambda \in \sigma_p(\widetilde{A}_\Theta) \Longleftrightarrow 0 \in \sigma_p(\Theta - M(\lambda))$.

## 2.3. Integral systems

Let us consider on a compact interval $[0, l]$ an integral system

$$J\vec{f}(x) - J\vec{a}(x) = \int_0^x dS(t) \cdot \vec{f}(t) \tag{2.13}$$

where $\vec{f}$ is a $n \times 1$ complex vector, $\vec{a}$ is a fixed complex vector valued function of bounded variation, $dS$ is a finite $n \times n$ measure, and $J$ is a constant $n \times n$ matrix such that $J^* = -J$.

**Definition 2.8.** We say that a vector valued function $\vec{f}$ is a solution of integral system (2.13) if (each component of) $\vec{f}$ is of bounded variation and the equality (2.13) holds for every point of $[0, l]$.

It is easy to see that if for some vector valued function $\vec{f}$ the right-hand part of equality (2.13) exists for all $x \in [0, l]$ then it is of bounded variation on $[0, l]$ and therefore inclusion $\vec{f} \in BV[0, l]$ is necessery for (2.13). The same condition is also sufficient for existence of the integral in the right-hand part of (2.13) (as a Lebesgue–Stietjes integral).

In general case measure $dS$ is not supposed to be absolutely continuous and may have mass points on $[0, l]$. Therefore in equality (2.13) and in the following we should understand $\int_a^b f d\mu$ as the Lebesgue–Stieltjes integral $\int f \chi_{[a,b)} d\mu$, where $\chi_{[a,b)}$ is the characteristic function of the half-open interval. Under this conventions integrals as functions of its limits of integration are left-continuous.

The following theorem was proved in [6].

**Theorem 2.9.** [6] *For any left-continuous vector-function* $\vec{a}(x) \in BV[0,l]$ *there exists a unique solution of* (2.13).

Further in this paper the integration by parts formula will be used in the following form (see [15]). If $u$ is a left-continuous function of bounded variation then we denote by $u_+$ the right-continuous function that coincides with $u$ in every continuity point. If $v$ is another left-continuous function of bounded variation then the following equality holds

$$\int_y^x v du = v(x)u(x) - v(y)u(y) - \int_y^x u_+ dv. \tag{2.14}$$

Now suppose that $n = 2$, matrices $J$ and $dS$ have the following form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad dS = \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix} \tag{2.15}$$

where $\lambda$ is a complex parameter, $P$, $Q$ and $W$ are of bounded variation and left-continuous on $[0,l]$ functions that satisfy the condition

$$P(0) = Q(0) = W(0) = 0 \tag{2.16}$$

and $W$ is nondecreasing. We assume that functions $P$, $Q$ and $W$ are defined on the whole real line and their values on the intervals $(-\infty, 0]$ and $[l, +\infty)$ are constant.

In the remaining part of this paper attention will be restricted to considering (2.13) when the matrices $J$ and $dS$ have the form (2.15).

Everywhere in the following we use

**Assumption 2.10.** *Functions* $Q$ *and* $W$ *have no common discontinuities with* $P$.

## 3. Green's identity and linear relation $A_{max}$

### 3.1. Green's identity

Let $\mathcal{L}^2(W)$ be an inner product space, which consists of complex valued functions $f$ such that

$$\int_0^l |f(t)|^2 dW(t) < \infty. \tag{3.1}$$

The inner product in $\mathcal{L}^2(W)$ is defined by

$$(f, g)_W = \int_0^l f(t)\overline{g(t)} dW(t). \tag{3.2}$$

Denote by $L^2(W)$ the corresponding quotient space, which consists of equivalence classes with respect to the measure $dW$. To avoid confusion we will denote elements of the space $L^2(W)$ by gothic letters $\mathfrak{f}, \mathfrak{g}$ etc.

Let us consider the inhomogeneous system

$$J\begin{pmatrix} f \\ f^{[1]} \end{pmatrix}\Bigg|_0^x = \int_0^x \begin{pmatrix} -dQ(t) & 0 \\ 0 & dP(t) \end{pmatrix}\begin{pmatrix} f \\ f^{[1]} \end{pmatrix} + \int_0^x \begin{pmatrix} dW(t) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{3.3}$$

**Definition 3.1.** A pair $\{\vec{f}, g\}$ that consists of a vector-function $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ and a scalar function $g$ is said to satisfy system (3.3) (or that $\vec{f}$ is a solution of this system with fixed $g$), if the following conditions hold:

(i) $g \in \mathcal{L}^2(W)$;

(ii) $\vec{f} \in BV[0,l]$;

(iii) the equality (3.3) holds for each $x \in [0,l]$.

*Remark.* It is clear that condition $\vec{f} \in BV[0,l]$ is automatically satisfied as equality (3.3) holds. In this case it follows from $\vec{f} \in BV[0,l]$ that $f \in \mathcal{L}^2(W)$.

The componentwise rewriting of system (3.3) gives

$$\begin{cases} f(x) - f(0) = \int\limits_0^x f^{[1]}(t)dP(t), \\ f^{[1]}(x) - f^{[1]}(0) = \int\limits_0^x (f(t)dQ(t) - g(t)dW(t)). \end{cases} \tag{3.4}$$

**Theorem 3.2** (The first Green's identity)**.** *Suppose that Assumption 2.10 holds and pairs* $\left\{\vec{f}, g\right\}$, $\{\vec{u}, v\}$ *satisfy system* (3.3) *(see Definition 3.1). Then for any* $\alpha, \beta \in [0,l]$ *the next equality holds*

$$\int_\alpha^\beta gu\ dW = \int_\alpha^\beta fu\ dQ + \int_\alpha^\beta f^{[1]}u^{[1]}\ dP - f^{[1]}u\Big|_\alpha^\beta. \tag{3.5}$$

*Proof.* From (3.4) we have:

$$du = u^{[1]}dP, \quad df^{[1]} = fdQ - gdW. \tag{3.6}$$

It follows from Assumtion 2.10 that functions $u$ and $f^{[1]}$ have no common discontinuities. Consider the measure $d\left(f^{[1]}u\right)$. Then

$$d\left(f^{[1]}u\right) = df^{[1]}u + f^{[1]}du = fu\ dQ + f^{[1]}u^{[1]}\ dP - gu\ dW, \tag{3.7}$$

hence

$$gu\ dW = fu\ dQ + f^{[1]}u^{[1]}\ dP - d\left(f^{[1]}u\right). \tag{3.8}$$

To conclude the proof it remains to note that function $f^{[1]}u$ is left-continuous and to integrate equality (3.8) over $[\alpha, \beta)$. □

For a pair of vector valued functions $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}$ we define the generalized Wronskian by

$$\left[\vec{f}, \vec{u}\right] := \left(fu^{[1]} - f^{[1]}u\right). \tag{3.9}$$

**Theorem 3.3.** *Suppose Assumption 2.10 holds and pairs* $\left\{\vec{f}, g\right\}$, $\{\vec{u}, v\}$ *satisfy system* (3.3)*. Then for any* $\alpha, \beta \in [0, l]$ *the next equality holds*

$$\int_\alpha^\beta (gu - fv)\ dW = \left[\vec{f}, \vec{u}\right]\Big|_\alpha^\beta. \tag{3.10}$$

*Proof.* Application of Theorem 3.2 gives

$$gu\ dW = fu\ dQ + f^{[1]}u^{[1]}\ dP - d\left(f^{[1]}u\right), \tag{3.11}$$

$$fv\ dW = fu\ dQ + f^{[1]}u^{[1]}\ dP - d\left(fu^{[1]}\right). \tag{3.12}$$

Subtraction of (3.12) from (3.11) proves the statement. □

**Corollary 3.4** (The second Green's identity)**.** *For any two pairs* $\left\{\vec{f}, g\right\}$ *and* $\{\vec{u}, v\}$ *satisfying* (3.3) *the generalized Green's identity holds*

$$(g, u)_W - (f, v)_W = \left(f^{[1]}\overline{u}|_0 - f^{[1]}\overline{u}|_l\right) - \left(f\overline{u^{[1]}}|_0 - f\overline{u^{[1]}}|_l\right). \tag{3.13}$$

### 3.2. Linear relation $A_{max}$

**Definition 3.5.** We shall say that a pair of classes $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in L^2(W) \times L^2(W)$ belongs to the linear relation $A_{max}$ if there exist functions $f, f^{[1]}$ and $g$ such that

(i) the pair $\left\{\vec{f}, g\right\}$, where $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$, satisfies (3.3) (in the sense of Definition 3.1);

(ii) $f \in \mathfrak{f}$, $g \in \mathfrak{g}$.

In the succeeding we require the following

**Assumption 3.6.** *For any $a, b, a_1, b_1 \in \mathbb{C}$ there exists a pair $\left\{\vec{f}, g\right\}$ satisfying* (3.3) *such that*

$$f(0) = a, \quad f^{[1]}(0) = a_1, \quad f(l) = b, \quad f^{[1]}(l) = b_1. \tag{3.14}$$

In particular, if $dQ \equiv 0$ then a sufficient condition for Assumption 3.6 to hold is the next

**Proposition 3.7.** *Suppose $dQ \equiv 0$. If there exist closed on the left and disjoint intervals $i_1$ and $i_2$ on $[0, l]$ such that*

$$\dim L^2(i_j, W) > 0 \quad (j \in \{1, 2\}), \tag{3.15}$$

$$\frac{1}{dW(i_2)} \int_{i_2} P(t)dW(t) > \frac{1}{dW(i_1)} \int_{i_1} P(t)dW(t), \tag{3.16}$$

*then Assumption 3.6 holds.*

*Proof.* Let $(a\ b\ a_1\ b_1)^T$ be an arbitrary vector from $\mathbb{C}^4$. It follows from condition (3.15) that there exist functions $u_j$ that equal to one on interval $i_j$ and equal zero on its complement, and $\|u_j\|_W = dW(i_j) \neq 0$ ($j \in \{1, 2\}$).

Put $g = c_1 u_1 + c_2 u_2$, where $c_1$ and $c_2$ are some constants from $\mathbb{C}$. We shall define vector-function $\vec{f}$ by the next system

$$\begin{cases} f(x) = a + \int\limits_0^x f^{[1]}(t)dP(t), \\ f^{[1]}(x) = a_1 - \int\limits_0^x g(t)dW(t). \end{cases} \tag{3.17}$$

It is clear that for any $c_1, c_2 \in \mathbb{C}$ we have $g \in \mathcal{L}^2(W)$. Further, it follows from system (3.17) that vector-function $\vec{f}$ is of bounded variation on $[0, l]$ and $\vec{f}(0) = (a\ a_1)^T$, i.e. the pair $\left\{\vec{f}, g\right\}$ satisfies system (3.3) with the initial conditions given in advance.

Let us show now that constants $c_1$ and $c_2$ may be chosen so that equality $\vec{f}(l) = (b\ b_1)^T$ holds. It is true if and only if there exists a solution of the next system (with respect to $c_1, c_2$)

$$\begin{cases} c_1 dW(i_1) + c_2 dW(i_2) = a_1 - b_1, \\ c_1 \int\limits_0^l dP(t) \int\limits_0^t u_1(s)dW(s) + c_2 \int\limits_0^l dP(t) \int\limits_0^t u_2(s)dW(s) = \\ \qquad\qquad a - b + a_1 P(l). \end{cases} \tag{3.18}$$

By Assumption 2.10 functions $P$ and $W$ have no common discontinuities, so using integration by parts formula (2.14) we get:

$$\int_0^l dP(t) \int_0^t u_j(s)dW(s) = P(l)dW(i_j) - \int_0^l P(t)u_j(t)dW(t) \tag{3.19}$$

where $j \in \{1, 2\}$. Multiplying the first equation of system (3.18) by $P(l)$ and subtracting it from the second one and combining the obtained equation with (3.19) we will have a system (with respect to $c_1, c_2$), whose determinant

$$\begin{vmatrix} dW(i_1) & dW(i_2) \\ \int_{i_1} P(t)dW(t) & \int_{i_2} P(t)dW(t) \end{vmatrix} \tag{3.20}$$

is strictly positive due to (3.16). This ensures the solvability of system (3.18). □

**Theorem 3.8.** *Let Assumption 2.10 and Assumption 3.6 be satisfied and let the mappings* $\Gamma_0, \Gamma_1 : A_{max} \to \mathbb{C}^2$ *be defined by*

$$\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f(0) \\ f(l) \end{pmatrix}, \quad \Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f^{[1]}(0) \\ -f^{[1]}(l) \end{pmatrix} \tag{3.21}$$

*where the pair* $\left\{\vec{f}, g\right\}$ *satisfies system* (3.3), $f \in \mathfrak{f}$, $g \in \mathfrak{g}$. *Then:*

*(i) the mappings* $\Gamma_0,\ \Gamma_1$ *are well-defined;*

*(ii) the tuple* $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ *is a boundary triple for the linear relation* $A_{max}$.

*Proof.* (i) Let us show first that the mappings $\Gamma_0, \Gamma_1$ from (3.21) are independent of the choice of $f, g$ from classes $\mathfrak{f}, \mathfrak{g}$ respectively. It is clear that if a pair $\left\{\vec{f}, g_1\right\}$ satisfies system (3.3) then a pair $\left\{\vec{f}, g_2\right\}$ also satisfies (3.3) if $g_1$ and $g_2$ are equivalet with respect to the measure $dW$. It means that the values of $\Gamma_0,\ \Gamma_1$ are independent of the choice of $g \in \mathfrak{g}$.

Further let us prove that the values of the mappings $\Gamma_0, \Gamma_1$ are independent of choosing an instance $f$ from the class $\mathfrak{f}$. Let pairs $\left\{\vec{f_1}, g\right\}$ and $\left\{\vec{f_2}, g\right\}$ satisfy system (3.3) such that $f_1, f_2 \in \mathfrak{f}$. The application of Green's identity in the form (3.10) for both of the pairs on $[0, l]$ gives us two equalities. Subtracting one from the other gives

$$0 = \int_0^l (f_2 - f_1)\overline{v}dW = \left[\vec{f_1} - \vec{f_2}, \vec{\overline{u}}\right]\Big|_0^l. \tag{3.22}$$

By Assumption 3.6 a pair of functions $\{\vec{u}, v\}$ satisfying system (3.3) can be chosen such that $u(0), u(l), u^{[1]}(0)$ and $u^{[1]}(l)$ may be arbitrary from $\mathbb{C}$. This means that we have

$$f_1(0) = f_2(0), \qquad f_1(l) = f_2(l), \tag{3.23}$$

$$f_1^{[1]}(0) = f_2^{[1]}(0), \qquad f_1^{[1]}(l) = f_2^{[1]}(l) \tag{3.24}$$

which proofs that the mappings $\Gamma_0, \Gamma_1$ are single-valued.

(ii) It follows directly from Corollary 3.4 and Assumption 3.6 that the requirements of Definition 2.2 are satisfied. □

*Remark* 3.9. Evidently, if Assumption 3.6 does not hold then the objects $\Gamma_0$ and $\Gamma_1$, defined by (3.21), in general are not operators but linear relations in $L^2(W)^2 \times \mathbb{C}^2$. Such boundary triples were considered in [9].

It is also possible that if Assumption 3.6 does not hold then the mapping $\Gamma = (\Gamma_0\ \Gamma_1)^T$ is not surjective. This happens, for example, if $dQ = 0,\ dP = dx$, and $W$ is piecewise with a single jump.

In the case of $dQ \equiv 0$ system (3.4) can be rewritten as follows

$$f(x) = f(0) + f^{[1]}(0)P(x) - \int_0^x \left\{ \int_0^t g(s)dW(s) \right\} dP(t). \tag{3.25}$$

Function $G(t) := \int_0^t g(s)dW(s)$ is of bounded variation on $[0, l]$ and the set of its jumps is a subset of jumps of function $W$. Hence, functions $G$ and $P$ have no common discontinuities. The application of integration by parts formula (2.14) to equality (3.25) gives us (cf. [17, p. 650, equality (1.1)])

$$f(x) = f(0) + f^{[1]}(0)P(x) - \int_0^x \{P(x) - P(t)\}\, g(t)dW(t). \tag{3.26}$$

This leads to the following

**Proposition 3.10.** *Suppose Assumption 2.10 holds and $dQ \equiv 0$. Then the kernel of the linear relation $A_{max} \subset L^2(W)^2$ is two-dimensional if and only if function $P$ is not equivalent to a constant in $L^2(W)$ and one-dimensional otherwise.*

*Proof.* Let $g$ be zero element of $L^2(W)$. Then equality (3.26) takes the form

$$f(x) = f(0) + f^{[1]}(0)P(x), \tag{3.27}$$

which is equivalent to $f \in \operatorname{span}\{1, P\}$. □

*Remark.* Further, in the proof of Theorem 3.12 it will be shown that the kernel of linear relation $A_{max}$ is always two-dimensional if in addition Assumption 3.6 holds.

**Definition 3.11.** We shall say that an element $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}$ of the linear relation $A_{max}$ belongs to the linear relation $A_{min}$, if

$$f(0) = f^{[1]}(0) = f(l) = f^{[1]}(l) = 0. \tag{3.28}$$

It follows from equality (3.13) that the linear relation $A_{min}$ is symmetric.

**Theorem 3.12.** *Linear relations $A_{min}$ and $A_{max}$ are closed, $A_{min}^* = A_{max}$, and deficiency indices of $A_{min}$ are* $(2,2)$.

*Proof.* We shall check that for linear relations $A_{min}$ and $A_{max}$ conditions of Theorem 2.4 are satisfied. It follows directly from Theorem 2.9 that $\operatorname{ran} A_{max} = L^2(W)$. Let $\mathfrak{g}$ be an arbitrary class from $L^2(W)$, $g$ be some instance of $\mathfrak{g}$. Then (for any fixed initial value) by Theorem 2.9 there exists a vector-function $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ such that pair $\left\{\vec{f}, g\right\}$ satisfies system (3.3) and, as a consequence, $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max}$.

Further, let us show that $\dim \ker A_{max} = 2$. By Theorem 2.9 if $g = 0$ then for any complex numbers $a, a_1$ there exists a unique vector-function $\vec{f}$ such that $f(0) = a$, $f^{[1]}(0) = a_1$ and $\mathfrak{f} \in \ker A_{max}$, where $\mathfrak{f}$ is the class from $L^2(W)$ generated by $f$. If Assumption 3.6 holds then similarly to the proof of Theorem 3.8 we get that $\dim \ker A_{max}$ is isomorphic to $\mathbb{C}^2$. By the same argument, we get $\ker A_{min} = \{0\}$. Now the statement of this theorem follows from Theorem 2.4. □

**Theorem 3.13.** [25] *The set of all self-adjoint extensions of the linear relation $A_{min}$ is described by the boundary conditions*

$$\widetilde{A} = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : C\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} + D\Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = 0 \right\} \tag{3.29}$$

*where $C, D$ are complex valued $2 \times 2$ matrices such that*

$$\det(CC^* + DD^*) \neq 0, \quad CD^* = DC^*. \tag{3.30}$$

In particular, linear relations $A_0$ and $A_1$ defined by equalities (2.11) are self-adjoint extensions of the linear relation $A_{min}$. Extensions $A_0$ and

$A_1$ corresponding to boundary triple (3.21) coincide with the Dirichlet extension and the Neumann extension

$$A_D := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f(l) = 0 \right\}, \tag{3.31}$$

$$A_N := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f^{[1]}(0) = f^{[1]}(l) = 0 \right\}, \tag{3.32}$$

respectively.

## 3.3. Functions $c(x,\lambda)$ and $s(x,\lambda)$. Weyl function of linear relation $A_{max}$

Let Assumption 2.10 and Assumption 3.6 hold. It follows from Theorem 2.9 that for each fixed $\lambda \in \mathbb{C}$ there exist unique vector-functions $\vec{c}(x,\lambda)$ and $\vec{s}(x,\lambda)$ satisfying the initial conditions

$$\begin{aligned} c(0,\lambda) &= 1, \quad c^{[1]}(0,\lambda) = 0, \\ s(0,\lambda) &= 0, \quad s^{[1]}(0,\lambda) = 1, \end{aligned} \tag{3.33}$$

such that the pairs $\{\vec{c}, \lambda c\}$ and $\{\vec{s}, \lambda s\}$ satisfy system (3.3). Here we have inclusions $c,\ s \in \mathcal{L}^2(W)$. Let $\mathfrak{c}(\lambda)$ and $\mathfrak{s}(\lambda)$ be classes from $L^2(W)$ generated by $c(x,\lambda)$ and $s(x,\lambda)$, respectively. Then

$$\begin{pmatrix} \mathfrak{c}(\lambda) \\ \lambda\mathfrak{c}(\lambda) \end{pmatrix}, \begin{pmatrix} \mathfrak{s}(\lambda) \\ \lambda\mathfrak{s}(\lambda) \end{pmatrix} \in A_{max}. \tag{3.34}$$

It is known (see [6]) that functions $c(x,\lambda)$ and $s(x,\lambda)$ are entire in $\lambda$ of order not greater that $1/2$.

By conditions (3.33) functions $c$ and $s$ are linearly independent, and it follows from Assumption 3.6 that classes $\mathfrak{c}(\lambda)$ and $\mathfrak{s}(\lambda)$ are linearly independent too. Any element $\mathfrak{f}_\lambda$ from the defect subspace $\mathfrak{N}_\lambda$ can be represented as

$$\mathfrak{f}_\lambda = a_1\mathfrak{c}(\lambda) + a_2\mathfrak{s}(\lambda), \quad a_1, a_2 \in \mathbb{C}. \tag{3.35}$$

**Theorem 3.14.** *The generalized Wronskian of the functions $\vec{c}(x,\lambda)$ and $\vec{s}(x,\lambda)$ is a constant:*

$$[\vec{c}, \vec{s}] = c(x,\lambda)s^{[1]}(x,\lambda) - c^{[1]}(x,\lambda)s(x,\lambda) = 1, \quad x \in [0,l]. \tag{3.36}$$

*Proof.* Note that both pairs $\begin{pmatrix} \mathfrak{s} \\ \lambda\mathfrak{s} \end{pmatrix}$ and $\begin{pmatrix} \bar{\mathfrak{s}} \\ \overline{\lambda\mathfrak{s}} \end{pmatrix}$ belong or do not belong to the linear relation $A_{max}$ simultaneously. The application of Green's identity in the form (3.10) to pairs $\begin{pmatrix} \mathfrak{c} \\ \lambda\mathfrak{c} \end{pmatrix}$ and $\begin{pmatrix} \bar{\mathfrak{s}} \\ \overline{\lambda\mathfrak{s}} \end{pmatrix}$ gives

$$[\vec{c}(t,\lambda), \vec{s}(t,\lambda)]|_0^x = 0. \tag{3.37}$$

□

**Theorem 3.15.** *The Weyl function and the $\gamma$-field of the linear relation $A_{max}$ corresponding to boundary triple $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ from* (3.21) *have the forms*

$$M(\lambda) = \frac{-1}{s(l,\lambda)} \begin{pmatrix} c(l,\lambda) & -1 \\ -1 & s^{[1]}(l,\lambda) \end{pmatrix}, \tag{3.38}$$

$$\gamma(\lambda) = \frac{1}{s(l,\lambda)} \begin{pmatrix} \mathfrak{c}(\lambda)s(l,\lambda) - c(l,\lambda)\mathfrak{s}(\lambda) & \mathfrak{s}(\lambda) \end{pmatrix}. \tag{3.39}$$

*Proof.* Let $\mathfrak{f}_\lambda = a_1\mathfrak{c}(\lambda) + a_2\mathfrak{s}(\lambda)$. Then

$$\begin{aligned} \Gamma_0 \begin{pmatrix} \mathfrak{f}_\lambda \\ \lambda\mathfrak{f}_\lambda \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ c(l,\lambda) & s(l,\lambda) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} =: Y_0 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \\ \Gamma_1 \begin{pmatrix} \mathfrak{f}_\lambda \\ \lambda\mathfrak{f}_\lambda \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -c^{[1]}(l,\lambda) & -s^{[1]}(l,\lambda) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} =: Y_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \end{aligned} \tag{3.40}$$

It follows from Definition 2.5 of the Weyl function and equality (3.36) that

$$\begin{aligned} M(\lambda) = Y_1Y_0^{-1} &= \frac{-1}{s(l,\lambda)} \begin{pmatrix} c(l,\lambda) & -1 \\ c^{[1]}(l,\lambda)s(l,\lambda) - c(l,\lambda)s^{[1]}(l,\lambda) & s^{[1]}(l,\lambda) \end{pmatrix} \\ &= \frac{-1}{s(l,\lambda)} \begin{pmatrix} c(l,\lambda) & -1 \\ -1 & s^{[1]}(l,\lambda) \end{pmatrix}. \end{aligned} \tag{3.41}$$

Finally, by definition of the $\gamma$-field we have

$$\begin{aligned} \gamma(\lambda) &= \begin{pmatrix} \mathfrak{c}(\lambda) & \mathfrak{s}(\lambda) \end{pmatrix} Y_0^{-1} \\ &= \frac{1}{s(l,\lambda)} \begin{pmatrix} \mathfrak{c}(\lambda)s(l,\lambda) - c(l,\lambda)\mathfrak{s}(\lambda) & \mathfrak{s}(\lambda) \end{pmatrix}. \end{aligned} \tag{3.42}$$

□

## 4. Weyl functions of intermediate extensions of linear relation $A_{min}$

In this section the boundary triples and the corresponding Weyl functions for intermediate extensions of the linear relation $A_{min}$ are constructed.

**Definition 4.1.** Let us set

$$A_{D0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_D : f^{[1]}(0) = 0 \right\}, \quad A_{Dl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_D : f^{[1]}(l) = 0 \right\},$$

$$A_{N0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_N : f(0) = 0 \right\}, \quad A_{Nl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_N : f(l) = 0 \right\}.$$

It follows from Definition 4.1, (3.31), and (3.32) that

$$A_{D0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f(l) = f^{[1]}(0) = 0 \right\}, \tag{4.1}$$

$$A_{Dl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f(l) = f^{[1]}(l) = 0 \right\}, \tag{4.2}$$

$$A_{N0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f^{[1]}(0) = f^{[1]}(l) = 0 \right\}, \tag{4.3}$$

$$A_{Nl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(l) = f^{[1]}(0) = f^{[1]}(l) = 0 \right\}. \tag{4.4}$$

**Theorem 4.2.** *Linear relation $A_{D0}$ is symmetric in $L^2(W)$ with deficiency indices $(1,1)$ and the following conditions hold:*

*(i) The adjoint linear relation $A_{D0}^*$ has the form*

$$A_{D0}^* = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(l) = 0 \right\}. \tag{4.5}$$

*(ii) The tuple $\{\mathbb{C}, \Gamma_0^{D0}, \Gamma_1^{D0}\}$, where*

$$\Gamma_0^{D0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(0), \quad \Gamma_1^{D0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = -f(0), \tag{4.6}$$

*is a boundary triple for $A_{D0}^*$.*

*(iii) The corresponding Weyl function and the $\gamma$-field have the form*

$$M_{D0}(\lambda) = \frac{s(l,\lambda)}{c(l,\lambda)}, \quad \gamma_{D0}(\lambda) = \mathfrak{s}(\lambda) - \frac{s(l,\lambda)}{c(l,\lambda)} \mathfrak{c}(\lambda). \tag{4.7}$$

*Proof.* (i) Suppose $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{D0}$. By definition $\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{D0}^*$ holds if and only if

$$(\mathfrak{g}, \mathfrak{u})_{L^2(W)} = (\mathfrak{f}, \mathfrak{v})_{L^2(W)}. \tag{4.8}$$

The last equality is equivalent to

$$\left( \Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}, \Gamma_0 \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \right)_{\mathcal{H}} = \left( \Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}, \Gamma_1 \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \right)_{\mathcal{H}}. \tag{4.9}$$

Since $f^{[1]}(l)$ is arbitrary, the last equality holds if and only if $u(l) = 0$.

(ii) Let us show that Green's identity (in the sense of Definition 2.2) holds for the mappings $\Gamma_0^{D0}, \Gamma_1^{D0}$, which are defined on $A_{D0}^*$. It is clear that $A_{D0}^* \subset A_{max}$. Hence, for any $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}, \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{D0}^*$ the equality (3.13) holds and taking into account (4.5) we have

$$(\mathfrak{g}, \mathfrak{u})_{L^2(W)} - (\mathfrak{f}, \mathfrak{v})_{L^2(W)} = f^{[1]}(0)\overline{u(0)} - f(0)\overline{u^{[1]}(0)}. \tag{4.10}$$

It remains to check that the mapping $\Gamma_{D0} = \begin{pmatrix} \Gamma_0^{D0} \\ \Gamma_1^{D0} \end{pmatrix} : A_{D0}^* \to \mathbb{C} \oplus \mathbb{C}$ is surjective, which follows directly from the subjectivity of the mapping $\Gamma$ on $A_{max}$.

(iii) The defect subspace of linear relation $A_{D0}^*$ has the form

$$\mathfrak{N}_\lambda(A_{D0}^*) = \operatorname{span}\{\mathfrak{c}(\lambda) + k\mathfrak{s}(\lambda)\} \tag{4.11}$$

where the coefficient $k$ is chosen to satisfy $f_\lambda(l) = 0$. Further

$$\Gamma_0^{D0}\hat{f}_\lambda = k = -\frac{c(l,\lambda)}{s(l,\lambda)}, \quad \Gamma_1^{D0}\hat{f}_\lambda = -1, \tag{4.12}$$

and finally

$$M_{D0}(\lambda) = \frac{s(l,\lambda)}{c(l,\lambda)}, \quad \gamma_{D0}(\lambda) = \mathfrak{s}(\lambda) - \frac{s(l,\lambda)}{c(l,\lambda)}\mathfrak{c}(\lambda). \tag{4.13}$$

□

Similar theorems for extensions $A_{Dl}$, $A_{N0}$, and $A_{Nl}$ are given below without proofs.

**Theorem 4.3.** *Linear relation $A_{Dl}$ is symmetric in $L^2(W)$ with deficiency indices $(1,1)$, and the following conditions hold:*

*(i) The adjoint linear relation $A_{Dl}^*$ has the form*

$$A_{Dl}^* = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = 0 \right\}. \tag{4.14}$$

*(ii) The tuple $\{\mathbb{C}, \Gamma_0^{Dl}, \Gamma_1^{Dl}\}$, where*

$$\Gamma_0^{Dl}\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(l), \quad \Gamma_1^{Dl}\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f(l), \tag{4.15}$$

*is a boundary triple for $A_{Dl}^*$.*

*(iii) The corresponding Weyl function and the $\gamma$-field have the form*

$$M_{Dl}(\lambda) = \frac{s(l,\lambda)}{s^{[1]}(l,\lambda)}, \quad \gamma_{Dl}(\lambda) = \frac{\mathfrak{s}(\lambda)}{s^{[1]}(l,\lambda)}. \tag{4.16}$$

**Theorem 4.4.** *Linear relation $A_{N0}$ is symmetric in $L^2(W)$ with deficiency indices $(1,1)$ and the following conditions hold:*

*(i) The adjoint linear relation $A^*_{N0}$ has the form*

$$A^*_{N0} = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f^{[1]}(l) = 0 \right\}. \tag{4.17}$$

*(ii) The tuple $\{\mathbb{C}, \Gamma_0^{Dl}, \Gamma_1^{Dl}\}$, where*

$$\Gamma_0^{N0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(0), \quad \Gamma_1^{N0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = -f(0), \tag{4.18}$$

*is a boundary triple for $A^*_{N0}$.*

*(iii) The corresponding Weyl function and $\gamma$-field have the form*

$$M_{N0}(\lambda) = \frac{s^{[1]}(l,\lambda)}{c^{[1]}(l,\lambda)}, \quad \gamma_{N0}(\lambda) = \mathfrak{s}(\lambda) - \frac{s^{[1]}(l,\lambda)}{c^{[1]}(l,\lambda)} c(\cdot,\lambda). \tag{4.19}$$

**Theorem 4.5.** *Linear relation $A_{Nl}$ is symmetric in $L^2(W)$ with deficiency indices $(1,1)$ and the following conditions hold:*

*(i) The adjoint linear relation $A^*_{Nl}$ has the form*

$$A^*_{N1} = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f^{[1]}(0) = 0 \right\}. \tag{4.20}$$

*(ii) The tuple $\{\mathbb{C}, \Gamma_0^{Nl}, \Gamma_1^{Nl}\}$ where*

$$\Gamma_0^{Nl} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(l), \quad \Gamma_1^{Nl} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f(l), \tag{4.21}$$

*is a boundary triple for $A^*_{Nl}$.*

*(iii) The corresponding Weyl function and $\gamma$-field have the form*

$$M_{Nl}(\lambda) = \frac{c(l,\lambda)}{c^{[1]}(l,\lambda)}, \quad \gamma_{Nl}(\lambda) = \frac{c(\cdot,\lambda)}{c^{[1]}(l,\lambda)}. \tag{4.22}$$

*Remark* 4.6. The Weyl functions $M_{D0}$, $M_{N0}$ in the case $dQ \equiv 0$ coincide with the functions $\Omega_0$, $\Omega_1$, see [18, p. 666, (2.40–41)].

## 5. Special cases

### 5.1. Absolutely continuous case. Sturm-Liouville operator

Let functions $P$, $Q$ and $W$ be absolutely continuous on $[0, l]$, i.e. there exist functions $p$, $q$ and $w$ from $L^1[0, l]$ such that

$$P(x) = \int_0^x p(t)dt, \quad Q(x) = \int_0^x q(t)dt, \quad W(t) = \int_0^x w(t)dt, \tag{5.1}$$

$p(t) \neq 0$ and $w(t) \geq 0$ almost everywhere with respect to Lebesgue measure on $[0, l]$. In addition, we require that the space $L^2(W)$ be nontrivial. The last requirement is equivalent to $W(l) > W(0)$.

In this special case system (1.1) can be written in the form

$$J\vec{f}'(x) = \lambda H(x)\vec{f}(x) + V(x)\vec{f}(x), \quad \vec{f}(0) = \vec{a}(0), \tag{5.2}$$

where

$$H(x) = \begin{pmatrix} w(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} -q(x) & 0 \\ 0 & p(x) \end{pmatrix}, \quad \vec{f}(x) = \begin{pmatrix} f(x) \\ f^{[1]}(x) \end{pmatrix}$$

or, equivalently,

$$\begin{cases} -(f^{[1]})'(x) = \lambda w(x)f(x) - q(x)f(x), \\ f'(x) = p(x)f^{[1]}(x). \end{cases} \tag{5.3}$$

System (5.3) is equivalent to the Sturm–Liouville equation (see [26])

$$-\frac{d}{dx}\left(\frac{1}{p(x)}\frac{d}{dx}f(x)\right) + q(x)f(x) = \lambda w(x)f(x). \tag{5.4}$$

with the initial conditions

$$f(0) = a_1, \quad f^{[1]}(0) = a_2.$$

More general canonical systems (5.2) were studied in [16, 20, 24], where, in particular, it was shown that the maximal and the minimal operators associated with such canonical systems can be linear relations with nontrivial multivalued part. In the 2-dimensional case the multivalued part of the maximal operator was calculated explicitly in terms of the so called $H$-indivisible intervals, [16]. Actually in the absolutely continuous case the results of the paper can be easily derived from the results of [4] and [23].

### 5.2. Discrete case. Stieltjes string

Let us consider system (3.4) in the case $dQ \equiv 0,\ dP = dx$, and $W$ be a left-continuous monotonically nondecreasing piecewise constant function on $[0, l]$ that has at least two growth points. Let $\{x_j\}_{j=0}^{n-1}$ be the growth points of $W$ such that

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n := l. \tag{5.5}$$

By $w_j$ denote

$$w_j := W(x_j + 0) - W(x_j) \quad (j \in \{1, 2, \ldots, n-1\}). \tag{5.6}$$

The distance between neighboring growth points is denoted by

$$l_j := x_j - x_{j-1} \quad (j \in \{1, 2, \ldots, n\}). \tag{5.7}$$

Finally for convenience denote

$$f_j := f(x_j), \quad g_j := g(x_j), \quad h_j := f^{[1]}(x_j). \tag{5.8}$$

With generating function $W$ the space $L^2(W)$ is isomorphic to $\mathbb{C}^n$, and each of its element is a vector $[f_0, f_1, \ldots, f_{n-1}]^T$.

It is easy to check that by the above assumptions one can choose closed on the left intervals $i_1, i_2$ such that they satisfy Proposition 3.7. For instance, it is sufficient to choose intervals $i_j$ with the only growth point $x_{j-1}$ ($j \in \{1, 2\}$). Theh the spaces $L^2(i_j, W)$ obviously are non-trivial and inequality (3.16) takes the following form

$$\frac{w_1 l_1}{w_2} > 0. \tag{5.9}$$

Combining (5.6), (5.7), and (5.8) we can rewrite system (3.4) as

$$\begin{cases} f_{j+1} - f_j = h_{j+1} l_{j+1}, \\ h_{j+1} - h_j = -w_j g_j \end{cases} \tag{5.10}$$

where $j \in \{0, 1, \ldots, n-1\}$.

**Proposition 5.1.** *In the assumptions of case 5.2 the multivalued part of the linear relation $A_{max}$ has the form*

$$\{(c_1, 0, 0, \ldots, 0, c_2)^T : c_1, c_2 \in \mathbb{C}\} \tag{5.11}$$

*and the linear relation $A_{min}$ is the graph of a single-valued linear operator.*

*Proof.* Let $\mathfrak{f}$ be the zero element of $L^2(W)$. Thus, in (5.10) we have $f_j = 0$ as $j \in \{0, 1, \ldots, n-1\}$, hence $h_j = 0$ as $j \in \{1, 2, \ldots, n-1\}$ and $g_j = 0$ as $j \in \{1, 2, \ldots, n-2\}$. The converse is also true: since we can choose $h_0 = w_0 g_0$ then for each vector $\mathfrak{g} \in L^2(W)$ of the form (5.11) the pair $\begin{pmatrix} 0 \\ \mathfrak{g} \end{pmatrix}$ belongs to the linear relation $A_{max}$.

If, in addition, $f_n = h_0 = h_n = 0$ then it follows from (5.10) that $g_j = 0$ as $j \in \{0, 1, \ldots n-1\}$. □

### 5.3. Mixed case. Krein–Feller string

A more general case can be obtained if we suppose $dQ \equiv 0,\ dP = dx$ and $W$ is an arbitrary mototonically nondecreasing function.

In this Proposition 3.7 holds if and only if $W$ has at least two distinct growth points on $[0, l]$:

$$0 < W(x_0) < W(x_1) \le W(l). \tag{5.12}$$

Now system (3.4) has the following form:

$$\begin{cases} f(x) - f(0) = \int\limits_0^x f^{[1]}(t)dt, \\ f^{[1]}(x) - f^{[1]}(0) = -\int\limits_0^x g(t)dW(t). \end{cases} \tag{5.13}$$

In particular we have the next

**Proposition 5.2.** *Suppose the assumptions of case 5.3 are satisfied. If a pair* $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in L^2(W)^2$ *belongs to linear relation* $A_{max}$ *then there exists* $f \in \mathfrak{f}$ *such that* $f$ *is absolutely continuous with respect to Lebesgue measure and its derivative coincides with* $f^{[1]}$ *almost everywhere.*

Let us rewrite system (5.13) as

$$f(x) = f(0) + x f^{[1]}(0) - \int\limits_0^x \left( \int\limits_0^t g(s)dW(s) \right) dt. \tag{5.14}$$

The function $\int\limits_0^t g(s)dW(s)$ is left-continuous and of bounded variation on $[0, l]$. It follows from (2.14) that equality (5.14) can be rewritten as

$$f(x) = f(0) + x f^{[1]}(0) - \int\limits_0^x (x - s) g(s)dW(s). \tag{5.15}$$

**Definition 5.3.** [17] A function $f$ is said to belong to the Stieltjes class $S^+$ if $f \in$R and $f$ admits a holomorphic non-negative continuation to $(-\infty, 0)$.

In the paper [18] the differential operation defined by (5.15) was investigated. I. S. Kats and M. G. Krein showed that under assumptions of the case 5.2 the Weyl functions $M_{D0}, M_{N0}$ and $M_{Nl}$ constructed in section 4, belong to the Stieltjes class $S^+$, see [18, p. 666, Lemma 2.3].

## References


[1] R. Arens, *Operational calculus of linear relations* // Pacific Journal of Mathematics, **11** (1961), No. 1, 9–23.

[2] D. Z. Arov, H. Dym, *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*, 2012.

[3] F. V. Atkinson, *Discrete and continuous boundary problems*, Academic Press, New York-London, Mathematics in Science and Engineering, Vol. 8, 1964.

[4] J. Behrndt, S. Hassi, H. de Snoo, R. Wietsma, *Square-integrable solutions and Weyl functions for singular canonical systems* // Mathematische Nachrichten, **284** (2011), No. 11-12, 1334–1384.

[5] C. Bennewitz, *Symmetric relations on a Hilbert space*, Lecture Notes in Math., Springer, Berlin, Heidelberg, **280** (1972), 212–218.

[6] C. Bennewits, *Spectral Asymptotics for Sturm-Liouville Equations* // Proceedings of the London Mathematical Society, (3) **59** (1989), No. 2, 294–338.

[7] V. M. Bruk, *On a class of problems with the spectral parameter in the boundary conditions* // Mat. Sb., **100** (1976), 210–216.

[8] J. W. Calkin, *Abstract symmetric boundary conditions* // TAMS, **45** (1939), No. 3, 369–442.

[9] V. Derkach, S. Hassi, M. Malamud, H. de Snoo, *Boundary relations and their Weyl families* // Transactions of the American Mathematical Society, **358** (2006), No. 12, 5351–5400.

[10] V. A. Derkach, M. M. Malamud, *Generalized resolvents and the boundary value problems for hermitian operators with gaps* // J. Funct. Anal., **95** (1991), 1–95.

[11] V. A. Derkach, M. M. Malamud, *The extension theory of hermitian operators and the moment problem* // J. Math. Sciences, **73** (1995), 141–242.

[12] V. A. Derkach, M. M. Malamud, *Extension theory of symmetric operators and boundary value problems*, Proceedings of Institute of Mathematics NAS of Ukraine, Kyiv, 2017, 573 p. (2017)

[13] W. Feller, *On Second Order Differential Operators* // Annals of Mathematics, **61** (1955), No. 1, 90–105.

[14] V. I. Gorbachuk, M. L. Gorbachuk, *Boundary problems for differential operator equations*, Naukova Dumka, Kiev, 1984.

[15] E. Hewitt, *Integration by Parts for Stieltjes Integrals* // The American Mathematical Monthly, **67** (1960), No. 5, 419–423.

[16] I. S. Kac, *Linear relations generated by a canonical differential equation of phase dimension 2 and decomposability in eigenfunctions* // Algebra i analiz, **14** (2002), No. 3, 86–120.

[17] I. S. Kac, M. G. Krein, *R-functions – analytic functions mapping the upper half plane into itself*, Supplement I to the Russian translation of F. V. Atkinson, *Discrete and continuous boundary problems*, Mir, Moscow, 1968, 629–647.

[18] I. S. Kac, M. G. Krein, *On the spectral functions of the string*, Supplement II to the Russian translation of F. V. Atkinson, *Discrete and continuous boundary problems*, Mir, Moscow, 1968, 648–737.

[19] A. N. Kochubey, *On extensions of symmetric operators and symmetric binary relations* // Math. Notes, **17** (1975), No. 1, 41–48.

[20] M. Lesch, M. Malamud, *On the deficiency indices and self-adjointness of symmetric Hamiltonian systems* //, J. Differential Equations, **189** (2003), No. 2, 556-615

[21] M. M. Malamud, *On the formula of generalized resolvents of a nondensely defined Hermitian operator* // Ukr. Mat. Zh., **44** (1992), No. 12, 1658–1688.

[22] V. Mogilevsky, *Boundary triplets and Titchmarsh - Weyl functions of differential operators with arbitrary deficiency indices* // Methods of Funct. Anal. Topology, **15** (2009), No. 3, 280–300.

[23] V. I. Mogilevskii, *Spectral and pseudospectral functions of Hamiltonian systems: development of the results by Arov-Dym and Sakhnovich* // Methods of Funct. Anal. Topology, **21** (2015), No. 4, 370–402.

[24] B. Orcutt, Canonical differential equations, Ph.D. thesis, University of Virginia, 1969.

[25] F. S. Rofe-Beketov, *On selfadjoint extensions of differential operators in a space of vector-functions* // Teor. Funkts., Funkts. Anal. i Prilozhen., **8** (1969), 3–24.

[26] E. C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, Part I, 2nd edn (Oxford University Press, 1962).

## Contact information

**Dmytro Strelnikov** Vasyl' Stus Donetsk National University,
Vinnitsya, Ukraine
*E-Mail:* `d.strelnikov@donnu.edu.ua`