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Bernstein-Walsh type inequalities in unbounded regions with piecewise asymptotically conformal curve in the weighted Lebesgue space

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Abstract. In this work, we obtain pointwise Bernstein–Walsh-type estimation for algebraic polynomials in the unbounded regions with piecewise asymptotically conformal boundary, having exterior and interior zero angles, in the weighted Lebesgue space.

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1. Introduction and Definitions

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded region, with $0 \in G$ and the boundary $L := \partial G$ be a Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = ext\, L$. Denote by $w = \Phi(z)$ the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z \to \infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$.

For $t \geq 1$, $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set:

$$L_t := \{z : \ |\Phi(z)| = t\} \ (L_1 \equiv L), \ G_t := int\, L_t, \ \Omega_t := ext\, L_t;$$

$$d(z, M) = dist(z, M) := \inf \{|z - \zeta| : \ \zeta \in M\}.$$

Let $\{\xi_j\}_{j=1}^m$ be a fixed system of distinct points on curve L located in the positive direction. For some fixed R_0, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z) := \prod_{j=1}^{m} |z - \xi_j|^{\gamma_j}, \tag{1.1}$$

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where $\gamma_j > -1$ for all $j = 1, 2, \dots, m$.

For a rectifiable Jordan curve L and for $0 < p \leq \infty$, let $\mathcal{L}_p(h, L)$ denote the weighted Lebesgue space of complex-valued functions on L. Specifically, $f \in \mathcal{L}_p(h, L)$ if f is measurable and the following quasinorm (a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$) is finite:

$$\begin{aligned} \|f\|_p : &= \|f\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left| f(z) \right|^p |dz| \right)^{1/p}, \ 0 < p < \infty; \quad (1.2) \\ \|f\|_\infty : &= \|f\|_{\mathcal{L}_\infty(1,L)} := ess \sup_{z \in L} |f(z)|, \ p = \infty. \end{aligned}$$

We denote by $\wp_n$, $n = 1, 2, \dots$, the set of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Bernstein–Walsh Lemma [28] says that for any $P_n \in \wp_n$ and $R > 1$, the following

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})} \quad (1.3)$$

holds. In [28] also was given some similar estimates for various norms on the right-hand side of (1.3). Analogously estimation with respect to the quasinorm (1.2) for $p > 0$ was obtained in [19] for $h(z) \equiv 1$ (i.e., $\gamma_j = 0$ for all $j = 1, 2, \dots, m$). Moreover, in [6, Lemma 2.4] this estimate has been generalized for $h(z) \neq 1$, defined as in (1.1) and was proved the following:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\{0; \gamma_j : j \leq \ m\}. \quad (1.4)$$

For any $p > 0$ we also introduce:

$$\|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \left| P_n(z) \right|^p d\sigma_z \right)^{1/p} < \infty, 0 < p < \infty, \quad (1.5)$$

where σ_z is the two-dimensional Lebesgue measure.

The Bernstein–Walsh type estimates for the quasinorm (1.5), for the regions with quasiconformal boundary (see, below) and weight function $h(z)$, defined in (1.1) with $\gamma_j > -2$, for all $p > 0$ as follows

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 \ {R^*}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \quad (1.6)$$

was found in [3] (see, also [2]), where $R^* := 1 + c_2(R-1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent from n and R. In [4, Theorem 1.1], analogously estimate was studied for $A_p(1, G)$-norm, $p > 0$, for arbitrary

Jordan region and was obtained: for any $P_n \in \wp_n$, $R_1 = 1 + \frac{1}{n}$ and arbitrary R, $R > R_1$, the following estimate

$$\|P_n\|_{A_p(G_R)} \le c \cdot R^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})},$$

is true, where $c = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$. Note that, the c is the sharp constant.

In [27] was given a new version of the Bernstein–Walsh Lemma: *For quasiconformal and rectifiable curve L there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \le c \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{1.7}$$

holds for every $P_n \in \wp_n$.

In this work, continue investigated pointwise estimations in unbounded region Ω of the type

$$|P_n(z)| \le c_2 \eta_n(G, h, p, d(z,L)) \|P_n\|_p \; |\Phi(z)|^{n+1}, \tag{1.8}$$

where $c_2 = c_2(G,p) > 0$ is a constant independent of n, h and P_n, and $\eta_n(G, h, p, d(z,L)) \to \infty$, $n \to \infty$, depending on the properties of the G and h.

Analogous results of (1.8)-type for some norms and for different unbounded regions were obtained by S. N. Bernstein [28], N. A. Lebedev, P. M. Tamrazov, V. K. Dzjadyk, I. A. Shevchuk (see, for example, [14]), N. Stylianopoulos [27] and others. Recent results (1.8) for some regions and the weight function $h(z)$ defined as in (1.1) with $\gamma_j > -1$ were also obtained: in [6] for $p > 1$ and in [22] for $p > 0$, for regions bounded by piecewise Dini-smooth boundary with interior and exterior zero angles; in [7] for $p > 0$ and for regions bounded by piecewise quasiconformal boundary with interior and exterior zero angles; in [5] for $p > 1$ and for regions bounded by piecewise smooth boundary with exterior zero angles (without interior zero angles); in [8] for $p > 0$ and for regions bounded by piecewise quasismooth boundary with interior and exterior zero angles and in others.

Now, we begin to give some definitions and notations.

Let z_1, z_2 be an arbitrary points on l and $l(z_1, z_2)$ denotes the subarc of l of shorter diameter with endpoints z_1 and z_2. The curve l is a quasicircle if and only if the quantity

$$\sup_{z_1, z_2 \in l;\; z \in l(z_1, z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \tag{1.9}$$

is bounded. Following to Lesley [21], the curve l to be said "*$c-$quasiconformal*", if the quantity (1.9) bounded by positive constant c, independent from points z_1, z_2 and z. At the literature it is possible to find various functional definitions of the quasiconformal curves (see, for example, Def. 3.1, [23, p. 286–294], [20, p. 105], [9, p. 81], [24, p. 107]).

The Jordan curve l is called asymptotically conformal [13, 24], *if*

$$\sup_{z_1,z_2\in l;\ z\in l(z_1,z_2)} \frac{|z_1-z|+|z-z_2|}{|z_1-z_2|}\to 1, \qquad |z_1-z_2|\to 0. \tag{1.10}$$

We will denote this class as AC, and will write $G\in AC$, if $L:=\partial G\in AC$.

The asymptotically conformal curves occupies a special place in the problems of the geometric theory of functions of a complex variable. These curves in various problems has been studied by J. M. Anderson, J. Becker and F. D. Lesley [10], E. M. Dyn'kin [15], Ch. Pommerenke, S. E. Warschawski [25], V. Ya. Gutlyanskii, V. I. Ryazanov [16–18] and others. According to the geometric criteria of quasiconformality of the curves ([9, p. 81], [24, p. 107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [12], [20, p. 104]). The same is true for asymptotically conformal curves.

We say that $L\in\widetilde{AC}$, if $L\in AC$ and L is rectifiable. A Jordan arc ℓ is called asymptotically conformal arc, when ℓ is a part of some asymptotically conformal curve.

Now, we define a new class of regions bounded by piecewise asymptotically conformal curves having interior and exterior cusps at the connecting points of boundary arcs.

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general and on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

For any $k\geq 0$ and $m>k$, notation $i=\overline{k,m}$ means $i=k, k+1, ..., m$. For any $i=1,2,...,$ $k=0,1,2$ and $\varepsilon_1>0$, we denote by $f_i:[0,\varepsilon_1]\to\mathbb{R}^+$ and $g_i:[0,\varepsilon_1]\to\mathbb{R}^+$ twice differentiable functions such that

$$f_i(0)=g_i(0)=0,\ f_i^{(k)}(x)>0,\ \ g_i^{(k)}(x)>0,\ 0<x\leq\varepsilon_1. \tag{1.11}$$

Definition 1.1. *We say that a Jordan region $G\in AC\,(f_i,\ g_i)$, for some $f_i=f_i(x)$, $i=\overline{1,m_1}$ and $g_i=g_i(x)$, $i=\overline{m_1+1,m}$, defined as in* (1.11), *if $L=\partial G=\bigcup\limits_{i=0}^{m} L_i$ is the union of the finite number of asymptotically conformal arcs L_i, connecting at the points $\{z_i\}_{i=0}^{m}\in L$ and such that L*

is a locally asymptotically conformal arc at the $z_0 \in L \setminus \{z_i\}_{i=1}^{m}$ *and, in the* (x, y) *local co-ordinate system with its origin at the* z_i*,* $1 \le i \le m$*, the following conditions are satisfied:*

a) for every $z_i \in L,\ i = \overline{1, m_1},\ m_1 \le m,$

$$\left\{ z = x + iy :\ |z| \le \varepsilon_1,\ \overset{i}{c}_{11} f_i(x) \le y \le \overset{i}{c}_{12} f_i(x),\ 0 \le x \le \varepsilon_1 \right\} \subset \overline{G},$$
$$\{ z = x + iy : |z| \le \varepsilon_1,\ |y| \ge \varepsilon_2 x,\ 0 \le x \le \varepsilon_1 \} \subset \overline{\Omega};$$

b) for every $z_i \in L,\ i = \overline{m_1 + 1, m},$

$$\{ z = x + iy : |z| < \varepsilon_3,\ \ c_{21}^{i} g_i(x) \le y \le c_{22}^{i} g_i(x),\ 0 \le x \le \varepsilon_3 \} \subset \overline{\Omega},$$
$$\{ z = x + iy : |z| < \varepsilon_3,\ \ |y| \ge \varepsilon_4 x,\ 0 \le x \le \varepsilon_3 \} \subset \overline{G},$$

for some constants $-\infty < c_{11}^{i} < c_{12}^{i} < \infty$*,* $-\infty < c_{21}^{i} < c_{22}^{i} < \infty$ *and* $\varepsilon_s > 0,\ s = \overline{1, 4}$*.*

Definition 1.2. *We say that a Jordan region* $G \in \widetilde{AC}(f_i, g_i)$*,* $f_i = f_i(x)$*,* $i = \overline{1, m_1}$*,* $g_i = g_i(x)$*,* $i = \overline{m_1 + 1, m}$*, if* $G \in AC(f_i, g_i)$ *and* $L := \partial G$ *is rectifiable.*

It is clear from Definitions 1.2 and 1.1, that each region $G \in \widetilde{AC}(f_i, g_i)$ may have m_1 interior and $m - m_1$ exterior zero angles (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^{m} \in L$. If a region G does not have interior zero angles ($m_1 = 0$) (exterior zero angles ($m_1 = m$)), then it is written as $G \in \widetilde{AC}(0, g_i)$ ($G \in \widetilde{AC}(f_i, 0)$). If a region G does not have such angles ($m = 0$), then we will assume that G is bounded by a asymptotically conformal curve and in this case we set $\widetilde{AC}(0, 0) \equiv \widetilde{AC}$.

Throughout this work, we will assume that the points $\{\xi_i\}_{i=1}^{m} \in L$ defined in (1.1) and the points $\{z_i\}_{i=1}^{m} \in L$ defined in Definition 1.2 and 1.1 coincide. Without loss of generality, we also will assume that the points $\{z_i\}_{i=0}^{m}$ are ordered in the positive direction on the curve L such that G has interior zero angles at the points $\{z_i\}_{i=1}^{m_1}$, if $m_1 \ge 1$ and exterior zero angles at the points $\{z_i\}_{i=m_1+1}^{m}$, if $m \ge m_1 + 1$.

2. Main Results

Now, we can state our new results. Our first result is related to the general case. Namely, let region G has $m_1 \ge 1$ interior zero angles at the points $\{z_i\}_{i=1}^{m_1}$ and $m - m_1$ exterior zero angles at the points $\{z_i\}_{i=m_1+1}^{m}$. In this case, we have the following estimate, i.e. with respect to each points $\{z_i\}_{i=1}^{m}$.

Theorem 2.1. *Let* $p>0$; $G\in\widetilde{AC}\left(f_i,g_i\right)$, *for some* $f_i(x)=c_ix^{1+\alpha_i}$, $\alpha_i\geq 0$, $i=\overline{1,m_1}$, *and* $g_i(x)=c_ix^{1+\beta_i}$, $\beta_i>0$, $i=\overline{m_1+1,m}$; $h(z)$ *defined as in* (1.1)*. Then, for any* $\gamma_i>-1$, $i=\overline{1,m}$, *and* $P_n\in\wp_n$, $n\in\mathbb{N}$, *there exists* $c_1=c_1(G,p,\varepsilon,\gamma_i,\beta_i)>0$ *such that:*

$$|P_n(z)|\leq c_1\frac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L_R)}\left(\sum_{i=1}^{m_1}B^i_{n,1}+\sum_{i=m_1+1}^{m}B^i_{n,2}\right)\|P_n\|_p\,,\ z\in\Omega_R, \tag{2.1}$$

where

$$B^i_{n,1}:=\begin{cases} n^{\frac{\gamma_i-1}{p}+\widetilde{\varepsilon}} & \gamma_i>\frac{2+\widetilde{\varepsilon}}{1+\widetilde{\varepsilon}},\\ (n\ln n)^{\frac{1}{p}}, & \gamma_i=\frac{2+\widetilde{\varepsilon}}{1+\widetilde{\varepsilon}},\\ n^{\frac{1}{p}}, & 0<\gamma_i<\frac{2+\widetilde{\varepsilon}}{1+\widetilde{\varepsilon}},\\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_i\leq 0;\end{cases}\qquad \widetilde{\varepsilon}:=\begin{cases}1, & \alpha_i\neq 0,\\ \varepsilon, & \alpha_i=0;\end{cases}\quad\text{and}$$

$$B^i_{n,2}:=\begin{cases} n^{\frac{\gamma_i-1}{p(1+\beta_i)}+\varepsilon}, & \gamma_i>2+\beta_i-\varepsilon,\\ (n\ln n)^{\frac{1}{p}}, & \gamma_i=2+\beta_i-\varepsilon,\\ n^{\frac{1}{p}}, & 0<\gamma_i<2+\beta_i-\varepsilon,\\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_i\leq 0.\end{cases} \tag{2.2}$$

Now, we assume that, $i=1,2;\ m_1=1,\ m=2$.

Theorem 2.2. *Let* $p>0$; $G\in\widetilde{AC}\left(f_1,g_2\right)$, *for some* $f_1(x)=c_1x^{1+\alpha_1}$, $\alpha_1\geq 0$, *and* $g_2(x)=c_2x^{1+\beta_2}$, $\beta_2>0$; $h(z)$ *defined as in* (1.1) *for* $m=2$. *Then, for any* $\gamma_1>-1$, $i=1,2$, *and* $P_n\in\wp_n$, $n\in\mathbb{N}$, *there exists* $c_2=c_2(G,p,\varepsilon,\gamma_i,\beta_2)>0$ *such that:*

$$|P_n(z)|\leq c_2\frac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L_R)}B_n\|P_n\|_p\,,\ z\in\Omega_R, \tag{2.3}$$

where

$$B_n:=\begin{cases} n^{\frac{2(\gamma_1-1)}{p}}, & \gamma_1>1+\frac{\gamma_2-1}{2(1+\beta_2)},\ \gamma_2>2+\beta_2-\varepsilon,\\ n^{\frac{\gamma_2-1}{p(1+\beta_2)}+\varepsilon}, & 0<\gamma_1\leq 1+\frac{\gamma_2-1}{2(1+\beta_2)},\ \gamma_2>2+\beta_2-\varepsilon,\\ n^{\frac{2(\gamma_1-1)}{p}}, & \gamma_1>\frac{3}{2},\ 0<\gamma_2<2+\beta_2-\varepsilon,\\ n^{\frac{1}{p}}, & 0<\gamma_1<\frac{3}{2},\ 0<\gamma_2<2+\beta_2-\varepsilon,\\ (n\ln n)^{\frac{1}{p}}, & \gamma_1=\frac{3}{2},\ \gamma_2=2+\beta_2-\varepsilon,\\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_1\leq 0,-1<\gamma_2\leq 0.\end{cases} \tag{2.4}$$

In particular, if $\alpha_1 = 0$, i.e. G has only exterior zero angle at the z_2, then we have:

Theorem 2.3. *Let $p > 0$; $G \in \widetilde{AC}(0, g_2)$, for some $g_2(x) = c_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ defined as in (1.1) for $m = 2$. Then, for any $\gamma_1 > -1$, $i = 1, 2$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_3 = c_3(G, p, \varepsilon, \gamma_i, \beta_2) > 0$ such that:*

$$|P_n(z)| \leq c_3 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_R)} B_n \|P_n\|_p, \quad z \in \Omega_R, \tag{2.5}$$

where

$$B_n := \begin{cases} n^{\frac{\gamma_1 - 1}{p} + \varepsilon}, & \gamma_1 > 1 + \frac{\gamma_2 - 1}{1+\beta_2}, \ \gamma_2 \geq 2 + \beta_2, \\ n^{\frac{\gamma_2 - 1}{p(1+\beta_2)} + \varepsilon} & 2 \leq \gamma_1 \leq 1 + \frac{\gamma_2 - 1}{1+\beta_2}, \ \gamma_2 \geq 2 + \beta_2, \\ n^{\frac{\gamma_1 - 1}{p} + \varepsilon}, & \gamma_1 \geq 2, \ 0 < \gamma_2 < 2 + \beta_2, \\ n^{\frac{1}{p}}, & 0 < \gamma_1 < 2, \ 0 < \gamma_2 < 2 + \beta_2, \\ (n \ln n)^{\frac{1}{p}}, & \gamma_1 = 2 - \varepsilon, \gamma_2 = 2 + \beta_2 - \varepsilon, \\ n^{\frac{\varepsilon}{p}}, & -1 < \gamma_1 \leq 0, -1 < \gamma_2 \leq 0. \end{cases} \tag{2.6}$$

Remark 2.1. In Theorems 2.1–2.3, in the right hand sides of estimations (2.1), (2.3), (2.5) and their corollaries there exist value $d^{2/p}(z, L_R)$. We can replace $d^{2/p}(z, L_R)$ with $d(z, L_R)$, if we consider only the values $p > 1$ instead of $p > 0$.

The sharpness of the estimations (2.1)–(2.6) for some special cases can be discussed by comparing them with the following:

Remark 2.2. For any $n \in \mathbb{N}$ there exist polynomials $P_n^* \in \wp_n$, regions $G^* \subset \mathbb{C}$ and constant $c_4 = c_4(G) > 0$, such that

$$|P_n^*(z)| \geq c_4 |\Phi(z)|^{n+1} \|P_n^*\|_{\mathcal{L}_2(\partial G^*)}, \ \forall z \in F \Subset C\overline{G^*}. \tag{2.7}$$

3. Some auxiliary results

For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well known (see, for example, [9], [20, p. 97] and [26]):

Definition 3.1. *The Jordan arc (or curve) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).*

Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let defines

$$K_L := \inf\left\{K(f) : f \in F(L)\right\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

Lemma 3.1. [1] *Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent.

So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon_1 < 1$, $c > 1$, $0 < r_0 < 1$ are constants, depending on G and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Lemma 3.2. [21, p. 342] *Let L be an asymptotically conformal curve. Then, Φ and Ψ are Lipα for all $\alpha < 1$ in $\overline{\Omega}$ and $\overline{\Delta}$, correspondingly.*

Lemma 3.3. *Let L be an asymptotically conformal curve. Then,*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\varepsilon},$$

for all $w_1, w_2 \in \overline{\Delta}$ and $\forall \varepsilon > 0$.

This fact follows from Lemma 3.2. We also will use the estimation for the Ψ' (see, for example, [11, Th. 2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.1}$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1.1).

Lemma 3.4. [8], [19, $h(z) \equiv 1$] *Let L be a rectifiable Jordan curve; $h(z)$ defined as in (1.1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\widetilde{\gamma}}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \; p > 0, \tag{3.2}$$

is true, where $\widetilde{\gamma} := \max\left\{0; \gamma_i : \; i = \overline{1,m}\right\}$.

4. Proof of Theorems

4.1. Proof of Theorems 2.1–2.3

Proof. Suppose that $G \in \widetilde{AC}(f_i, g_i)$, for some $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, m_1}$, and $g_i(x) = c_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{m_1 + 1, m}$; $h(z)$ be defined as in (1.1). Let $\left\{\zeta_j^*\right\}$, $1 \leq j \leq m \leq n$, be zeros of $P_n(z)$ lying on Ω and let

$$B_m(z) := \prod_{j=1}^{m} \widetilde{B}_j(z) = \prod_{j=1}^{m} \frac{\Phi(z) - \Phi(\zeta_j^*)}{1 - \overline{\Phi(\zeta_j^*)}\Phi(z)}$$

denote a Blaschke function with respect to zeros $\left\{\zeta_j^*\right\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$. For any $p > 0$ and $z \in \Omega$, let us set:

$$G_n(z) := \left[\frac{P_n(z)}{B_m(z)\ \Phi^{n+1}(z)}\right]^{p/2}. \tag{4.1}$$

Cauchy integral representation for the unbounded region Ω gives:

$$G_n(z) = -\frac{1}{2\pi i} \int\limits_{L_R} G_n(\zeta) \frac{d\zeta}{\zeta - z}\ ,\ z \in \Omega_R. \tag{4.2}$$

Since $|B_m(\zeta)| = 1$, for $\zeta \in L$, then, for arbitrary ε, $0 < \varepsilon < \varepsilon_1$, there exists a circle $|w| = 1 + \frac{\varepsilon_1}{n}$, such that for any $j = \overline{1, m}$ the following is satisfied:

$$\left|\widetilde{B}_j(\Psi(w)\right| > 1 - \varepsilon.$$

Then, $|B_m(\zeta)| > (1-\varepsilon)^m \succeq 1$ for each $\varepsilon \leq n^{-1}$. On the other hand, $|\Phi(\zeta)| = R > 1$, for $\zeta \in L_R$. Therefore, for any $z \in \Omega_R$, we have:

$$\left|\left[\frac{P_n(z)}{B_m(z)\ \Phi^{n+1}(z)}\right]^{p/2}\right| \leq \frac{1}{2\pi} \int\limits_{L_R} \left|\frac{P_n(\zeta)}{B_m(\zeta)\ \Phi^{n+1}(\zeta)}\right|^{p/2} \frac{|d\zeta|}{|\zeta - z|} \tag{4.3}$$

$$\preceq \frac{1}{d(z, L_R)} \int\limits_{L_R} |P_n(\zeta)|^{p/2} |d\zeta| =: \frac{1}{d(z, L_R)} A_n.$$

To estimate the integral A_n, we introduce:

$$w_j := \Phi(z_j),\ \varphi_j := \arg w_j,\ L_R^j := L_R \cap \overline{\Omega}^j,\ j = \overline{1, m},$$

where $\Omega^j := \Psi(\Delta_j^{'})$;

$$\Delta_1^{'} \ :\ = \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_m + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},$$

$$\Delta_m^{'} \ :\ = \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{m-1} + \varphi_m}{2} \leq \theta < \frac{\varphi_m + \varphi_1}{2}\right\}$$

and, for $j=\overline{2,m-1}$

$$\Delta_j^{'} := \left\{ t = Re^{i\theta} : R > 1, \ \frac{\varphi_{j-1}+\varphi_j}{2} \ \leq \theta < \frac{\varphi_j+\varphi_{j+1}}{2} \right\}.$$

Then, we have:

$$A_n = \sum_{i=1}^{m} \int\limits_{L_R^i} \left| P_n\left(\zeta\right) \right|^{p/2} \left| d\zeta \right|. \tag{4.4}$$

Multiplying the numerator and denominator of the integrand by $h^{1/2}(\zeta)$, after applying the Hölder inequality, we obtain:

$$\begin{aligned} A_n \quad &\leq \quad \sum_{i=1}^{m} \left(\int\limits_{L_R^i} h(\zeta) \left| P_n(\zeta) \right|^p \left| d\zeta \right| \right)^{1/2} \times \left(\int\limits_{L_R^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{m} |\zeta - z_j|^{\gamma_j}} \right)^{1/2} \\ &= \quad : \sum_{i=1}^{m} \widetilde{J}_{n,1}^i \cdot \widetilde{J}_{n,2}^i. \end{aligned} \tag{4.5}$$

According to Lemma 3.4, for the $\widetilde{J}_{n,1}^i$ we get:

$$\widetilde{J}_{n,1}^i \preceq \left\| P_n \right\|_p^{p/2}, \ i = \overline{1,m}. \tag{4.6}$$

Then, from (4.5) and (4.6) we have:

$$A_n \preceq \left\| P_n \right\|_p^{p/2} \sum_{i=1}^{m} \widetilde{J}_{n,2}^i.$$

For the integral $J_{n,2}^i$ we obtain:

$$\left(\widetilde{J}_{n,2}^i \right)^2 := \int\limits_{L_R^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{m} |\zeta - z_j|^{\gamma_i}} \asymp \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}}, \ i = 1, 2, \tag{4.7}$$

since the points $\{z_j\}_{j=1}^m$ are distinct on L. Then, from (4.7), we have:

$$A_n \preceq \left\| P_n \right\|_p^{p/2} \sum_{i=1}^{2} \widetilde{J}_{n,2}^i, \tag{4.8}$$

where

$$\widetilde{J}_{n,2}^1 = \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} \ ; \ \widetilde{J}_{n,2}^2 = \int\limits_{L_R^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}}. \tag{4.9}$$

It remains to estimate these integrals for each $i=\overline{1,m}$. For simplicity of our next calculations, we assume that:

$$i=1,2;\ m_1=1,\ m=2;\quad z_1=-1,\ z_2=1;\ (-1,1)\subset G;\ R=1+\frac{\varepsilon_0}{n},$$

and let local co-ordinate axis in Definitions 1.1 and 1.2 is parallel to OX and OY in the OXY co-ordinate system; $L=L^+\cup L^-$, where $L^+ := \{z\in L:\operatorname{Im} z\geq 0\}$, $L^- := \{z\in L:\operatorname{Im} z<0\}$. Let $w^\pm := \left\{w=e^{i\theta}:\theta=\frac{\varphi_1\pm\varphi_2}{2}\right\}$, $z^\pm\in\Psi(w^\pm)$ and L^i an arcs, connecting the points $z^+,\ z_i,\ z^-\in\ L$; $L^{i,\pm}:=L^i\cap L^\pm,\ i=1,2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0=z^+$ $(z_0=z^-)$. Analogously to the previous notations, we introduce the following: $L_R=L_R^+\cup L_R^-$, where $L_R^+:=\{z\in L_R:\operatorname{Im}z\geq 0\}$, $L_R^-:=\{z\in L_R:\operatorname{Im}z<0\}$; Let $w_R^\pm:=\left\{w=Re^{i\theta}:\theta=\frac{\varphi_1\pm\varphi_2}{2}\right\}$, $z_R^\pm\in\Psi(w_R^\pm)$. We set: $z_{i,R}\in L_R$, such that $d_{i,R}=|z_i-z_{i,R}|$ and $\zeta^\pm\in L^\pm$, such that $d(z_{2,R},L^2\cap L^\pm):=d(z_{2,R},L^\pm)$; $z_i^\pm:=\left\{\zeta\in L^i:\ |\zeta-z_i|=c_id(z_i,L_R)\right\}$, $z_{i,R}^\pm:=\left\{\zeta\in L_R^i:\ |\zeta-z_{i,R}|=c_id(z_{i,R},L_R)\right\}$, $w_{i,R}^\pm=\Phi(z_{i,R}^\pm)$. Let $L_R^i,\ i=1,2$, denote arcs, connecting the points $z_R^+,\ z_{i,R},\ z_R^-\in\ L_R,\ L_R^{i,\pm}:=L_R^i\cap L_R^\pm$ and $l_{i,R}^\pm(z_{i,R}^\pm,z_R^\pm)$ denote arcs, connecting the points $z_{i,R}^\pm$ with $z_R^\pm$, respectively and $\left|\ l_{i,R}^\pm\right|:=mes\ l_{i,R}^\pm(z_{i,R}^\pm,z_R^\pm)$, $i=1,2$. We denote:

$$\begin{aligned}
S_{1,R}^{i,\pm}\quad &:\quad =\left\{\zeta\in L_R^{i,\pm}:\ |\zeta-z_i|<c_id_{i,R}\right\},\\
S_{2,R}^{i,\pm}\quad &:\quad =\left\{\zeta\in L_R^{i,\pm}:c_id_{i,R}\leq|\zeta-z_i|\leq\left|\ l_{i,R}^\pm\right|\right\},\ \mathcal{F}_{j,R}^{i,\pm}:=\Phi(S_{j,R}^{i,\pm});\\
S_1^{i,\pm}\quad &:\quad =\left\{\zeta\in L^{i,\pm}:\ |\zeta-z_i|<c_id_{i,R}\right\},\\
S_2^{i,\pm}\quad &:\quad =\left\{\zeta\in L^{i,\pm}:c_id_{i,R}\leq|\zeta-z_i|\leq\left|\ l_{i,R}^\pm\right|\right\},\\
&\mathcal{F}_j^{i,\pm}:=\Phi(S_j^{i,\pm}),\ i,j=1,2.
\end{aligned}$$

Taking into consideration above notations, replacing the variable $\tau=\Phi(\zeta)$, according to (3.1), we have:

$$\begin{aligned}
\widetilde{J}_{n,2}^i\ &\asymp\ \sum_{i,j=1}^{2}\ \int\limits_{\mathcal{F}_{j,R}^{i,+}\cup\mathcal{F}_{j,R}^{i,-}}\frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i}}\\
&\asymp\ \sum_{i,j=1}^{2}\ \int\limits_{\mathcal{F}_{j,R}^{i,+}\cup\mathcal{F}_{j,R}^{i,-}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,(|\tau|-1)}\\
&=\ :\sum_{i,j=1}^{2}\left[\widetilde{J}(\mathcal{F}_{j,R}^{i,+})+\widetilde{J}(\mathcal{F}_{j,R}^{i,-})\right]
\end{aligned}$$

and, from (4.8), we have:

$$A_n \preceq \|P_n\|_p^{p/2} \sum_{i=1}^{2} \widetilde{J}_{n,2}^{i} \tag{4.10}$$
$$= : \|P_n\|_p^{p/2} \sum_{i=1}^{2} \left[I_{n,1}^{i}(S_{1,R}^{i,+}) + I_{n,2}^{i}(S_{2,R}^{i,-}) \right]$$
$$= : \|P_n\|_p^{p/2} \sum_{i=1}^{2} \left[I_{n,1}^{i,+} + I_{n,2}^{i,-} \right], \ i = 1, 2,$$

where

$$I_{n,k}^{i,\pm} := I_{n,k}^{i}(S_{k,R}^{i,\pm}) := \int\limits_{\mathcal{F}_{k,R}^{i,\pm}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{\gamma_i}\,(|\tau| - 1)}; \ i, \ k = 1, 2. \tag{4.11}$$

According to (4.3) and (4.4), it is sufficient to estimate the integrals $I_{n,k}^{i,\pm}$ for each $i = 1, 2$ and $k = 1, 2$.

Given the possible values of γ_i $(-1 < \gamma_i \leq 0, \ \gamma_i > 0, \ i = 1, 2)$, we will consider the estimates for the $I_{n,k}^{i,\pm}$ separately.

1. Let $i = 1$.

1.1. For the integral $I_{n,1}^{1,+} + I_{n,1}^{1,-}$, we get:

$$I_{n,1}^{1,+} + I_{n,1}^{1,-} = \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}\,(|\tau| - 1)} \tag{4.12}$$

$$\preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1 - 1}} \preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1 - 1)(1 + \widetilde{\varepsilon})}}$$
$$\preceq \begin{cases} n^{(\gamma_1 - 1)(1 + \widetilde{\varepsilon})}, & (\gamma_1 - 1)(1 + \widetilde{\varepsilon}) > 1, \\ n \ln n, & (\gamma_1 - 1)(1 + \widetilde{\varepsilon}) = 1, \\ n, & (\gamma_1 - 1)(1 + \widetilde{\varepsilon}) < 1, \end{cases}$$

for $\gamma_1 > 0$ and

$$I_{n,1}^{1,+} + I_{n,1}^{1,-} = \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}\,(|\tau| - 1)} \tag{4.13}$$

$$\preceq n d_{1,R}^{(-\gamma_1)+1} \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} |d\tau| \preceq n \left(\frac{1}{n} \right)^{[(-\gamma_1)+1](1-\varepsilon)} \cdot mes \left(\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-} \right)$$
$$\preceq n^{[\gamma_1 - 1](1-\varepsilon)} \preceq 1,$$

for $-1<\gamma_1\leq 0$.

1.2. Analogously to the (4.12) and (4.13), for the integral $I_{n,2}^{1,+}+I_{n,2}^{1,-}$, we get:

$$I_{n,2}^{1,+}+I_{n,2}^{1,-}=\int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,(|\tau|-1)} \tag{4.14}$$

$$\preceq\ n\int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1-1}}\preceq n\int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}}\frac{|d\tau|}{|\tau-w_1|^{(\gamma_1-1)(1+\widetilde{\varepsilon})}}$$

$$\preceq\ \begin{cases} n^{(\gamma_1-1)(1+\widetilde{\varepsilon})}, & (\gamma_1-1)\,(1+\widetilde{\varepsilon})>1,\\ n\ln n, & (\gamma_1-1)\,(1+\widetilde{\varepsilon})=1,\\ n, & (\gamma_1-1)\,(1+\widetilde{\varepsilon})<1,\end{cases}$$

for $\gamma_1>0$, and

$$I_{n,2}^{1,+}+I_{n,2}^{1,-}=\int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,(|\tau|-1)} \tag{4.15}$$

$$\preceq n\left(\frac{1}{n}\right)^{1-\varepsilon}\int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}}|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)}\,|d\tau|\preceq n^{\varepsilon},$$

for $-1<\gamma_1\leq 0$.

2. Let $i=2$. Analogously to the previous case, we obtain:

2.1.

$$I_{n,1}^{2,+}+I_{n,1}^{2,-}=\int\limits_{\mathcal{F}_{1,R}^{2,+}\cup\mathcal{F}_{1,R}^{2,-}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,(|\tau|-1)} \tag{4.16}$$

$$\preceq\ n\int\limits_{\mathcal{F}_{1,R}^{2,+}\cup\mathcal{F}_{1,R}^{2,-}}\frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^{+})\right|^{\frac{\gamma_2-1}{1+\beta_2}}}\preceq n\int\limits_{\mathcal{F}_{1,R}^{2,+}\cup\mathcal{F}_{1,R}^{2,-}}\frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{\frac{\gamma_2-1}{1+\beta_2}(1+\varepsilon)}}$$

$$\preceq\ \begin{cases} n^{\frac{\gamma_2-1}{1+\beta_2}+\varepsilon}, & \frac{\gamma_2-1}{1+\beta_2}>1-\varepsilon,\\ n\ln n, & \frac{\gamma_2-1}{1+\beta_2}=1-\varepsilon,\\ n, & \frac{\gamma_2-1}{1+\beta_2}<1-\varepsilon,\end{cases}$$

for $\gamma_2 > 0$ and

$$\begin{aligned} I_{n,1}^{2,+} + I_{n,1}^{2,-} &= \int\limits_{\mathcal{F}_{1,R}^{2,+}\cup\mathcal{F}_{1,R}^{2,-}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,(|\tau|-1)} \\ &\preceq n d_{2,R}^{(-\gamma_2)+1} \int\limits_{\mathcal{F}_{1,R}^{2,+}\cup\mathcal{F}_{1,R}^{2,-}} |d\tau| \preceq n\cdot mes\left(\mathcal{F}_{1,R}^{1,+}\cup\mathcal{F}_{1,R}^{1,-}\right) \preceq 1, \end{aligned} \tag{4.17}$$

for $\gamma_2 \le 0$.

2.2.

$$I_{n,2}^{2,+} + I_{n,2}^{2,-} = \int\limits_{\mathcal{F}_{2,R}^{2,+}\cup\mathcal{F}_{2,R}^{2,-}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,(|\tau|-1)} \tag{4.18}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+}\cup\mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^{+})\right|^{\frac{\gamma_2-1}{1+\beta_2}(1+\varepsilon)}} \preceq \begin{cases} n^{\frac{\gamma_2-1}{1+\beta_2}+\varepsilon}, & \frac{\gamma_2-1}{1+\beta_2} > 1-\varepsilon, \\ n\ln n, & \frac{\gamma_2-1}{1+\beta_2} = 1-\varepsilon, \\ n, & \frac{\gamma_2-1}{1+\beta_2} < 1-\varepsilon, \end{cases}$$

for $\gamma_2 > 0$, and

$$I_{n,2}^{2,+} + I_{n,2}^{2,-} = \int\limits_{\mathcal{F}_{2,R}^{2,+}\cup\mathcal{F}_{2,R}^{2,-}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,(|\tau|-1)}, \tag{4.19}$$

$$\preceq n\left(\frac{1}{n}\right)^{1-\varepsilon} \int\limits_{\mathcal{F}_{2,R}^{2,+}\cup\mathcal{F}_{2,R}^{2,-}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma_2)}\,|d\tau| \preceq n^{\varepsilon},$$

for $-1 < \gamma_2 \le 0$. Therefore, from (4.10)–(4.19), for any $p > 0$, we obtain

$$\begin{aligned} A_n^{2/p} \preceq \|P_n\|_{\mathcal{L}_p(h,L)} &\begin{cases} n^{\frac{2(\gamma_1-1)}{p}}, & \gamma_1 > \frac{3}{2}, \\ (n\ln n)^{\frac{1}{p}}, & \gamma_1 = \frac{3}{2}, \\ n^{\frac{1}{p}}, & 0 < \gamma_1 < \frac{3}{2}, \\ n^{\frac{\varepsilon}{p}}, & -1 < \gamma_1 \le 0 \end{cases} \\ &+ \begin{cases} n^{\frac{\gamma_2-1}{p(1+\beta_2)}+\varepsilon}, & \gamma_2 > 2+\beta_2-\varepsilon, \\ (n\ln n)^{\frac{1}{p}}, & \gamma_2 = 2+\beta_2-\varepsilon, \\ n^{\frac{1}{p}}, & 0 < \gamma_2 < 2+\beta_2-\varepsilon, \\ n^{\frac{\varepsilon}{p}}, & -1 < \gamma_2 \le 0, \end{cases} \end{aligned}$$

if $\alpha_1 \neq 0$, and

$$A_n^{2/p} \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} n^{\frac{\gamma_1-1}{p}+\varepsilon}, & \gamma_1 > 2-\varepsilon, \\ (n\ln n)^{\frac{1}{p}}, & \gamma_1 = 2-\varepsilon, \\ n^{\frac{1}{p}}, & 0<\gamma_1<2-\varepsilon, \\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_1\leq 0 \end{cases}$$
$$+ \begin{cases} n^{\frac{\gamma_2-1}{p(1+\beta_2)}+\varepsilon}, & \gamma_2 > 2+\beta_2-\varepsilon, \\ (n\ln n)^{\frac{1}{p}}, & \gamma_2 = 2+\beta_2-\varepsilon, \\ n^{\frac{1}{p}}, & 0<\gamma_2<2+\beta_2-\varepsilon, \\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_2\leq 0, \end{cases}$$

if $\alpha_1 = 0$. So, for A_n we get

$$A_n^{2/p} \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} n^{\frac{2(\gamma_1-1)}{p}}, & \gamma_1 > 1+\frac{\gamma_2-1}{2(1+\beta_2)}, \gamma_2 > 2+\beta_2-\varepsilon, \\ n^{\frac{\gamma_2-1}{p(1+\beta_2)}+\varepsilon}, & 0<\gamma_1 \leq 1+\frac{\gamma_2-1}{2(1+\beta_2)}, \gamma_2 > 2+\beta_2-\varepsilon, \\ n^{\frac{2(\gamma_1-1)}{p}}, & \gamma_1 > \frac{3}{2}, 0<\gamma_2<2+\beta_2-\varepsilon, \\ n^{\frac{1}{p}}, & 0<\gamma_1<\frac{3}{2}, 0<\gamma_2<2+\beta_2-\varepsilon, \\ (n\ln n)^{\frac{1}{p}}, & \gamma_1=\frac{3}{2}, \gamma_2 = 2+\beta_2-\varepsilon, \\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_1\leq 0, -1<\gamma_2\leq 0, \end{cases}$$

if $\alpha_1 \neq 0$, and

$$A_n^{2/p} \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} n^{\frac{\gamma_1-1}{p}+\varepsilon}, & \gamma_1 > 1+\frac{\gamma_2-1}{1+\beta_2},\ \gamma_2 \geq 2+\beta_2, \\ n^{\frac{\gamma_2-1}{p(1+\beta_2)}+\varepsilon} & 2\leq\gamma_1 \leq 1+\frac{\gamma_2-1}{1+\beta_2},\ \gamma_2 \geq 2+\beta_2, \\ n^{\frac{\gamma_1-1}{p}+\varepsilon}, & \gamma_1 \geq 2,\ 0<\gamma_2<2+\beta_2, \\ n^{\frac{1}{p}}, & 0<\gamma_1<2,\ 0<\gamma_2<2+\beta_2, \\ (n\ln n)^{\frac{1}{p}}, & \gamma_1=2-\varepsilon, \gamma_2 = 2+\beta_2-\varepsilon, \\ n^{\frac{\varepsilon}{p}}, & -1<\gamma_1\leq 0, -1<\gamma_2\leq 0, \end{cases} \tag{4.20}$$

if $\alpha_1 = 0$.

Comparing (4.3) and (4.20), we get:

$$|P_n(z)| \preceq \left[\frac{A_n}{d(z,L_R)}\right]^{2/p} \left|B_m(z)\ \Phi^{n+1}(z)\right|,$$

where A_n taken from (4.20). The function $B_m(z)$ is analytic in Ω, continuous on $\overline{\Omega}$ and $|B_m(z)| = 1$ on L. Then, according to the maximum modulus principle, we get

$$|B_m(z)| < 1, \; z \in \Omega_R,$$

and, so the proof is complete. □

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