Український математичний вісник
Том 14 **(2017)**, № 4, 496 – 514

Automorphisms of semigroups of k-linked upfamilies

Volodymyr M. Gavrylkiv

(*Presented by V. O. Derkach*)

Abstract. A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any set $B \supset A$ belongs to $\mathcal{A}$. An upfamily $\mathcal{L}$ is called *k-linked* if $\bigcap \mathcal{F} \neq \emptyset$ for any subfamily $\mathcal{F} \subset \mathcal{L}$ of cardinality $|\mathcal{F}| \leq k$. The extension $N_k(X)$ consists of all k-linked upfamilies on X. Any associative binary operation $*: X \times X \to X$ can be extended to an associative binary operation $*: N_k(X) \times N_k(X) \to N_k(X)$. In the paper, we study automorphisms of the extensions of groups, finite monogenic semigroups and describe the automorphism groups of extensions of null semigroups, almost null semigroups, right zero semigroups and left zero semigroups.

2010 MSC. 18B40, 37L05, 22A15, 20D45, 20M15, 20B25.

Key words and phrases. Semigroup, k-linked upfamily, automorphism group.

Introduction

In this paper, we investigate the automorphism groups of the extensions $N_k(S)$ of a semigroup S. The thorough study of various extensions of semigroups was started in [13] and continued in [1–10, 14–19]. The largest among these extensions is the semigroup $\upsilon(S)$ of all upfamilies on S. A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any subset $B \supset A$ of X belongs to $\mathcal{A}$. Each family $\mathcal{B}$ of non-empty subsets of X generates the upfamily $\langle \mathcal{B} \rangle := \{A \subset X : \exists B \in \mathcal{B}\ (B \subset A)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called *the Stone-Čech compactification of* X, see [20, 24]. An ultrafilter $\langle \{x\} \rangle$, generated by a singleton $\{x\}$, $x \in X$,

Received 05.12.2017

ISSN 1810 – 3200.

is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle\{x\}\rangle$ generated by the singleton $\{x\}$, and hence we can consider $X \subset \beta(X) \subset \upsilon(X)$. It was shown in [13] that any associative binary operation $*: S \times S \to S$ can be extended to an associative binary operation $*: \upsilon(S) \times \upsilon(S) \to \upsilon(S)$ by the formula

$$\mathcal{A} * \mathcal{B} = \Big\langle \bigcup_{a \in A} a * B_a : A \in \mathcal{A},\ \ \{B_a\}_{a \in A} \subset \mathcal{B} \Big\rangle$$

for upfamilies $\mathcal{A}, \mathcal{B} \in \upsilon(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $\upsilon(S)$.

The semigroup $\upsilon(S)$ contains many other important extensions of S. In particular, it contains the semigroups $N_k(S)$ of k-linked upfamilies for $k \in \mathbb{N} \setminus \{1\}$. An upfamily $\mathcal{L} \in \upsilon(S)$ is called *k-linked* if $\bigcap \mathcal{F} \neq \emptyset$ for any subfamily $\mathcal{F} \subset \mathcal{L}$ of cardinality $|\mathcal{F}| \le k$. The space $N_k(S)$ is well-known in General and Categorial Topology, see [22–25].

For a finite set X the cardinality of the set $N_k(X)$ growth very quickly as $|X|$ tends to infinity. The calculation of the cardinality of $N_k(X)$ seems to be a difficult combinatorial problem related to the still unsolved Dedekind's problem of calculation of the number $M(n)$ of all monotone Boolean functions of n Boolean variable, see [11].

We were able to calculate the cardinalities of $N_k(X)$ only for sets X of cardinality $|X| \le 5$, see [12]. The results of (computer) calculations are presented in Table 1.

$\lvert X\rvert$	$\lvert N_2(X)\rvert$	$\lvert N_3(X)\rvert$	$\lvert N_4(X)\rvert$
1	1	1	1
2	3	3	3
3	11	10	10
4	80	54	53
5	2645	762	687

Table 1: The cardinalities of $N_k(X)$ for sets X of cardinality $|X| \le 5$

Each map $f : X \to Y$ for each $k \in \mathbb{N} \setminus \{1\}$ induces the map

$$N_k f : N_k(X) \to N_k(Y), \quad N_k f : \mathcal{M} \mapsto \big\langle f(M) : M \in \mathcal{M} \big\rangle, \ \text{ see [12].}$$

If $\varphi : S \to S'$ is a homomorphism of semigroups, then for each $k \in \mathbb{N} \setminus \{1\}$, the map $N_k\varphi : N_k(S) \to N_k(S')$ is a homomorphism as well, see [13].

Recall that an *isomorphism* between semigroups S and S' is bijective function $\psi : S \to S'$ such that $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in S$. If there exist an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted $S \cong S'$. An isomorphism $\psi : S \to S$ is called an *automorphism* of a semigroup S. By $\mathrm{Aut}\,(S)$ we denote the automorphism group of a semigroup S.

A non-empty subset I of a semigroup S is called an *ideal* if $IS \cup SI \subset I$. An ideal I of a semigroup S is said to be *proper* if $I \neq S$. A proper ideal M of S is *maximal* if M coincides with each proper ideal I of S that contains M. An element z of a semigroup S is called a *zero* (resp. a *left zero*, a *right zero*) in S if $az = za = z$ (resp. $za = z$, $az = z$) for any $a \in S$. An element e of a semigroup S is called an *idempotent* if $ee = e$. By $E(S)$ we denote the set of all idempotents of a semigroup S.

1. Extending automorphisms from a semigroup to its extensions

In this section we show that for each $k \in \mathbb{N} \setminus \{1\}$ any automorphism of a semigroup S can be extended to an automorphism of its extension $N_k(S)$ and the automorphism group $\mathrm{Aut}\,(N_k(S))$ of the extension $N_k(S)$ of a semigroup S contains a subgroup, isomorphic to the group $\mathrm{Aut}\,(S)$.

The following propositions are corollaries of the functoriality of N_k in the category of semigroups, see [3, 24].

Proposition 1.1. *If $\psi : S \to S$ is an automorphism of a semigroup S, then for each $k \in \mathbb{N} \setminus \{1\}$ the map $N_k\psi : N_k(S) \to N_k(S)$ is an automorphism of the extension $N_k(S)$.*

Proposition 1.2. *For each $k \in \mathbb{N} \setminus \{1\}$ the automorphism group $\mathrm{Aut}\,(N_k(S))$ of the extension $N_k(S)$ of a semigroup S contains a subgroup, isomorphic to the automorphism group $\mathrm{Aut}\,(S)$ of S.*

2. The automorphism groups of the extensions $N_k(G)$ of a group G

In this section we shall study automorphisms of extensions $N_k(G)$ of a group G.

Proposition 2.1. *Let G be a group, $k \in \mathbb{N} \setminus \{1\}$. If $\psi : N_k(G) \to N_k(G)$ is an automorphism, then $\psi(G) = G$.*

Proof. It was shown in [8, Proposition 4.2] that $N_k(G) \setminus G$ is an ideal of $N_k(G)$. Let us prove that $N_k(G) \setminus G$ is the unique maximal ideal of $N_k(G)$.

Indeed, let I be any ideal of $N_k(G)$. If $g \in G \cap I$, then $N_k(G) = gN_k(G) \subset I$, and hence $I = N_k(G)$. Consequently, $N_k(G) \setminus G$ contains each proper ideal of $N_k(G)$. Taking into account that the set of maximal ideals of a semigroup is preserved by isomorphisms and $N_k(G) \setminus G$ is the unique maximal ideal of $N_k(G)$, we conclude that $\psi(N_k(G) \setminus G) = N_k(G) \setminus G$. Therefore, $\psi(G) = G$. □

Corollary 2.2. *Each automorphism of $N_k(G)$ is an extension of an automorphism of a group G.*

Next we shall describe the structure of the automorphism groups of extensions $N_k(G)$ of finite groups G of cardinality $|G| \le 3$.

Before describing the structure of extensions of finite groups, let us make some remarks concerning the structure of a semigroup S containing a group G with the identity element which also is a left identity of S. In this case S can be thought as a G-space endowed with the left action of the group G. So we can consider the orbit space $S/G = \{Gs : s \in S\}$ and the projection $\pi : S \to S/G$. If G lies in the center of the semigroup S (which means that the elements of G commute with all the elements of S), then the orbit space S/G admits a unique semigroup operation turning S/G into a semigroup and the orbit projection $\pi : S \to S/G$ into a semigroup homomorphism. If $s \in S$ is an idempotent, then the orbit Gs is a group isomorphic to a quotient group of G. A subsemigroup $T \subset S$ will be called a *transversal semigroup* if the restriction $\pi : T \to S/G$ is an isomorphism of the semigroups. If S admits a transversal semigroup T, then it is a homomorphic image of the product $G \times T$ under the semigroup homomorphism

$$h : G \times T \to S, \quad h : (g, t) \mapsto gt.$$

This helps to recover the algebraic structure of S from the structure of a transversal semigroup.

2.1. The semigroups $N_k(C_1)$

For the cyclic group C_1 the semigroups $N_k(C_1)$, $k \ge 2$, are isomorphic to C_1. Therefore, $\mathrm{Aut}\,(N_k(C_1)) \cong \mathrm{Aut}\,(C_1) \cong C_1$.

2.2. The semigroups $N_k(C_2)$

For the cyclic group C_2 the semigroups $N_k(C_2)$, $k \ge 2$, contain two principal ultrafilters and the k-linked upfamily $\{C_2\}$ which is the zero in $N_k(C_2)$. The semigroups $N_k(C_2)$ are isomorphic to the semigroup $\{-1, 0, 1\}$. Since the zero is preserved by automorphisms of semigroups, each automorphism of C_2 is extended to the unique automorphism of $N_k(C_2)$ by Corollary 2.2. Therefore, $\mathrm{Aut}\,(N_k(C_2)) \cong \mathrm{Aut}\,(C_2) \cong C_1$.

2.3. The semigroups $N_k(C_3)$

Consider the cyclic group $C_3 = \langle a \rangle = \{e, a, a^2 : a^3 = e\}$ generated by $a = e^{2\pi i/3} \in \mathbb{C}$.

Let us introduce the notations

$$
\begin{aligned}
&|^x_y = |^y_x = \langle \{x, y\} \rangle, \\
&\vee_x = \{F \subset C_3 : |F| \geq 2, x \in F\}, \\
&\triangle = \{F \subset C_3 : |F| \geq 2\}, \\
&\bigcirc = \{C_3\}.
\end{aligned}
$$

In these notations $N_k(C_3) = \{e, a, a^3, |^a_e, |^{a^2}_e, |^{a^2}_a, \vee_e, \vee_a, \vee_{a^2}, \bigcirc\}$ for $k \geq 3$ and $N_2(C_3) = N_k(C_3) \cup \{\triangle\}$.

$*$	a	a^2	e	$\vert^{a^2}_a$	$\vert^a_e$	$\vert^{a^2}_e$	$\vee_{a^2}$	$\vee_a$	$\vee_e$	$\bigcirc$	$\triangle$
a	a^2	e	a	$\vert^{a^2}_e$	$\vert^{a^2}_a$	$\vert^a_e$	$\vee_e$	$\vee_{a^2}$	$\vee_a$	$\bigcirc$	$\triangle$
a^2	e	a	a^2	$\vert^a_e$	$\vert^{a^2}_e$	$\vert^{a^2}_a$	$\vee_a$	$\vee_e$	$\vee_{a^2}$	$\bigcirc$	$\triangle$
e	a	a^2	e	$\vert^{a^2}_a$	$\vert^a_e$	$\vert^{a^2}_e$	$\vee_{a^2}$	$\vee_a$	$\vee_e$	$\bigcirc$	$\triangle$
$\vert^{a^2}_a$	$\vert^{a^2}_e$	$\vert^a_e$	$\vert^{a^2}_a$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\vert^a_e$	$\vert^{a^2}_e$	$\vert^{a^2}_a$	$\bigcirc$	$\triangle$
$\vert^a_e$	$\vert^{a^2}_a$	$\vert^{a^2}_e$	$\vert^a_e$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\vert^{a^2}_e$	$\vert^{a^2}_a$	$\vert^a_e$	$\bigcirc$	$\triangle$
$\vert^{a^2}_e$	$\vert^a_e$	$\vert^{a^2}_a$	$\vert^{a^2}_e$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\vert^{a^2}_a$	$\vert^a_e$	$\vert^{a^2}_e$	$\bigcirc$	$\triangle$
$\vee_{a^2}$	$\vee_e$	$\vee_a$	$\vee_{a^2}$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\vee_a$	$\vee_e$	$\vee_{a^2}$	$\bigcirc$	$\triangle$
$\vee_a$	$\vee_{a^2}$	$\vee_e$	$\vee_a$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\vee_e$	$\vee_{a^2}$	$\vee_a$	$\bigcirc$	$\triangle$
$\vee_e$	$\vee_a$	$\vee_{a^2}$	$\vee_e$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\vee_{a^2}$	$\vee_a$	$\vee_e$	$\bigcirc$	$\triangle$
$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\triangle$
$\triangle$	$\triangle$	$\triangle$	$\triangle$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\triangle$	$\triangle$	$\triangle$	$\bigcirc$	$\triangle$

Table 2: The Cayley table for the semigroups $N_k(C_3)$

Analyzing the Cayley Table 2 for the semigroups $N_k(C_3)$ we can establish the following properties.

The semigroup $N_2(C_3)$ contains 11 elements among them there are 4 idempotents: e, $\vee_e$, $\bigcirc$, $\triangle$. Two idempotents are right zeros. The orbit semigroup $N_2(C_3)/C_3$ contains 5 elements. The semigroup $N_2(C_3)$ contains a transversal semigroup $T(N_2(C_3)) = \{e, |^a_e, \vee_e, \bigcirc, \triangle\}$.

For $k \geq 3$ the semigroups $N_k(C_3)$ contain 10 elements among them there are 3 idempotents e, $\vee_e$, $\bigcirc$ which commute. The set $E(N_k(C_3))$ of idempotents of $N_k(C_3)$ is isomorphic to the semilattice $3 = \{0, 1, 2\}$ endowed with the operation of minimum. The orbit semigroups $N_k(C_3)/C_3$ contain 4 elements. The semigroups $N_k(C_3)$ contain transversal semigroups $T(N_k(C_3)) = \{e, |_e^a, \vee_e, \bigcirc\}$.

Therefore, $N_k(C_3) = \{x, x \cdot |_e^a, x \cdot \vee_e, \bigcirc \mid x \in C_3\}$ for $k \geq 3$ and $N_2(C_3) = N_k(C_3) \cup \{\triangle\}$.

We shall prove that the automorphism groups $\mathrm{Aut}\,(N_k(C_3))$ of the semigroups $N_k(C_3)$ are isomorphic to the holomorph $\mathrm{Hol}(C_3)$ of the group C_3.

We recall that the *holomorph* $\mathrm{Hol}(G)$ of a group G (see [23]) is the semi-direct product $G \rtimes \mathrm{Aut}\,(G) := (G \times \mathrm{Aut}\,(G), \star)$ of the group G with its automorphism group $\mathrm{Aut}\,(G)$, endowed with the group operation

$$(x, f) \star (y, g) = (x \cdot f(y), f \circ g).$$

It is known[1] that for the cyclic group C_3 its holomorph $\mathrm{Hol}(C_3)$ is isomorphic to the symmetric group S_3.

Proposition 2.3. *For each $k \in \mathbb{N} \setminus \{1\}$, the automorphism group $\mathrm{Aut}\,(N_k(C_3))$ is isomorphic to the holomorph $\mathrm{Hol}(C_3)$ of the cyclic group C_3 and hence is isomorphic to the symmetric group S_3.*

Proof. Let $\psi : N_k(C_3) \to N_k(C_3)$ be an automorphism. Then the restriction of ψ to C_3 is an automorphism of C_3 by Proposition 2.1, and hence $\psi(e) = e$.

Since the semigroup $N_2(C_3)$ contains two right zeros and the set of right zeros is preserved by automorphisms of semigroups, $\psi(\{\triangle, \bigcirc\}) = \{\triangle, \bigcirc\}$ for any automorphism $\psi : N_2(C_3) \to N_2(C_3)$. Assume that $\psi(\triangle) = \bigcirc$ and $\psi(\bigcirc) = \triangle$, then $\psi(\triangle \cdot |_e^a) = \psi(\bigcirc) = \triangle$ but $\psi(\triangle) \cdot \psi(|_e^a) = \bigcirc \cdot \psi(|_e^a) \in \bigcirc \cdot (N_2(C_3) \setminus \{\triangle, \bigcirc\}) = \{\bigcirc\}$. So we arrive to a contradiction with $\psi \in \mathrm{Aut}\,(N_2(C_3))$. Therefore, $\psi(\triangle) = \triangle$ and $\psi(\bigcirc) = \bigcirc$ for each automorphism $\psi : N_2(C_3) \to N_2(C_3)$.

The k-linked upfamily $\bigcirc$ is the zero of semigroups $N_k(C_3)$ for $k \geq 3$. Since the zero is preserved by automorphisms of semigroups, $\psi(\bigcirc) = \bigcirc$ for any automorphism $\psi : N_k(C_3) \to N_k(C_3)$, $k \geq 3$.

Taking into account that $\psi(E(N_k(C_3))) = E(N_k(C_3))$, we conclude that $\psi(\vee_e) = \vee_e$ for each $\psi \in \mathrm{Aut}\,(N_k(C_3))$, $k \geq 2$. Therefore, $\psi(x \cdot \vee_e) = \psi(x) \cdot \psi(\vee_e) = \psi(x) \cdot \vee_e$ for any $x \in C_3$.

Let $\psi(|_e^a) = c \cdot |_e^a$ for some $c \in C_3$. Then $\psi(x \cdot |_e^a) = \psi(x) \cdot c \cdot |_e^a$ for any $x \in C_3$. It follows that an element c can be chosen from C_3 in

[1] https://groupprops.subwiki.org/wiki/Holomorph_of_a_group

one of three ways and hence analyzing the Cayley Table 2 one can check that each automorphism of C_3 can be extended to an automorphism of $N_k(C_3)$ exactly in three different ways.

For any pair $(c, f) \in C_3 \times \mathrm{Aut}\,(C_3)$ consider the automorphism $\psi_{(c,f)}$ of $N_k(C_3)$ defined by

$$\psi_{c,f}(x) = f(x), \quad \psi_{c,f}(x \cdot \vee_e) = f(x) \cdot \vee_e, \quad \psi_{c,f}(x \cdot |_e^a) = f(x) \cdot c \cdot |_e^a$$

$$\text{for } \ x \in C_3, \quad \psi_{c,f}(\bigcirc) = \bigcirc,$$

and $\psi_{c,f}(\triangle) = \triangle$ for the semigroup $N_2(C_3)$.

It follows that each automorphism of $N_k(C_3)$ is of the form $\psi_{c,f}$ for some $(c, f) \in C_3 \times \mathrm{Aut}\,(C_3)$.

Observe that for any $(b, f), (c, g) \in C_3 \times \mathrm{Aut}\,(C_3)$ and $x \in C_3$ we get:

$$\begin{aligned}
&\psi_{b,f} \circ \psi_{c,g}(x) = \psi_{b,f}(g(x)) = f \circ g(x),\\
&\psi_{b,f} \circ \psi_{c,g}(x \cdot \vee_e) = \psi_{b,f}(g(x) \cdot \vee_e) = f \circ g(x) \cdot \vee_e,\\
&\psi_{b,f} \circ \psi_{c,g}(x \cdot |_e^a) = \psi_{b,f}(g(x) \cdot c \cdot |_e^a) = f \circ g(x) \cdot f(c) \cdot b \cdot |_e^a,\\
&\psi_{b,f} \circ \psi_{c,g}(\bigcirc) = \psi_{b,f}(\bigcirc) = \bigcirc,\\
&\psi_{b,f} \circ \psi_{c,g}(\triangle) = \psi_{b,f}(\triangle) = \triangle.
\end{aligned}$$

Consequently, $\psi_{b,f} \circ \psi_{c,g} = \psi_{b \cdot f(c), f \circ g}$ and hence for each $k \in \mathbb{N} \setminus \{1\}$ the group $\mathrm{Aut}\,(N_k(C_3))$ is isomorphic to the holomorph $\mathrm{Hol}(C_3) = C_3 \rtimes \mathrm{Aut}\,(C_3) \cong C_3 \rtimes C_2$ of the group C_3, which is known to be isomorphic to the symmetric group S_3. □

3. The automorphism groups of the extensions of finite monogenic semigroups

A semigroup $\langle a \rangle = \{a^n\}_{n \in \mathbb{N}}$ generated by a single element a is called *monogenic* or *cyclic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$. A finite monogenic semigroup $S = \langle a \rangle$ also has the simple structure, see [21]. There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{r+m-1}\}$ and $r + m - 1 = |S|$;
- $a^{r+m} = a^r$;
- $\{a^r, a^{r+1}, \dots, a^{r+m-1}\}$ is a cyclic and the maximal subgroup of S with the identity element $e = a^n$ and generator a^{n+1}, where $n \in m \cdot \mathbb{N} \cap \{r, \dots, r + m - 1\}$.

From now on we denote by $\mathrm{M}_{r,m}$ a finite monogenic semigroup of index r and period m, and the maximal subgroup of $\mathrm{M}_{r,m}$ is denoted by C_m. Note that $|\mathrm{M}_{r,m}| = r + m - 1$.

A homomorphism $\varphi : S \to I$ from a semigroup S into an ideal $I \subset S$ is called a *homomorphic retraction* if $\varphi(a) = a$ for any element $a \in I$.

Let e be the identity element of the maximal subgroup C_m of a monogenic semigroup $\mathrm{M}_{r,m}$. The following lemma was proved in [15, Lemma 1.3].

Lemma 3.1. *The map $\varphi : \mathrm{M}_{r,m} \to C_m$, $\varphi(x) = ex$, is a homomorphic retraction and $\varphi(x)y = xy$ for any $x \in \mathrm{M}_{r,m}$ and $y \in C_m$.*

Proposition 3.2. *Let $\mathrm{M}_{r,m}$ be a monogenic semigroup of index $r \geq 3$. If $\psi : N_k(\mathrm{M}_{r,m}) \to N_k(\mathrm{M}_{r,m})$ is an automorphism, then $\psi(s) = s$ for any $s \in \mathrm{M}_{r,m}$.*

Proof. Let $\mathrm{M}_{r,m} = \langle a \rangle = \{a, \dots, a^r, \dots, a^{r+m-1}\}$ and assume that $\psi(a) = \mathcal{A} \in N_k(\mathrm{M}_{r,m}) \setminus \{a\}$. Since ψ is an automorphism of $N_k(\mathrm{M}_{r,m})$, $\psi(a * N_k(\mathrm{M}_{r,m})) = \psi(a) * \psi(N_k(\mathrm{M}_{r,m})) = \psi(a) * N_k(\mathrm{M}_{r,m})$. Hence the semigroups $a * N_k(\mathrm{M}_{r,m})$ and $\mathcal{A} * N_k(\mathrm{M}_{r,m})$ are isomorphic. It is easy to see that $\mathcal{A} * N_k(\mathrm{M}_{r,m}) \subset a * N_k(\mathrm{M}_{r,m})$. Taking into account that in the extensions $N_k(\mathrm{M}_{r,m})$ of a monogenic semigroup of index $r \geq 3$ the equality $\mathcal{L} * \mathcal{M} = a^2$ implies $\mathcal{L} = \mathcal{M} = a$, we conclude that $a^2 \in a * N_k(\mathrm{M}_{r,m}) \setminus \mathcal{A} * N_k(\mathrm{M}_{r,m})$, and hence $|\mathcal{A} * N_k(\mathrm{M}_{r,m})| < |a * N_k(\mathrm{M}_{r,m})|$. This contradiction proves that $\psi(a) = a$, and therefore, $\psi(a^i) = (\psi(a))^i = a^i$ for any $i \in \{2, \dots, r + m - 1\}$. □

Proposition 3.3. *If $\psi : N_k(\mathrm{M}_{1,m}) \to N_k(\mathrm{M}_{1,m})$ is an automorphism, then $\psi(\mathrm{M}_{1,m}) = \mathrm{M}_{1,m}$.*

Proof. Since a monogenic semigroup $\mathrm{M}_{1,m}$ is isomorphic to the cyclic group C_m, we conclude that $\psi(\mathrm{M}_{1,m}) = \mathrm{M}_{1,m}$ according to Proposition 2.1. □

The following theorem shows that there are exist automorphisms of the semigroups $N_k(S)$ that are not extensions of automorphisms of a semigroup S.

Consider the monogenic semigroup $\mathrm{M}_{2,m} = \langle a \rangle = \{a, \dots, a^{m+1} \mid a^{m+2} = a^2\}$ and let

$$\mathbb{X} = \{\mathcal{M} \in N_k(\mathrm{M}_{2,m}) \mid \{a, a^{m+1}\} \in \mathcal{M}\} \setminus \{a^{m+1}\}.$$

Theorem 3.4. *A homomorphism $\psi : \mathrm{M}_{2,m} \to N_k(\mathrm{M}_{2,m})$, $\psi(a) = \mathcal{A}$, can be extended to an automorphism $\psi : N_k(\mathrm{M}_{2,m}) \to N_k(\mathrm{M}_{2,m})$ if and only if $\mathcal{A} \in \mathbb{X}$. The automorphism group $\mathrm{Aut}\,(N_k(\mathrm{M}_{2,m}))$ contains as a subgroup the symmetric group $S_{\mathbb{X}}$.*

Proof. Let $\mathcal{A} \in \mathbb{X}$. Since for $m \geq 2$, a^m is the identity of the cyclic group $C_m = \{a^2, \ldots, a^{m+1}\}$ and $ay \in C_m$, then $ay = \varphi(ay) = a^m(ay) = a^{m+1}y$ for any $y \in \mathrm{M}_{2,m}$ according to Lemma 3.1. The monogenic semigroup $\mathrm{M}_{2,1} = \{a, a^2\}$ is a null semigroup with the zero a^2 and hence $ay = a^2 = a^2y$ for any $y \in \mathrm{M}_{2,1}$. This implies that $a*\mathcal{M} = \mathcal{A}*\mathcal{M}$ and $\mathcal{M}*a = \mathcal{M}*\mathcal{A}$ for any $\mathcal{M} \in N_k(\mathrm{M}_{2,m})$, $m \in \mathbb{N}$. Indeed, $aM = \{a, a^{m+1}\}M \in \mathcal{A} * \mathcal{M}$ and $Ma = M\{a, a^{m+1}\} \in \mathcal{M} * \mathcal{A}$ for any $M \in \mathcal{M}$, and hence $a * \mathcal{M} \subset \mathcal{A} * \mathcal{M}$ and $\mathcal{M} * a \subset \mathcal{M} * \mathcal{A}$. Since $\mathcal{A} \ni \{a, a^{m+1}\}$ is k-linked, $a \in A$ or $a^{m+1} \in A$ for any $A \in \mathcal{A}$. Taking into account that $\bigcup_{s\in A} sM_s \supset aM_a \in a * \mathcal{M}$ or $\bigcup_{s\in A} sM_s \supset a^{m+1}M_{a^{m+1}} = aM_{a^{m+1}} \in a * \mathcal{M}$ for any basic set $\bigcup_{s\in A} sM_s \in \mathcal{A} * \mathcal{M}$, we conclude that $\mathcal{A} * \mathcal{M} \subset a * \mathcal{M}$. Let $\bigcup_{s\in M} sA_s \in \mathcal{M} * \mathcal{A}$. Since $\bigcup_{s\in M} sA_s \supset \bigcup_{s\in M}(s\{a\} = s\{a^{m+1}\}) = Ma$, $\mathcal{M} * \mathcal{A} \subset \mathcal{M} * a$. Therefore, $\mathcal{M} * \mathcal{A} = \mathcal{M} * a$ and $\mathcal{A} * \mathcal{M} = a * \mathcal{M}$.

Note that $\{a^2\} = \{a, a^{m+1}\}\{a, a^{m+1}\} \in \mathcal{A}*\mathcal{A}$. Then the linkedness of $\mathcal{A}*\mathcal{A}$ implies that $\mathcal{A}*\mathcal{A} = a^2$. By the same arguments $a*\mathcal{A} = \mathcal{A}*a = a^2$. Therefore, $\psi(a^i) = \mathcal{A}^i = a^i$ for any $i \geq 2$.

Let us put $\psi(\mathcal{A}) = a$ and $\psi(\mathcal{M}) = \mathcal{M}$ for any $\mathcal{M} \in N_k(\mathrm{M}_{2,m})\setminus\{a, \mathcal{A}\}$. Then above proved equalities imply that $\mathcal{M} * \mathcal{L} = \psi(\mathcal{M}) * \psi(\mathcal{L})$ for any $\mathcal{L}, \mathcal{M} \in N_k(\mathrm{M}_{2,m})$. Since $\mathcal{M} * \mathcal{L} \in N_k(\mathrm{M}_{2,m}) \setminus \{a, \mathcal{A}\}$ for any $\mathcal{L}, \mathcal{M} \in N_k(\mathrm{M}_{2,m})$, we have $\psi(\mathcal{M} * \mathcal{L}) = \mathcal{M} * \mathcal{L} = \psi(\mathcal{M}) * \psi(\mathcal{L})$, and hence ψ is an automorphism of $\mathrm{M}_{2,m}$.

Let $\mathcal{A} \notin \mathbb{X}$. If $\psi(a) = a^i$ for some $i \in \{2, \ldots, m+1\}$, then $\psi(\mathrm{M}_{2,m}) = \mathrm{M}_{2,m} \setminus \{a\}$, and thus ψ is not one-to-one. Therefore, $\mathcal{A} \notin \mathbb{X} \cup \mathrm{M}_{2,m}$, and thus $\mathrm{M}_{2,m} \subset a*N_k(\mathrm{M}_{2,m})\setminus\mathcal{A}*N_k(\mathrm{M}_{2,m})$. Consequently, $|a*N_k(\mathrm{M}_{2,m})| > |\mathcal{A} * N_k(\mathrm{M}_{2,m})|$, and hence ψ can not be a bijection.

Let us prove that the automorphism group $\mathrm{Aut}\,(N_k(\mathrm{M}_{2,m}))$ contains as a subgroup the symmetric group $S_{\mathbb{X}}$. Let us extend any bijection ψ of a set $\mathbb{X}$ to $N_k(\mathrm{M}_{2,m})$ putting $\psi(\mathcal{L}) = \mathcal{L}$ for any $\mathcal{L} \in N_k(\mathrm{M}_{2,m}) \setminus \mathbb{X}$. As we have shown above $\mathcal{M}*\mathcal{A} = \mathcal{M}*a = \mathcal{M}*\mathcal{B}$ and $\mathcal{A}*\mathcal{M} = a*\mathcal{M} = \mathcal{B}*\mathcal{M}$ for any $\mathcal{A}, \mathcal{B} \in \mathbb{X}$, $\mathcal{M} \in N_k(\mathrm{M}_{2,m})$. Also $\mathcal{A}*\mathcal{B} \ni \{a, a^{m+1}\}\{a, a^{m+1}\} = \{a^2\}$, and the linkedness of $\mathcal{A} * \mathcal{B}$ implies that $\mathcal{A} * \mathcal{B} = a^2$ for any $\mathcal{A}, \mathcal{B} \in \mathbb{X}$. Taking into account that $\mathcal{M} * \mathcal{L} \in N_k(\mathrm{M}_{2,m}) \setminus \{a, \mathcal{A}\}$ for any $\mathcal{L}, \mathcal{M} \in N_k(\mathrm{M}_{2,m})$, we conclude that $\psi(\mathcal{M} * \mathcal{L}) = \mathcal{M} * \mathcal{L}$. Therefore, $\psi(\mathcal{M} * \mathcal{L}) = \mathcal{M} * \mathcal{L} = \psi(\mathcal{M}) * \psi(\mathcal{L})$ for any $\mathcal{L}, \mathcal{M} \in N_k(\mathrm{M}_{2,m})$, and hence ψ is an automorphism of $N_k(\mathrm{M}_{2,m})$. □

Now we shall describe the structure of the automorphism groups of semigroups of k-linked upfamilies on monogenic semigroups $\mathrm{M}_{r,m}$ of order $|\mathrm{M}_{r,m}| \leq 3$.

It is well-known that $\mathrm{Aut}\,(\mathrm{M}_{r,m}) \cong C_1$ for $r \geq 2$ and $\mathrm{Aut}\,(\mathrm{M}_{1,m}) \cong \mathrm{Aut}\,(C_m) \cong C_{\varphi(m)}$, where $\varphi(m)$ is the value of Euler's function for $m \in \mathbb{N}$.

3.1. The semigroups $N_k(\mathrm{M}_{1,1})$, $N_k(\mathrm{M}_{1,2})$ and $N_k(\mathrm{M}_{2,1})$

For the trivial monogenic semigroup $M_{1,1}$ the semigroups $N_k(\mathrm{M}_{1,1})$ are trivial as well. Therefore, $\mathrm{Aut}\,(N_k(\mathrm{M}_{1,1})) \cong \mathrm{Aut}\,(\mathrm{M}_{1,1}) \cong \mathrm{Aut}\,(C_1) \cong C_1$.

For a semigroup $\mathrm{M}_{r,m} = \langle a\rangle$ with $m+r = 3$ the semigroups $N_k(\mathrm{M}_{r,m})$ contain the two principal ultrafilters a, a^2 and the k-linked upfamily $\{M_{r,m}\}$.

Taking into account that $\mathrm{M}_{1,2}$ is isomorphic to C_2, we conclude that $\mathrm{Aut}\,(N_k(\mathrm{M}_{1,2})) \cong \mathrm{Aut}\,(N_k(C_2)) \cong C_1$.

Consider the semigroup $N_k(\mathrm{M}_{2,1})$. The proof of Theorem 3.4 implies that $\psi(a^2) = a^2$ for any $\psi \in \mathrm{Aut}\,(N_k(\mathrm{M}_{2,1}))$. Then except for the identity automorphism the group $\mathrm{Aut}\,(N_k(\mathrm{M}_{2,1}))$ contains the automorphism ψ with $\psi(a) = \{M_{2,1}\}$, $\psi(\{M_{2,1}\}) = a$ according to Theorem 3.4. Consequently, $\mathrm{Aut}\,(N_k(\mathrm{M}_{2,1})) \cong C_2$.

3.2. The semigroups $N_k(\mathrm{M}_{1,3})$

The semigroup $\mathrm{M}_{1,3}$ is isomorphic to the cyclic group C_3. Therefore,

$$\mathrm{Aut}\,(N_k(\mathrm{M}_{1,3})) \cong \mathrm{Aut}\,(N_k(C_3)) \cong S_3.$$

3.3. The semigroups $N_k(\mathrm{M}_{2,2})$

Consider the semigroup $\mathrm{M}_{2,2} = \{a, a^2, a^3 \mid a^4 = a^2\}$. The semigroup $N_2(\mathrm{M}_{2,2})$ contains 11 elements while the semigroups $N_k(\mathrm{M}_{2,2})$ for $k \geq 3$ have 10 elements.

In the Cayley Table 3 for $N_k(\mathrm{M}_{2,2})$ we denote by a^i the principal ultrafilter generated by $\{a^i\}$ and introduce the notations

$$|^x_y = |^y_x = \langle\{a^x, a^y\}\rangle, \quad \vee_x = \{F \subset \mathrm{M}_{2,2} : |F| \geq 2, a^x \in F\}, \ \bigcirc = \{\mathrm{M}_{2,2}\}.$$

In these notations

$$N_k(\mathrm{M}_{2,2}) = \{a, a^2, a^3, |^2_1, |^3_1, |^3_2, \vee_1, \vee_2, \vee_3, \bigcirc\}$$

for $k \geq 3$ and $N_2(\mathrm{M}_{2,2}) = N_k(\mathrm{M}_{2,2}) \cup \{\triangle\}$.

Let $\psi \in \mathrm{Aut}\,(N_k(\mathrm{M}_{2,2}))$. Then $\psi(a^i) = a^i$ for $i \in \{2,3\}$ according to proof of Theorem 3.4. Since $|^3_2$ is the unique idempotent in $N_k(\mathrm{M}_{2,2}) \setminus \mathrm{M}_{2,2}$, we conclude that $\psi(|^3_2) = |^3_2$.

Consider the semigroups $N_k(\mathrm{M}_{2,2}) = \{a, a^2, a^3, |^2_1, |^3_1, |^3_2, \vee_1, \vee_2, \vee_3, \bigcirc\}$ for $k \geq 3$. Let $\mathbb{X} = \{a, |^3_1, \vee_1, \vee_3\}$, $\mathbb{Y} = \{|^2_1, \vee_2, \bigcirc\}$ and $x \in \mathbb{X}$, $y \in \mathbb{Y}$. Assume that $\psi(y) = x \in \mathbb{X}$. Then $\psi(y * y) = \psi(|^3_2) = |^3_2$ and $\psi(y) * \psi(y) = x * x = a^2$ which contradicts that ψ is an automorphism. Therefore, $\psi(x) \in \mathbb{X}$ and $\psi(y) \in \mathbb{Y}$ for any $x \in \mathbb{X}$, $y \in \mathbb{Y}$.

$*$	a	a^2	a^3	$\|_1^2$	$\|_1^3$	$\|_2^3$	$\vee_1$	$\vee_2$	$\vee_3$	$\bigcirc$	$\triangle$
a	a^2	a^3	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2
a^2	a^3	a^2	a^3	$\|_2^3$	a^3	$\|_2^3$	a^3	$\|_2^3$	a^3	$\|_2^3$	a^3
a^3	a^2	a^3	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2
$\|_1^2$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$
$\|_1^3$	a^2	a^3	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2
$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$
$\vee_1$	a^2	a^3	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2
$\vee_2$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$
$\vee_3$	a^2	a^3	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2
$\bigcirc$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$	$\|_2^3$
$\triangle$	a^2	a^3	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2	$\|_2^3$	a^2

Table 3: The Cayley table for the semigroups $N_k(\mathrm{M}_{2,2})$

Analyzing the Cayley Table 3 for the semigroups $N_k(\mathrm{M}_{2,2})$ one can establish that the semigroups $N_k(\mathrm{M}_{2,2})$ are commutative and $x_1 * s = x_2 * s$, $y_1 * s = y_2 * s$, $x_1 * y_1 = x_2 * y_2$, $x_1 * x_2 = a^2$, $y_1 * y_2 = |_2^3$ for any $x_1, x_2 \in \mathbb{X}$, $y_1, y_2 \in \mathbb{Y}$, $s \in \mathrm{M}_{2,2}$. Consequently, each permutation of $\mathbb{X}$ and each permutation of $\mathbb{Y}$ define the automorphism of the semigroup $N_k(\mathrm{M}_{2,2})$. Therefore, $\mathrm{Aut}\,(N_k(\mathrm{M}_{2,2})) \cong S_{\mathbb{X}} \times S_{\mathbb{Y}} \cong S_4 \times S_3$ for $k \geq 3$.

Consider the semigroup $N_2(\mathrm{M}_{2,2}) = N_k(\mathrm{M}_{2,2}) \cup \{\triangle\}$. Let $\mathbb{X}' = \mathbb{X} \cup \{\triangle\}$. By the same arguments $\mathrm{Aut}\,(N_2(\mathrm{M}_{2,2})) \cong S_{\mathbb{X}'} \times S_{\mathbb{Y}} \cong S_5 \times S_3$.

3.4. The semigroups $N_k(\mathrm{M}_{3,1})$

Consider the semigroup $\mathrm{M}_{3,1} = \{a, a^2, a^3 \mid a^4 = a^3\}$. In Cayley Table 4 for the semigroups $N_k(\mathrm{M}_{3,1})$ we use the similar notations as for the semigroups $N_k(\mathrm{M}_{2,2})$.

Let $\psi \in \mathrm{Aut}\,(N_k(\mathrm{M}_{3,1}))$. Then $\psi(a^i) = a^i$ for $i \in \{1,2,3\}$ according to Proposition 3.2.

Consider the semigroups $N_k(\mathrm{M}_{3,1}) = \{a, a^2, a^3, |_1^2, |_1^3, |_2^3, \vee_1, \vee_2, \vee_3, \bigcirc\}$ for $k \geq 3$. Let $\mathbb{X} = \{|_1^2, |_1^3, \vee_1, \bigcirc\}$ and $\mathbb{Y} = \{\vee_2, \vee_3\}$.

We claim that $\psi(|_2^3) = |_2^3$. Assume that $\psi(|_2^3) = x \in \mathbb{X}$. Then $\psi(a * |_2^3) = \psi(a^3) = a^3$ and $\psi(a) * \psi(|_2^3) = a * x = |_2^3$, and we arrive to a contradiction with $\psi \in \mathrm{Aut}\,(N_k(\mathrm{M}_{3,1}))$. In the same way assumming that $\psi(|_2^3) = y \in \mathbb{Y}$ we have $\psi(a * |_1^2) = \psi(|_2^3) = y$ but $\psi(a) * \psi(|_1^2) =$

$*$	a	a^2	a^3	$\vert^2_1$	$\vert^3_1$	$\vert^3_2$	$\vee_1$	$\vee_2$	$\vee_3$	$\bigcirc$	$\triangle$
a	a^2	a^3	a^3	$\vert^3_2$	$\vert^3_2$	a^3	$\vert^3_2$	a^3	a^3	$\vert^3_2$	a^3
a^2	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\vert^2_1$	$\vert^3_2$	a^3	a^3	$\vert^3_2$	$\vert^3_2$	3	$\vert^3_2$	a^3	a^3	$\vert^3_2$	a^3
$\vert^3_1$	$\vert^3_2$	a^3	a^3	$\vert^3_2$	$\vert^3_2$	3	$\vert^3_2$	a^3	a^3	$\vert^3_2$	a^3
$\vert^3_2$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\vee_1$	$\vert^3_2$	a^3	a^3	$\vert^3_2$	$\vert^3_2$	3	$\vert^3_2$	a^3	a^3	$\vert^3_2$	a^3
$\vee_2$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\vee_3$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\bigcirc$	$\vert^3_2$	a^3	a^3	$\vert^3_2$	$\vert^3_2$	3	$\vert^3_2$	a^3	a^3	$\vert^3_2$	a^3
$\triangle$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3

Table 4: The Cayley table for the semigroups $N_k(\mathrm{M}_{3,1})$

$a * \psi(|^2_1) \notin \mathbb{Y}$. These contradictions show that $\psi(|^3_2) = |^3_2$.

Then in the same way as for the semigroups $N_k(\mathrm{M}_{2,2})$ we establish that $\mathrm{Aut}\,(N_k(\mathrm{M}_{3,1})) \cong S_{\mathbb{X}} \times S_{\mathbb{Y}} \cong S_4 \times S_2$ for $k \geq 3$.

Consider the semigroup $N_2(\mathrm{M}_{3,1}) = N_k(\mathrm{M}_{3,1}) \cup \{\triangle\}$. Let $\mathbb{Y}' = \mathbb{Y} \cup \{\triangle\}$. By the same arguments $\mathrm{Aut}\,(N_2(\mathrm{M}_{3,1})) \cong S_{\mathbb{X}} \times S_{\mathbb{Y}'} \cong S_4 \times S_3$.

4. The automorphism groups of the semigroups $N_k(O_X)$, $N_k(LO_X)$, $N_k(RO_X)$, $N_k(AO_X)$ and $N_k((O_X)^{+0})$

A semigroup S is said to be a *left (right) zero semigroup* if $ab = a$ $(ab = b)$ for any $a, b \in S$. By LO_X and RO_X we denote the left zero semigroup and the right zero semigroup on a set X, respectively. If X is finite of cardinality $|X| = n$, then instead of LO_X and RO_X we use LO_n and RO_n, respectively.

Proposition 4.1. *If S is a left (right) zero semigroup, then for each $k \in \mathbb{N} \setminus \{1\}$ the extension $N_k(S)$ is a left (right) zero semigroup as well.*

Proof. Let S be a left zero semigroup. Then

$$\mathcal{L}*\mathcal{M} = \Big\langle \bigcup_{a\in L} a*M_a : L \in \mathcal{L},\ \ \{M_a\}_{a\in L} \subset \mathcal{M} \Big\rangle = \Big\langle \bigcup_{a\in L} \{a\} : L \in \mathcal{L} \Big\rangle = \mathcal{L}$$

for any $\mathcal{L}, \mathcal{M} \in N_k(S)$. Therefore, $N_k(S)$ is a left zero semigroup as well.

For a right zero semigroup the proof is similar. □

Proposition 4.2. *If X is a left zero semigroup or a right zero semigroup, then for each $k \in \mathbb{N} \setminus \{1\}$,* $\mathrm{Aut}\,(N_k(X))$ *is isomorphic to the symmetric group* $S_{N_k(X)}$.

Proof. In Proposition 4.1 it was shown that the extensions $N_k(S)$ of a left (right) zero semigroup S are left (right) zero semigroups as well. Each permutation on a left (right) zero semigroup is an automorphism. Indeed, $\psi(x * y) = \psi(x) = \psi(x) * \psi(y)$ and $\psi(x * y) = \psi(y) = \psi(x) * \psi(y)$ for any elements x and y of the left zero semigroup and the right zero semigroup, respectively. Therefore, $\mathrm{Aut}\,(N_k(X)) \cong S_{N_k(X)}$. □

Using the results of Table 1 and Proposition 4.2 in Table 5 we present the automorphism groups of the semigroups $N_k(LO_n)$ and $N_k(RO_n)$ for $k \in \{2,3,4\}$ and $n \leq 5$.

n	$\mathrm{Aut}\,(N_2(LO_n))$	$\mathrm{Aut}\,(N_3(LO_n))$	$\mathrm{Aut}\,(N_4(LO_n))$
1	C_1	C_1	C_1
2	S_3	S_3	S_3
3	S_{11}	S_{10}	S_{10}
4	S_{80}	S_{54}	S_{53}
5	S_{2645}	S_{762}	S_{687}

Table 5: The automorphism groups of the semigroups $N_k(LO_n)$ for $k \in \{2,3,4\}$ and $n \leq 5$

A semigroup S is called a *null semigroup* if there exists an element $z \in S$ such that $xy = z$ for any $x, y \in S$. In this case the element z is the zero of S. All null semigroups on the same set are isomorphic. By O_X we denote a null semigroup on a set X. If X is finite of cardinality $|X| = n$, then instead of O_X we use O_n.

Proposition 4.3. *If S is a null semigroup, then for each $k \in \mathbb{N} \setminus \{1\}$ the extension $N_k(S)$ is a null semigroup as well.*

Proof. Let S be a null semigroup. Then there exists $z \in S$ such that $xy = z$ for all $x, y \in S$. Therefore,

$$\mathcal{L} * \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle$$
$$= \Big\langle \bigcup_{a \in L} \{z\} : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle = z$$

for any $\mathcal{L}, \mathcal{M} \in N_k(S)$. Consequently, $N_k(S)$ is a null semigroup with the zero $z \in S \subset N_k(S)$. □

Since $N_k(O_1) \cong O_1$, the automorphism groups of the semigroups $N_k(O_1)$ are trivial. In the following proposition we describe the automorphism group of the semigroups $N_k(O_X)$ on a set X of cardinality $|X| \geq 2$.

Proposition 4.4. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$. For each $k \in \mathbb{N} \setminus \{1\}$ the automorphism group of the semigroup $N_k(O_X)$ is isomorphic to the symmetric group $S_{N_k(O_X)\setminus\{z\}}$.*

Proof. In Proposition 4.3 it was proved that the semigroups $N_k(O_X)$ are null semigroups with the zero z. Taking into account that z is the zero of the semigroups $N_k(O_X)$, we conclude that $\psi(z) = z$ for any $\psi \in \mathrm{Aut}\,(N_k(O_X))$. Each permutation on the set $N_k(O_X) \setminus \{z\}$ defines an automorphism. Indeed, $\psi(x * y) = z = \psi(x) * \psi(y)$ for any elements $x, y \in N_k(O_X)$. Therefore, $\mathrm{Aut}\,(N_k(O_X)) \cong S_{N_k(O_X)\setminus\{z\}}$. □

Using the results of Table 1 and Proposition 4.4 in Table 6 we present the automorphism groups of the semigroups $N_k(O_n)$ for $k \in \{2, 3, 4\}$ and $n \leq 5$.

n	$\mathrm{Aut}\,(N_2(O_n))$	$\mathrm{Aut}\,(N_3(O_n))$	$\mathrm{Aut}\,(N_4(O_n))$
1	C_1	C_1	C_1
2	C_2	C_2	C_2
3	S_{10}	S_9	S_9
4	S_{79}	S_{53}	S_{52}
5	S_{2644}	S_{761}	S_{686}

Table 6: The automorphism groups of the semigroups $N_k(O_n)$ for $k \in \{2, 3, 4\}$ and $n \leq 5$

A semigroup S is said to be an *almost null semigroup* if there exist the distinct elements $a, z \in S$ such that $aa = a$ and $xy = z$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$. In this case the element z is the zero of S and a is the unique idempotent in $S \setminus \{z\}$. All almost null semigroups on the same set are isomorphic. By AO_X we denote an almost null semigroup on a set X. If X is finite of cardinality $|X| = n$, then instead of AO_X we use AO_n.

It easy to check that the automorphism groups of the semigroups $N_k(AO_2)$ are trivial. In the following theorem we describe the automorphism groups of the semigroups $N_k(AO_X)$ on a set X of cardinality $|X| \geq 3$.

Theorem 4.5. *Let z be the zero of the almost null semigroup AO_X on a set X of cardinality $|X| \geq 3$, $\mathbb{A} = \{\mathcal{L} \in N_k(OA_X) \mid X \setminus \{a\} \in \mathcal{L}\}$, $\mathbb{B} = N_k(OA_X) \setminus \mathbb{A} = \{\mathcal{L} \in N_k(OA_X) \mid a \in L$ for any $L \in \mathcal{L}\}$, where a is the idempotent in $OA_X \setminus \{z\}$. For each $k \in \mathbb{N} \setminus \{1\}$ the automorphism group of the semigroup $N_k(AO_X)$ is isomorphic to the group $S_{\mathbb{A}\setminus\{z\}} \times S_{\mathbb{B}\setminus\{a,|^{z}_{a}\}}$.*

Proof. Let $\mathcal{A} \in \mathbb{A}$, $\mathcal{L} \in N_k(OA_X)$. Then $\mathcal{A}*\mathcal{L} \ni (X \setminus \{a\}) \cdot L = \{z\} = L \cdot (X\setminus\{a\}) \in \mathcal{L}*\mathcal{A}$ for any $L \in \mathcal{L}$, and hence the linkedness of $\mathcal{A}*\mathcal{L}$ and $\mathcal{L}*\mathcal{A}$ implies that $\mathcal{A} * \mathcal{L} = \mathcal{L} * \mathcal{A} = z$ for any $\mathcal{A} \in \mathbb{A}$, $\mathcal{L} \in N_k(OA_X)$. Consider any $(\mathcal{B}_1, \mathcal{B}_2) \in \mathbb{B} \times \mathbb{B} \setminus \{(a, a)\}$. Taking into account that $a \in B_1 \cap B_2$ and $|B_1| \geq 2$ or $|B_2| \geq 2$ for any $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$, we conclude that $\mathcal{B}_1 * \mathcal{B}_2 = \Big\langle \bigcup_{x\in B_1} x * B_x : B_1 \in \mathcal{B}_1,\ \ \{B_x\}_{x\in B_1} \subset \mathcal{B}_2 \Big\rangle = |^{z}_{a}$.

Consequently, the semigroups $N_k(AO_X)$ contain three idempotents: a, z and $|^{z}_{a}$. Taking into account that z is the zero of $N_k(OA_X)$ and the set of idempotents of a semigroup is preserved by automorphisms, we conclude that $\psi(z) = z$, and hence $\psi(\{a, |^{z}_{a}\}) = \{a, |^{z}_{a}\}$ for any $\psi \in \mathrm{Aut}\,(N_k(AO_X))$. Since $\psi(|^{z}_{0}) = \psi(z * |^{z}_{0}) = \psi(z) * \psi(^{z}_{0}) \in \{z * |^{z}_{0}, |^{z}_{0} * z\} = \{|^{z}_{0}\}$, we conclude $\psi(|^{z}_{0}) = |^{z}_{0}$, and hence $\psi(z) = z$ for any $\psi \in \mathrm{Aut}\,(N_k((O_X)^{+0}))$.

Let us show that $\psi(\mathbb{A}\setminus\{z\}) = \mathbb{A}\setminus\{z\}$ and $\psi(\mathbb{B}\setminus\{a, |^{z}_{a}\}) = \mathbb{B}\setminus\{a, |^{z}_{a}\}$ for any $\psi \in \mathrm{Aut}\,(N_k(AO_X))$. Assume that $\psi(\mathcal{B}) = \mathcal{A}$ for some $\mathcal{A} \in \mathbb{A} \setminus \{z\}$, $\mathcal{B} \in \mathbb{B} \setminus \{a, |^{z}_{a}\}$. Then $\psi(\mathcal{B}*\mathcal{B}) = \psi(|^{z}_{a}) = |^{z}_{a}$ but $\psi(\mathcal{B}) * \psi(\mathcal{B}) = \mathcal{A}*\mathcal{A} = z$. This contradiction show that $\psi(\mathcal{A}) \in \mathbb{A} \setminus \{z\}$, $\psi(\mathcal{B}) \in \mathbb{B} \setminus \{a, |^{z}_{a}\}$ for any $\mathcal{A} \in \mathbb{A} \setminus \{z\}$, $\mathcal{B} \in \mathbb{B} \setminus \{a, |^{z}_{a}\}$ and $\psi \in \mathrm{Aut}\,(N_k(AO_X))$.

Each permutation on the set $\mathbb{A} \setminus \{z\}$ and each permutation on the set $\mathbb{B} \setminus \{a, |^{z}_{a}\}$ define the automorphism $\psi : N_k(AO_X) \to N_k(AO_X)$. Indeed, $\psi(a * a) = \psi(a) = a = a * a = \psi(a) * \psi(a)$, $\psi(\mathcal{A} * \mathcal{L}) = \psi(z) = z = \psi(\mathcal{A}) * \psi(\mathcal{L})$, $\psi(\mathcal{L} * \mathcal{A}) = \psi(z) = z = \psi(\mathcal{L}) * \psi(\mathcal{A})$ for any $\mathcal{A} \in \mathbb{A}$, $\mathcal{L} \in N_k(OA_X)$, and $\psi(\mathcal{B}_1 * \mathcal{B}_2) = \psi(|^{z}_{a}) = |^{z}_{a} = \psi(\mathcal{B}_1) * \psi(\mathcal{B}_2)$ for any any $(\mathcal{B}_1, \mathcal{B}_2) \in \mathbb{B} \times \mathbb{B} \setminus \{(a, a)\}$.

Therefore, $\mathrm{Aut}\,(N_k(OA_X)) \cong S_{\mathbb{A}\setminus\{z\}} \times S_{\mathbb{B}\setminus\{a,|^{z}_{a}\}}$. □

Let us note that for a subsemigroup T of a semigroup S the map $i : N_k(T) \to N_k(S)$, $i : \mathcal{A} \to \{L \subset S \mid L \supset A \in \mathcal{A}\}$, is injective homomophism, and thus we can identify the semigroup $N_k(T)$ with the subsemigroup $i(N_k(T)) \subset N_k(S)$. Therefore, the set $\mathbb{A}$ from Theorem 4.5 can be identified with the subsemigroup $N_k(X \setminus \{a\})$ of the semigroup

$N_k(X)$. Consequently, for finite almost null semigroups we have the following corollary.

Corollary 4.6. *For each $k \geq 2$ and $n \geq 3$ the automorphism group of the semigroup $N_k(OA_n)$ is isomorphic to the group $S_{|N_k(n-1)|-1} \times S_{|N_k(n)|-|N_k(n-1)|-3}$.*

Using the results of Table 1 and Corollary 4.6 in Table 7 we present the automorphism groups of the semigroups $N_k(AO_n)$ for $k \in \{2,3,4\}$ and $n \in \{2,3,4,5\}$.

n	$\mathrm{Aut}\,(N_2(AO_n))$	$\mathrm{Aut}\,(N_3(AO_n))$	$\mathrm{Aut}\,(N_4(AO_n))$
2	C_1	C_1	C_1
3	$C_2 \times S_5$	$C_2 \times S_4$	$C_2 \times S_4$
4	$S_{10} \times S_{66}$	$S_9 \times S_{41}$	$S_9 \times S_{40}$
5	$S_{79} \times S_{2562}$	$S_{53} \times S_{705}$	$S_{52} \times S_{631}$

Table 7: The automorphism groups of the semigroups $N_k(AO_n)$ for $k \in \{2,3,4\}$ and $n \in \{2,3,4,5\}$

Let S be a semigroup and $0 \notin S$. The binary operation defined on S can be extended to $S \cup \{0\}$ putting $0s = s0 = 0$ for all $s \in S \cup \{0\}$. The notation S^{+0} denotes a semigroup $S \cup \{0\}$ obtained from S by adjoining the extra zero 0 (regardless of whether S has or has not the zero).

Theorem 4.7. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$, $\mathbb{A} = \{\mathcal{L} \in N_k((O_X)^{+0}) \mid X \in \mathcal{L}\}$, $\mathbb{B} = N_k((O_X)^{+0}) \setminus \mathbb{A} = \{\mathcal{L} \in N_k((O_X)^{+0}) \mid 0 \in L$ for any $L \in \mathcal{L}\}$, where 0 is the extra zero adjointed to O_X. For each $k \geq 2$ the automorphism group of the semigroup $N_k((O_X)^{+0})$ is isomorphic to the group $S_{\mathbb{A}\setminus\{z\}} \times S_{\mathbb{B}\setminus\{0,|^z_0\}}$.*

Proof. It is easy to see that 0 is the zero of $N_k((O_X)^{+0})$.

Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}$. Then $\mathcal{A}_1 * \mathcal{A}_2 \ni X \cdot X = \{z\}$. The linkedness of $\mathcal{A}_1 * \mathcal{A}_2$ implies that $\mathcal{A}_1 * \mathcal{A}_2 = z$ for any $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}$. Consider any $\mathcal{B} \in \mathbb{B} \setminus \{0\}$, $\mathcal{L} \in N_k((O_X)^{+0}) \setminus \{0\}$. Taking into account that $0 \in B$ and $|B| \geq 2$, $|L| \geq 2$ for any $B \in \mathcal{B}$, $L \in \mathcal{L}$, we conclude that $\mathcal{B} * \mathcal{L} = \left\langle \bigcup_{x\in B} x * L_x : B \in \mathcal{B},\ \ \{L_x\}_{x\in B} \subset \mathcal{L} \right\rangle = |^z_0$ and $\mathcal{L} * \mathcal{B} = \left\langle \bigcup_{x\in L} x * B_x : L \in \mathcal{L},\ \ \{B_x\}_{x\in L} \subset \mathcal{B} \right\rangle = |^z_0$.

Consequently, the semigroup $N_k((O_X)^{+0})$ contains three idempotents: 0, z and $|^z_0$. Taking into account that 0 is the zero of $N_k((O_X)^{+0})$

and the set of idempotents of a semigroup is preserved by automorphisms, we conclude that $\psi(0)=0$, and hence $\psi(\{z,|^z_0\})=\{z,|^z_0\}$ for any $\psi\in \mathrm{Aut}\,(N_k((O_X)^{+0}))$. Since $\psi(|^z_0)=\psi(z*|^z_0)=\psi(z)*\psi(|^z_0)\in\{z*|^z_0,|^z_0*z\}=\{|^z_0\}$, we conclude $\psi(|^z_0)=|^z_0$, and hence $\psi(z)=z$ for any $\psi\in \mathrm{Aut}\,(N_k((O_X)^{+0}))$.

Let us show that $\psi(\mathbb{A}\setminus\{z\})=\mathbb{A}\setminus\{z\}$ and $\psi(\mathbb{B}\setminus\{0,|^z_0\})=\mathbb{B}\setminus\{0,|^z_0\}$ for any $\psi\in \mathrm{Aut}\,(N_k((O_X)^{+0}))$. Assume that $\psi(\mathcal{A})=\mathcal{B}$ for some $\mathcal{A}\in\mathbb{A}\setminus\{z\}$, $\mathcal{B}\in\mathbb{B}\setminus\{0,|^z_0\}$. Then $\psi(\mathcal{A}*\mathcal{A})=\psi(z)=z$ but $\psi(\mathcal{A})*\psi(\mathcal{A})=\mathcal{B}*\mathcal{B}=|^z_0$. This contradiction show that $\psi(\mathcal{A})\in\mathbb{A}\setminus\{z\}$, $\psi(\mathcal{B})\in\mathbb{B}\setminus\{0,|^z_0\}$ for any $\mathcal{A}\in\mathbb{A}\setminus\{z\}$, $\mathcal{B}\in\mathbb{B}\setminus\{0,|^z_0\}$ and $\psi\in \mathrm{Aut}\,(N_k((O_X)^{+0}))$.

Each permutation on the set $\mathbb{A}\setminus\{z\}$ and each permutation on the set $\mathbb{B}\setminus\{0,|^z_0\}$ define the automorphism $\psi: N_k((O_X)^{+0})\to N_k((O_X)^{+0})$. Indeed, $\psi(0*\mathcal{L})=\psi(0)=0=0*\psi(\mathcal{L})=\psi(0)*\psi(\mathcal{L})$, $\psi(\mathcal{L}*0)=\psi(0)=0=\psi(\mathcal{L})*0=\psi(\mathcal{L})*\psi(0)$, $\psi(\mathcal{A}_1*\mathcal{A}_2)=\psi(z)=z=\psi(\mathcal{A}_1)*\psi(\mathcal{A}_2)$ for any $\mathcal{L}\in N_k((O_X)^{+0})$, $\mathcal{A}_1,\mathcal{A}_2\in\mathbb{A}$, and $\psi(\mathcal{B}*\mathcal{L})=\psi(|^z_0)=|^z_0=\psi(\mathcal{B})*\psi(\mathcal{L})$, $\psi(\mathcal{L}*\mathcal{B})=\psi(|^z_0)=|^z_0=\psi(\mathcal{L})*\psi(\mathcal{B})$ for any $\mathcal{B}\in\mathbb{B}\setminus\{0\}$, $\mathcal{L}\in N_k((O_X)^{+0})\setminus\{0\}$.

Therefore, $\mathrm{Aut}\,(N_k((O_X)^{+0}))\cong S_{\mathbb{A}\setminus\{z\}}\times S_{\mathbb{B}\setminus\{0,|^z_0\}}$. □

The set $\mathbb{A}$ from Theorem 4.7 can be identified with the subsemigroup $N_k(O_X)$ of the semigroup $N_k((O_X)^{+0})$. Consequently, for finite null semigroups O_n we have the following corollary.

Corollary 4.8. *For each $k\geq 2$ and $n\geq 2$ the automorphism group of the semigroup $N_k((O_n)^{+0})$ is isomorphic to the group $S_{|N_k(O_n)|-1}\times S_{|N_k((O_n)^{+0})|-|N_k(O_n)|-3}$.*

Using the results of Table 1 and Corollary 4.8 in Table 8 we present the automorphism groups of the semigroups $N_k((O_n)^{+0}))$ for $k\in\{2,3,4\}$ and $n\leq 4$.

n	$\mathrm{Aut}\,(N_2((O_n)^{+0}))$	$\mathrm{Aut}\,(N_3((O_n)^{+0}))$	$\mathrm{Aut}\,(N_4((O_n)^{+0}))$
1	C_1	C_1	C_1
2	$C_2\times S_5$	$C_2\times S_4$	$C_2\times S_4$
3	$S_{10}\times S_{66}$	$S_9\times S_{41}$	$S_9\times S_{40}$
4	$S_{79}\times S_{2562}$	$S_{53}\times S_{705}$	$S_{52}\times S_{631}$

Table 8: The automorphism groups of the semigroups $N_k((O_n)^{+0}))$ for $k\in\{2,3,4\}$ and $n\leq 4$

5. Acknowledgment

The author would like to express his sincere thanks to the anonymous referees for a very careful reading of the paper and for all their insightful comments and valuable suggestions, which improve considerably the presentation of this paper.

References

[1] T. Banakh, V. Gavrylkiv, *Algebra in superextension of groups, II: cancelativity and centers* // Algebra Discrete Math., **4** (2008), 1–14.

[2] T. Banakh, V. Gavrylkiv, *Algebra in superextension of groups: minimal left ideals* // Mat. Stud., **31** (2009), No. 2, 142–148.

[3] T. Banakh, V. Gavrylkiv, *Extending binary operations to functor-spaces* // Carpathian Math. Publ., **1** (2009), No. 2, 113–126.

[4] T. Banakh, V. Gavrylkiv, *Algebra in the superextensions of twinic groups* // Dissertationes Math., **473** (2010), 3–74.

[5] T. Banakh, V. Gavrylkiv, *Algebra in superextensions of semilattices* // Algebra Discrete Math., **13** (2012), No. 1, 26–42.

[6] T. Banakh, V. Gavrylkiv, *Algebra in superextensions of inverse semigroups* // Algebra Discrete Math., **13** (2012), No. 2, 147–168.

[7] T. Banakh, V. Gavrylkiv, *Characterizing semigroups with commutative superextensions* // Algebra Discrete Math., **17** (2014), No. 2, 161–192.

[8] T. Banakh, V. Gavrylkiv, *On structure of the semigroups of k-linked upfamilies on groups* // Asian-European J. Math., **10** (2017), No. 4, https://doi.org/10.1142/S1793557117500838.

[9] T. Banakh, V. Gavrylkiv, *Automorphism groups of superextensions of groups* // Mat. Stud., **48** (2017), No. 2.

[10] T. Banakh, V. Gavrylkiv, O. Nykyforchyn, *Algebra in superextensions of groups, I: zeros and commutativity* // Algebra Discrete Math., **3** (2008), 1–29.

[11] R. Dedekind, *Über Zerlegungen von Zahlen durch ihre grüssten gemeinsammen Teiler*, in *Gesammelte Werke*, Vol. **1**, Springer, 1897, 103–148.

[12] V. Gavrylkiv, *The spaces of inclusion hyperspaces over noncompact spaces* // Mat. Stud., **28** (2007), No. 1, 92–110.

[13] V. Gavrylkiv, *Right-topological semigroup operations on inclusion hyperspaces* // Mat. Stud., **29** (2008), No. 1, 18–34.

[14] V. Gavrylkiv, *On representation of semigroups of inclusion hyperspaces* // Carpathian Math. Publ., **2** (2010), No. 1, 24–34.

[15] V. Gavrylkiv, *Superextensions of cyclic semigroups* // Carpathian Math. Publ., **5** (2013), No. 1, 36–43.

[16] V. Gavrylkiv, *Semigroups of centered upfamilies on finite monogenic semigroups* // J. Algebra, Number Theory: Adv. App., **16** (2016), No. 2, 71–84.

[17] V. Gavrylkiv, *Semigroups of centered upfamilies on groups* // Lobachevskii J. Math., **38** (2017), No. 3, 420–428.

[18] V. Gavrylkiv, *Superextensions of three-element semigroups* // Carpathian Math. Publ., **9** (2017), No. 1, 28–36.

[19] V. Gavrylkiv, *On the automorphism group of the superextension of a semigroup* // Mat. Stud., **48** (2017), No. 1, 3–13.

[20] N. Hindman, D. Strauss, *Algebra in the Stone-Čech compactification*, de Gruyter, Berlin, New York, 1998.

[21] J.M. Howie, *Fundamentals of semigroup theory*, The Clarendon Press, Oxford University Press, New York, 1995.

[22] J. van Mill, *Supercompactness and Wallman spaces*, Mathematical Centre Tracts, Vol. **85**, Amsterdam, 1977.

[23] D. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1996.

[24] A. Teleiko, M. Zarichnyi, *Categorical Topology of Compact Hausdofff Spaces*, VNTL, Vol. **5**, Lviv, 1999.

[25] A. Verbeek, *Superextensions of topological spaces*, Mathematical Centre Tracts, Vol. **41**, Amsterdam, 1972.

Contact information

Volodymyr M. Gavrylkiv

Vasyl Stefanyk Precarpathian National University,
Ivano–Frankivsk, Ukraine
E-Mail: vgavrylkiv@gmail.com