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Partial logarithmic derivatives and distribution of zeros of analytic functions in the unit ball of bounded L-index in joint variables

Andriy I. Bandura, Oleh B. Skaskiv

(*Presented by V. Ya. Gutlyanskii*)

Abstract. In this paper, we obtain sufficient conditions of boundedness of **L**-index in joint variables for analytic functions in the unit ball, where $L : \mathbb{C}^n \to \mathbb{R}^n_+$ is a continuous positive vector-function. They give an estimate of maximum modulus of analytic function by its minimum modulus on a skeleton in a polydisc and describe the behavior of all partial logarithmic derivatives outside some exceptional set and the distribution of zeros. The deduced results are also new for analytic functions in the unit disc of bounded index and l-index. They generalize known results of G. H. Fricke, M. M. Sheremeta, A. D. Kuzyk and V. O. Kushnir.

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1. Introduction

A concept of an entire function of bounded index arose in analytic theory of differential equations. It appeared in paper of B. Lepson [24]. An entire function f is said to be of bounded index if there exists an integer $N > 0$ such that

$$(\forall z \in \mathbb{C})(\forall n \in \{0, 1, 2, \ldots\}) \colon \ \frac{|f^{(n)}(z)|}{n!} \le \max\left\{\frac{|f^{(j)}(z)|}{j!} \colon 0 \le j \le N\right\}. \tag{1.1}$$

The least such integer N is called the index of f. There was proved that every entire solution of ordinary n-th order linear differential equation with constant coefficients has bounded index. B. Lepson conjectured that every entire solution of the linear differential equation of infinite order with constant coefficients has the same property. In a general case

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the hypothesis is not correct because an entire function of bounded index has exponential type [20, 32] and there exist entire solutions with order greater than one. Therefore, it is natural to pose the same question about l-index boundedness of entire solutions (see definition below). This assumption has not yet been proven even in this extended formulation. However, there are many papers of various authors with different applications. Namely, theory of functions of bounded index has many applications in value distribution theory, differential equations and its system (see bibliography in [7, 33]). This concept is applicable as for entire functions of one and several variables [4, 10, 22] so for analytic functions in a domain [1, 11, 21, 23, 35]. In a comparison with traditional approaches (for example, Wiman–Valiron's method [15, 16, 31, 37] or value distribution theory [18, 19, 25, 29, 34]) it is more flexible to investigate analytic solutions of ordinary and partial differential equations [2, 3, 9, 27]. Particularly, if an entire solution has bounded index [22, 32, 36] then it immediately yields its growth estimates, an uniform distribution of its zeros in a sense, a certain regular behavior of the solution, etc. Similar conclusions are valid for functions of one variable which are analytic in a domain [13, 23, 33, 35].

To study more general entire functions, A. D. Kuzyk and M. M. Sheremeta [22] introduced a boundedness of the l-index, replacing $\frac{|f^{(p)}(z)|}{p!}$ on $\frac{|f^{(p)}(z)|}{p!l^p(|z|)}$ in (1.1), where $l:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function. In view of results from [14] it allows to study an arbitrary entire function f with bounded multiplicity of zeros. Besides, there are papers about bounded l-index for analytic function of one variable [23, 35].

In a multidimensional case a situation is more difficult and interesting. Recently we [11, 12] proposed approach to consider bounded $\mathbf{L}$-index in joint variables for analytic functions in a polydisc, where $\mathbf{L}(z)=(l_1(z),\ldots,l_n(z))$, $l_j:\mathbb{C}^n\to\mathbb{R}_+$ is a positive continuous functions, $j\in\{1,\ldots,n\}$. Although J. Gopala Krishna and S. M. Shah [21] introduced an analytic in a domain (a nonempty connected open set) $\Omega\subset\mathbb{C}^n$ ($n\in\mathbb{N}$) function of bounded index for $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n_+$. But analytic function of bounded index in a domain by Krishna and Shah is an entire function. It follows from necessary condition of the l-index boundedness for analytic in the unit disc function ([33, Th.3.3,p.71]): $\int_0^r l(t)dt\to\infty$ as $r\to 1$ (we take $l(t)\equiv\alpha_1$). Thus, there arises necessity to introduce and to investigate bounded $\mathbf{L}$-index in joint variables for analytic functions in polydisc domain. Besides a polydisc, other example of polydisc domain in $\mathbb{C}^n$ is a ball.

For analytic functions in the unit ball we introduced a concept of bounded $\mathbf{L}$-index in joint variables and deduced properties [1–3]. Also, there was presented an application of the concept to study properties of analytic solutions for systems of partial differential equations and to estimate its growth. But in one-dimensional case there is known logarithmic criterion of index boundedness [17,36]. It is very important to investigate infinite products and holomorphic solutions of differential equations. The assertion describes behavior of logarithmic derivative outside some exceptional set consisting with zeros of the function with some neighborhoods. The second condition of the criterion is uniform distribution of zeros in a sense. Above 20 years it has not been possible to obtain an analog of this criterion for entire functions of several variables by technical difficulties. Using other approach [4, 6, 8] we deduced the analog for a class of entire functions of bounded L-index in a direction. But for entire functions of bounded $\mathbf{L}$-index in joint variables the analog of logarithmic criterion remained unknown. Recently as sufficient conditions some analog of the characterization have been obtained for the class [5].

In view of importance of logarithmic criterion for analytic functions of one variable it is naturally to pose the following problem: *What is an analog of logarithmic criterion for analytic functions in the unit ball of bounded* $\mathbf{L}$*-index in joint variables?*

A complete solution to this problem may give new applications of bounded $\mathbf{L}$-index in joint variables for analytic functions in the unit ball. For example, this can be useful to investigate properties of multidimensional analogs of Blaschke products. or analytic solutions of partial differential equations system.

In this paper we will try to give some answer to the question.

2. Main definitions and notations

We need some standard notations. Denote $\mathbb{R}_+ = [0, +\infty)$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n_+$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n_+$, $\mathbf{1}_j = (0, \ldots, 0, \underbrace{1}_{j\text{–th place}}, 0, \ldots, 0) \in \mathbb{R}^n_+$, $R = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $|z| = \sqrt{\sum_{j=1}^n |z_j|^2}$. For $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions $AB = (a_1 b_1, \cdots, a_n b_n)$, $A/B = (a_1/b_1, \ldots, a_n/b_n)$, $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\|A\| = a_1 + \cdots + a_n$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \ldots, n\}$; the relation $A \leq B$ is defined similarly. For $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ denote $K! = k_1! \cdot \ldots \cdot k_n!$. Addition, scalar multiplication, and conjugation are defined on $\mathbb{C}^n$ componentwise. The polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j, \ j = 1, \ldots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j = 1, \ldots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| \le r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$, $\mathbb{D}^n = \mathbb{D}^n(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. The open ball $\{z \in \mathbb{C}^n : \ |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, the closed ball $\{z \in \mathbb{C}^n : \ |z - z^0| \le r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : \ |z| < 1\}$.

For $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and the partial derivatives of an analytic in $\mathbb{B}^n$ function $F(z) = F(z_1, \dots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n): \ l_j(z) > \beta/(1 - |z|), \ j \in \{1, \dots, n\}, \tag{2.1}$$

where $\beta > \sqrt{n}$ is a some constant.

S. N. Strochyk, M. M. Sheremeta, V. O. Kushnir [23, 33, 35] imposed a similar condition for a function $l : \mathbb{D} \to \mathbb{R}_+$ and $l : G \to \mathbb{R}_+$, where G is arbitrary domain in $\mathbb{C}$.

An analytic function $F \colon \mathbb{B}^n \to \mathbb{C}$ is said to be of *bounded* $\mathbf{L}$*-index (in joint variables)* [1–3], if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \ K \in \mathbb{Z}_+^n, \ \|K\| \le n_0 \right\}. \tag{2.2}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F, \mathbf{L}, \mathbb{B}^n)$ (for entire functions see [10, 25, 30]).

By $Q(\mathbb{B}^n)$ we denote the class of functions $\mathbf{L}$, which satisfy (2.1) and the following condition

$$(\forall R \in \mathbb{R}_+^n, |R| \le \beta, \ j \in \{1, \dots, n\}): \ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty,$$

where

$$\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{B}^n} \inf \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n \left[z^0, R/\mathbf{L}(z^0)\right] \right\},$$

$$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{B}^n} \sup \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n \left[z^0, R/\mathbf{L}(z^0)\right] \right\},$$

$$\Lambda_1(R) = (\lambda_{1,1}(R), \dots, \lambda_{1,n}(R)), \ \Lambda_2(R) = (\lambda_{2,1}(R), \dots, \lambda_{2,n}(R)).$$

We need the following assertion.

Theorem 2.1 ([2]). *Let $\mathbf{L} \in Q(\mathbb{B}^n)$, $F : \mathbb{B}^n \to \mathbb{C}$ be analytic function. If there exist R', $R'' \in \mathbb{R}^n_+$, $R' < R''$, $|R''| < \beta$ and $p_1 = p_1(R', R'') \ge 1$ such that for every $z^0 \in \mathbb{B}^n$*

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R''}{\mathbf{L}(z^0)}\right)\right\} \le p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R'}{\mathbf{L}(z^0)}\right)\right\} \tag{2.3}$$

then F is of bounded $\mathbf{L}$-index in joint variables.

3. Estimate maximum modulus on a skeleton in polydisc

Let Z_F be a zero set of analytic in $\mathbb{B}^n$ function F. We denote

$$G_R(F) = \bigcup_{z^0 \in Z_F} \left\{z \in \mathbb{C}^n\colon |z_j - z_j^0| < \frac{r_j}{l_j(z^0)} \ \forall j \in \{1, 2, \dots, n\}\right\} = \bigcup_{z^0 \in Z_F} \mathbb{D}^n\left(z^0, \frac{R}{\mathbf{L}(z^0)}\right).$$

Theorem 3.1. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function. If $\exists R \in \mathbb{R}^n_+$ with $r_j \in (0, \frac{\beta}{2\sqrt{n}})$, $\exists p_2 \ge 1$ $\exists \Theta \in \mathbb{R}^n_+$, $\mathbf{0} < \Theta < R$, $\exists R' > \mathbf{0}$, $(R' = \mathbf{0}$ for $Z_F = \emptyset)$ such that $\forall z^0 \in \mathbb{B}^n$ $\exists R^0 = R^0(z^0) \in \mathbb{R}^n_+$, $\Theta \le R^0 \le R$, for which*

$$\operatorname{meas}\left\{\mathbb{T}^n\left(z^0, \frac{R^0}{\mathbf{L}(z^0)}\right) \cap G_{R'}(F)\right\} < \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j}{\lambda_{2,j}(\boldsymbol{\beta}) l_j(z^0)} \tag{3.1}$$

and $$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R^0}{\mathbf{L}(z^0)}\right)\right\} \le p_2 \min\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R^0}{\mathbf{L}(z^0)}\right) \setminus G_{R'}(F)\right\} \tag{3.2}$$

*then the function F has bounded $\mathbf{L}$-index in joint variables (*meas *is the Lebesgue measure on the skeleton in the polydisc).*

Proof. Denote $\boldsymbol{\beta} = \frac{\beta}{\sqrt{n}}\mathbf{1}$. By Theorem 2.1 we will show that $\exists p_1 > 0$ $\forall z^0 \in \mathbb{B}^n$

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{\boldsymbol{\beta} - R}{\mathbf{L}(z^0)}\right)\right\} \le p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R}{\mathbf{L}(z^0)}\right)\right\}.$$

Denote $l_j^* = \max\left\{l_j(z) : z \in \mathbb{D}^n\left[z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right]\right\}$, $\rho_{j,0} = \frac{r_j}{l_j(z^0)}$, $\rho_{j,k} = \rho_{j,0} + \frac{k\cdot\theta_j}{l_j^*}$, $k \in \mathbb{N}$, $j \in \{1,\ldots,n\}$. The following estimate holds

$$\frac{\theta_j}{l_j^*} < \frac{r_j}{l_j^*} \le \frac{r_j}{l_j(z^0)} = \frac{\beta/\sqrt{n}}{l_j(z^0)} - \frac{\beta/\sqrt{n} - r_j}{l_j(z^0)}.$$

Hence, there exists $S^* = (s_1^*, \ldots, s_n^*) \in \mathbb{N}^n$ independent of z^0 such that

$$\rho_{j,m_j-1} < \frac{\beta/\sqrt{n} - r_j}{l_j(z^0)} \le \rho_{j,m_j} \le \frac{\beta/\sqrt{n}}{l_j(z^0)}$$

for some $m_j = m_j(z^0) \le s_j^*$ because $\mathbf{L} \in Q(\mathbb{B}^n)$. Indeed,

$$\left(\frac{\beta/\sqrt{n}}{l_j(z^0)} - \rho_{j,0}\right) / \frac{\theta_j}{l_j^*} = \left(\beta/\sqrt{n} - r_j\right) \frac{l_j^*}{\theta_j l_j(z^0)}$$
$$= \frac{\beta/\sqrt{n} - r_j}{\theta_j} \max\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n\left[z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right]\right\} \le \frac{\beta/\sqrt{n} - r_j}{\theta_j}\lambda_{2,j}(\boldsymbol{\beta}).$$

Thus, $s_j^* = \left[\frac{\beta/\sqrt{n}-r_j}{\theta_j}\lambda_{2,j}(\boldsymbol{\beta})\right]$, where $[x]$ is the integer part of $x \in \mathbb{R}$.

Let $M_0 = (m_1, \ldots, m_n)$ and τ_K^{**} be such a point in $\mathbb{B}^n$ that

$$|F(\tau_K^{**})| = \max\{|F(z)| \colon z \in \mathbb{T}^n(z^0, \mathcal{R}_K)\},$$

where $K = (k_1, \ldots, k_n)$, $\mathcal{R}_K = (\rho_{1,k_1}, \ldots, \rho_{n,k_n})$ and $\tau_{j,K}^*$ be the intersection point in $\mathbb{C}$ of the segment $[z_j^0, \tau_{j,K}^{**}]$ with $|z_j - z_j^0| = \rho_{j,k_j-1}$. We construct a sequence of polydisc $\mathbb{D}^n(z^0, \mathcal{R}_K)$ with $K \le M_0$, $\mathcal{R}_{\mathbf{0}} = R/\mathbf{L}(z^0) = (\rho_{1,0}, \ldots, \rho_{n,0})$ and $\Theta/\mathbf{L}(z^0) = (\theta_1/l_1^*, \ldots, \theta_n/l_n^*)$ (see Figures 1 and 2).

Denote $\alpha_K^{(j)} = (\tau_{1,K}^{**}, \ldots, \tau_{j-1,K}^{**}, \tau_{j,K}^{*}, \tau_{j+1,K}^{**}, \ldots, \tau_{n,K}^{**})$. Hence, for every $r_j > \theta_j$ and $K \le S^*$: $|\tau_{j,K}^* - \tau_{j,K}^{**}| = \frac{\theta_j}{l_j^*} \le \frac{r_j}{l_j(\alpha_K^{(j)})}$. Thus, for some $R^0 = R^0(\alpha_K^{(j)}) \in \mathbb{R}_+^n$, $\Theta \le R^0 \le R$, we deduce

$$|F(\tau_K^{**})| \le \max\left\{|F(z)| \colon z \in \mathbb{T}^n\left(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\right)\right\}$$
$$\le p_2 \min\left\{|F(z)| \colon z \in \mathbb{T}^n\left(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\right) \setminus G_{R'}(F)\right\}$$
$$\le p_2 \min\left\{|F(z)| \colon z \in \mathbb{T}^n\left(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\right) \setminus G_{R'}(F), z \in \mathbb{D}^n[z^0, \mathcal{R}_{K-\mathbf{1}_j}]\right\}$$
$$\le p_2 \max\{|F(z)| \colon z \in \mathbb{T}^n(z^0, \mathcal{R}_{K-\mathbf{1}_j})\}. \tag{3.3}$$

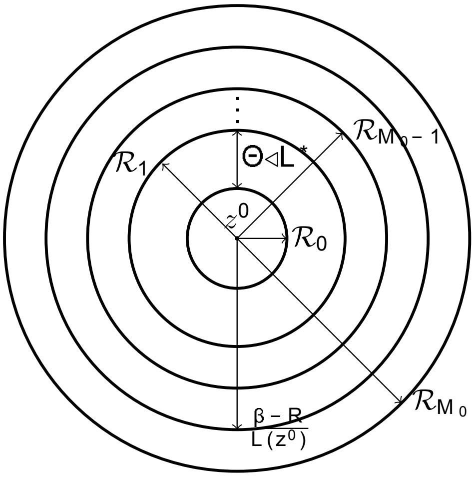

Figure 1

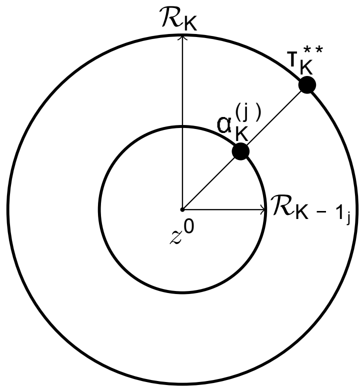

Figure 2

To deduce (3.3) we implicitly used that

$$\left(\mathbb{T}^n\left(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\right) \setminus G_{R'}(F)\right) \cap \mathbb{D}^n[z^0, \mathcal{R}_{K-\mathbf{1}_j}] \neq \emptyset. \tag{3.4}$$

Condition (3.1) provides (3.4). Indeed, we will find a lower estimate of n-dimensional Lebesgue measure of the set $\mathbb{T}^n\big(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\big) \cap \mathbb{D}^n[z^0, \mathcal{R}_{K-\mathbf{1}_j}]$ and will show that the measure is not lesser than a left-hand side of inequality (3.1).

The set $\mathbb{T}^n\left(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\right) \cap \mathbb{D}^n[z^0, \mathcal{R}_{K-\mathbf{1}_j}]$ is the Cartesian product of the following arcs on circles: for every $m \in \{1, \dots, n\}$, $m \neq j$ (see Figure 3)

$$\left\{z_m \in \mathbb{C}\colon\ |z_m - \tau_{m,K}^{**}| = \frac{r_m^0}{l_m(\alpha_K^{(j)})}\right\} \bigcap \left\{z_m \in \mathbb{C}\colon |z_m - z_m^0| \le \rho_{m,k_m}\right\}$$

and for $m = j$ (see Figure 4)

$$\left\{z_j \in \mathbb{C}\colon\ |z_j - \tau_{j,K}^{*}| = \frac{r_j^0}{l_j(\alpha_K^{(j)})}\right\} \bigcap \left\{z_j \in \mathbb{C}\colon |z_j - z_j^0| \le \rho_{j,k_j-1}\right\}.$$

It is easy to prove that the length of arc equals

$$\frac{2r_m^0}{l_m(\alpha_K^{(j)})} \cdot \arccos \frac{r_m^0}{2l_m(\alpha_K^{(j)})\rho_{m,k_m}} \quad \text{for } m \neq j \tag{3.5}$$

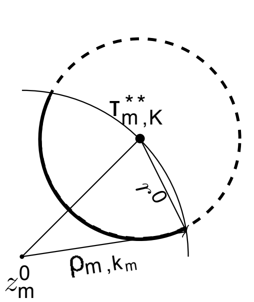

Figure 3 with $r^0 = \frac{r^0_m}{l_m(\alpha^{(j)}_K)}$

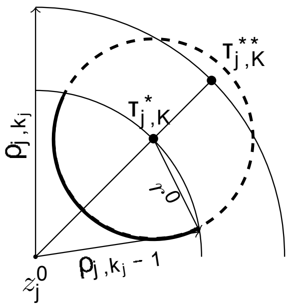

Figure 4 with $r^0 = \frac{r^0_j}{l_j(\alpha^{(j)}_K)}$

and

$$\frac{2r_j^0}{l_j(\alpha_K^{(j)})}\cdot \arccos \frac{r_j^0}{2l_j(\alpha_K^{(j)})\rho_{j,k_j-1}} \quad \text{for } m=j. \tag{3.6}$$

But for $m\neq j$ $\frac{r_m^0}{l_m(\alpha_K^{(j)})}\leq \rho_{m,k_m}$ and $\frac{r_j^0}{l_j(\alpha_K^{(j)})}\leq \rho_{j,k_j-1}$ then the argument in arccosine from (3.6) and (3.5) does not exceed $\frac{1}{2}$. This means that the length of arc is not lesser than

$$\frac{2r_m^0}{l_m(\alpha_K^{(j)})}\arccos\frac{1}{2}\geq \frac{2\theta_m\pi}{3l_m(z^0)\lambda_{2,m}(\boldsymbol{\beta})} \quad \text{for every } m\in\{1,2,\ldots,n\},$$

because $\mathbf{L}\in Q(\mathbb{B}^n)$. Accordingly, the measure of the set $\mathbb{T}^n\big(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}(\alpha_K^{(j)})}\big)\cap \mathbb{D}^n[z^0,\mathcal{R}_{K-\mathbf{1}_j}]$ on the skeleton of polydisc is always not lesser than $\prod_{m=1}^n \frac{2\theta_m\pi}{3l_m(z^0)\lambda_{2,m}(\boldsymbol{\beta})}$. Assuming a strict inequality in (3.1), we deduce that (3.4) is valid.

Applying (3.3) m_j-th times in every variable z_j, we obtain

$$\begin{aligned}
\max\left\{|F(z)|: z\in\mathbb{T}^n\left(z^0,\frac{\boldsymbol{\beta}-R}{\mathbf{L}(z^0)}\right)\right\} &\leq \max\{|F(z)|\colon z\in\mathbb{T}^n(z^0,\mathcal{R}_{M_0})\}\\
&\leq p_2\max\{|F(z)|\colon z\in\mathbb{T}^n(z^0,\mathcal{R}_{M_0-\mathbf{1}_n})\}\\
&\leq p_2^{m_n}\max\{|F(z)|\colon z\in\mathbb{T}^n(z^0,\mathcal{R}_{M_0-m_n\mathbf{1}_n})\}\leq\ldots\leq\\
&\leq p_2^{m_n+1}\max\{|F(z)|\colon z\in\mathbb{T}^n(z^0,\mathcal{R}_{M_0-m_n\mathbf{1}_n-\mathbf{1}_{n-1}})\}\\
&\leq p_2^{m_n+m_{n-1}}\max\{|F(z)|\colon z\in\mathbb{T}^n(z^0,\mathcal{R}_{M_0-m_n\mathbf{1}_n-m_{n-1}\mathbf{1}_{n-1}})\}\ldots\\
&\leq p_2^{\|M_0\|}\max\{|F(z)|\colon z\in\mathbb{T}^n(z^0,\mathcal{R}_{\mathbf{0}})\}\\
&\leq p_2^{\|S^*\|}\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)\right\}.
\end{aligned}$$

By Theorem 2.1 the function F has bounded $\mathbf{L}$-index in joint variables. □

Let us to denote $c(z',r) = \{z \in \mathbb{D}\colon |z-z'| = \frac{r}{l(z')}\}$. For $n=1$ Theorem 3.1 implies the following corollary.

Corollary 3.1. *Let $l \in Q(\mathbb{D})$, $f : \mathbb{D} \to \mathbb{C}$ be an analytic function. If $\exists r \in (0, \beta/2)$, $\exists r' \geq 0$, $\exists p_2 \geq 1$ $\exists \theta \in (0,r)$, such that $\forall z^0 \in \mathbb{D}$ $\exists r^0 = r^0(z^0) \in [\theta; r]$, and* meas $\left\{c(z^0,r^0) \cap G_{R'}(F)\right\} < \frac{2\pi\theta}{3l(z^0)\lambda_2(2r+2)}$ *and*

$$\max\left\{|f(z)|\colon z \in c(z^0,r^0)\right\} \leq p_2 \min\left\{|f(z)|\colon z \in c(z^0,r^0) \setminus G_{r'}(f)\right\} \tag{3.7}$$

then the function f has bounded l-index (here meas *means the Lebesgue measure on the circle).*

In a some sense, this corollary is new even for analytic functions of one variable because the circle $c(z^0,r^0)$ can contain zeros of the function f. Meanwhile, in corresponding theorems from [23,33] the circle $c(z^0,r^0)$ is chosen such that $f(z) \neq 0$ for all $z \in c(z^0,r^0)$.

4. Behavior of partial logarithmic derivatives

Theorem 4.1. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. If an analytic function $F : \mathbb{B}^n \to \mathbb{C}$ satisfies the following conditions:*

1) for every $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, there exists $p_1 = p_1(R) > 0$ such that for all $z \in \mathbb{B}^n \setminus G_R(F)$ and for all $j \in \{1,\dots,n\}$

$$\frac{1}{|F(z)|}\left|\frac{\partial F(z)}{\partial z_j}\right| \leq p_1 l_j(z), \tag{4.1}$$

2) for every $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, and $R' \geq \mathbf{0}$ there exists $p_2 = p_2(R,R') \geq 1$ that for all $z^0 \in \mathbb{B}^n$ such that $\mathbb{T}^n(z^0, \frac{R}{\mathbf{L}(z^0)}) \setminus G_{R'}(F) = \bigcup_i C_i \neq \emptyset$, where the sets C_i are connected disjoint sets, and either
a) $\max\limits_i \min\limits_{z \in C_i} |F(z)| \leq p_2 \min\limits_i \min\limits_{z \in C_i} |F(z)|$, or
b) $\max\limits_i \max\limits_{z \in C_i} |F(z)| \leq p_2 \min\limits_i \max\limits_{z \in C_i} |F(z)|$, or
c) $|F(z^*)| = \max_i \max_{z \in C_i} |F(z)|$, $|F(z^{**})| = \min_i \min_{z \in C_i} |F(z)|$, and z^*, z^{**} belong to the same set C_{i_0}

3) for every $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, there exists $\Theta, R' \in \mathbb{R}^n_+$, $0 < \theta_j < \frac{2r_j}{2+3\lambda_{2,j}(\boldsymbol{\beta})}$, such that for all $z \in \mathbb{B}^n$

$$meas\left\{G_{R'}(F) \cap \mathbb{D}^n\left[z, R/\mathbf{L}(z)\right]\right\} < \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)}{\lambda_{2,j}(\boldsymbol{\beta}) l_j^2(z)}, \tag{4.2}$$

then F has bounded $\mathbf{L}$-index in joint variables (here meas is $2n$-dimensional the Lebesgue measure).

Proof. Let $z^0 \in \mathbb{B}^n$. In view of Theorem 3.1, we need to prove that

$$\text{meas}\left\{\mathbb{T}^n\left(z^0, \frac{R^0}{\mathbf{L}(z^0)}\right) \cap G_{R'}(F)\right\} < \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j}{\lambda_{2,j}(\boldsymbol{\beta}) l_j(z^0)}$$

for some $R^0 = R^0(z^0)$.

Let $dS = ds_1 \cdot \ldots \cdot ds_n$, $S = (s_1, \ldots, s_n)$, ω_z be a volume measure in $\mathbb{R}^{2n}$. By the Fubini–Tonelli theorem we have

$$\int_{\mathbb{D}^n[z^0,R]} u(z) d\omega_z = \int_0^{r_1} \ldots \int_0^{r_n} s_1 \ldots s_n \Big(\int_0^{2\pi} \ldots \int_0^{2\pi} u(z^0 + Se^{i\Theta}) d\theta_1 \ldots d\theta_n\Big)$$
$$ds_1 \ldots ds_n = \int_0^R \left(\int_0^{2\pi} \ldots \int_0^{2\pi} u(z^0 + Se^{i\Theta}) d(s_1\theta_1) \ldots d(s_n\theta_n)\right) dS,$$

where u is measurable function. Let $u(z) = \chi_F(z)$ be a characteristic function of the set $G_{R'}(F)$ for the function F. We substitute $R/\mathbf{L}(z^0)$ instead R. Hence,

$$\begin{aligned} \text{meas}\left\{\mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right] \cap G_{R'}(F)\right\} &= \int_{\mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]} \chi_F(z) d\omega_z \\ &= \int_{\mathbf{0}}^{R/\mathbf{L}(z^0)} \int_{\mathbb{T}^n(z^0, S/\mathbf{L}(z^0))} \chi_F(z) d\mu_z dS \\ &= \int_{\mathbf{0}}^{R/\mathbf{L}(z^0)} \text{meas}\left\{\mathbb{T}^n(z^0, S) \cap G_{R'}(F)\right\} dS, \end{aligned} \tag{4.3}$$

where μ_z is the measure on the skeleton of polydisc in $\mathbb{C}^n$.

Combining (4.2) and (4.3), we obtain

$$\int_{\mathbf{0}}^{R/\mathbf{L}(z^0)} \text{meas}\left\{\mathbb{T}^n(z^0, S) \cap G_{R'}(F)\right\} dS$$
$$= \text{meas}\left\{\mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right] \cap G_{R'}(F)\right\} < \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j - \theta_j)}{\lambda_{2,j}(\boldsymbol{\beta}) l_j^2(z^0)}. \tag{4.4}$$

Besides, we have

$$\int_{\mathbf{0}}^{\Theta/\mathbf{L}(z^0)} \text{meas}\left\{\mathbb{T}^n(z^0, S) \cap G_{R'}(F)\right\} dS$$
$$= \text{meas}\left\{\mathbb{D}^n\left[z^0, \frac{\Theta}{\mathbf{L}(z^0)}\right] \cap G_{R'}(F)\right\} \le \pi^n \prod_{j=1}^n \frac{\theta_j^2}{l_j^2(z^0)}.$$

Thus, the following difference is positive

$$\left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)}{\lambda_{2,j}(\boldsymbol{\beta})l_j^2(z^0)} - \int_0^{\Theta/\mathbf{L}(z^0)} \operatorname{meas}\left\{\mathbb{T}^n(z^0,S)\cap G_{R'}(F)\right\} dS$$
$$\geq \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)}{\lambda_{2,j}(\boldsymbol{\beta})l_j^2(z^0)} - \pi^n \prod_{j=1}^n \frac{\theta_j^2}{l_j^2(z^0)}$$
$$= \pi^n \prod_{j=1}^n \frac{\theta_j}{l_j^2(z^0)} \frac{2r_j-\theta_j(2+3\lambda_{2,j}(\boldsymbol{\beta}))}{3\lambda_{2,j}(\boldsymbol{\beta})} > 0$$

because $\theta_j < \frac{2r_j}{2+3\lambda_{2,j}(\boldsymbol{\beta})}$. From (4.4) it follows that

$$\int_{\Theta/\mathbf{L}(z^0)}^{R/\mathbf{L}(z^0)} \operatorname{meas}\left\{\mathbb{T}^n(z^0,S)\cap G_{R'}(F)\right\} dS < \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)}{\lambda_{2,j}(\boldsymbol{\beta})l_j^2(z^0)}$$
$$-\int_0^{\Theta/\mathbf{L}(z^0)} \operatorname{meas}\left\{\mathbb{T}^n(z^0,S)\cap G_{R'}(F)\right\} dS \leq \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)}{\lambda_{2,j}(\boldsymbol{\beta})l_j^2(z^0)}. \tag{4.5}$$

By mean value theorem there exists $R^0=R^0(z^0)$ with $r_j^0\in[\theta_j,r_j]$ such that

$$\int_{\Theta/\mathbf{L}(z^0)}^{R/\mathbf{L}(z^0)} \operatorname{meas}\{\mathbb{T}^n(z^0,S)\cap G_{R'}(F)\} dS$$
$$= \operatorname{meas}\{\mathbb{T}^n(z^0,R^0/\mathbf{L}(z^0))\cap G_{R'}(F)\} \prod_{j=1}^n \frac{r_j-\theta_j}{l_j(z^0)}.$$

Hence, in view of (4.5) we obtain a desired inequality

$$\operatorname{meas}\left\{\mathbb{T}^n(z^0,R^0/\mathbf{L}(z^0))\cap G_{R'}(F)\right\} < \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j}{\lambda_{2,j}(\boldsymbol{\beta})l_j(z^0)}.$$

Clearly, that for every point $z^0\in\mathbb{B}^n$ we have $\mathbb{T}^n\left(z^0,\frac{R^0}{\mathbf{L}(z^0)}\right)\setminus Z_F = \bigcup_i C_i'$, where C_i' are connected disjoint sets, $C_i'\supset C_i$ and C_i is defined in condition 2). Without loss of generality we assume that two any points from C_i' can be connected by a segment of line lying inside in C_i'. Otherwise we can split C_i' by the sets with the property. Let $z^*\in \mathbb{T}^n(z^0,R/\mathbf{L}(z^0))$ be such that $|F(z^*)|=\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{R^0}{\mathbf{L}(z^0)}\right)\right\}$. Then there exists i_0 that $z^*\in C_{i_0}'$. Let $z^{**}\in C_{i_0}\subset C_{i_0}'$ be such that $|F(z^{**})| = \min_{z\in C_{i_0}} |F(z)|$. We connect the points z^* and z^{**} by a piecewise-analytic curve $z = z(t) = (z_1(t),\dots,z_n(t))$, $t\in[0;1]$. The curve is chosen such that $\int_0^1 |z_j'(t)|dt \le \frac{2\pi r_j}{l_j(z^0)}$. Integrating from z^* to z^{**}, we obtain

$$\ln\left|\frac{F(z^*)}{F(z^{**})}\right| \le \left|\ln\frac{F(z^*)}{F(z^{**})}\right| = \left|\int_{z^{**}}^{z^*} d\ln F(z)\right| = \left|\int_{z^{**}}^{z^*} \sum_{j=1}^n \frac{1}{F(z)}\frac{\partial F(z)}{\partial z_j}dz_j\right|$$

$$\le \left|\int_{z^{**}}^{z^*} \sum_{j=1}^n p_1 l_j(z)|dz_j|\right| \le \sum_{j=1}^n p_1 l_j(z^0)\lambda_{2,j}(R)\frac{2\pi r_j}{l_j(z_0)} = 2p_1\pi\sum_{j=1}^n r_j\lambda_{2,j}(R).$$

Hence,

$$\max\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{R^0}{\mathbf{L}(z^0)}\right)\right\} = |F(z^*)|$$

$$\le \exp\{2p_1\pi\sum_{j=1}^n r_j\lambda_{2,j}(R)\}|F(z^{**})| = \exp\{2p_1\pi\sum_{j=1}^n r_j\lambda_{2,j}(R)\}\min_{z\in C_{i_0}}|F(z)|$$

$$\le \exp\{2p_1\pi\sum_{j=1}^n r_j\lambda_{2,j}(R)\}p_2\min_i\min_{z\in C_i}|F(z)|$$

$$= \exp\{2p_1\pi\sum_{j=1}^n r_j\lambda_{2,j}(R)\}p_2\min\left\{|F(z)|\colon z\in\mathbb{T}^n\left(z^0,\frac{R^0}{\mathbf{L}(z^0)}\right)\setminus G_{R'}(F)\right\}.$$

By Theorem 3.1 the function F has bounded $\mathbf{L}$-index in joint variables. $\square$

Let us to denote Δ as Laplace operator. We will consider $\Delta\ln|F|$ as generalized function. Using some known results from potential theory, we can rewrite Theorem (4.1) in the following way

Theorem 4.2. *Let* $\mathbf{L}\in Q(\mathbb{B}^n)$. *If an analytic function* $F:\mathbb{B}^n\to\mathbb{C}$ *satisfies the following conditions*

1) *for every* $R\in\mathbb{R}^n_+$, $|R|\le\beta$, *there exists* $p_1 = p_1(R) > 0$ *such that for all* $z\in\mathbb{B}^n\setminus G_R(F)$ *and for every* $j\in\{1,\dots,n\}$

$$\left|\frac{\partial\ln F(z)}{\partial z_j}\right| \le p_1 l_j(z). \tag{4.6}$$

2) *for every* $R\in\mathbb{R}^n_+$, $|R|\le\beta$, *and* $R'>\mathbf{0}$ *there exists* $p_2 = p_2(R,R')\ge 1$ *such that for all* $z^0\in\mathbb{C}^n$ *such that* $\mathbb{T}^n(z^0,\frac{R}{\mathbb{L}(z^0)})\setminus G_{R'}(F) = \bigcup_i C_i \ne\emptyset$, *where the sets* C_i *are connected disjoint sets, and either*

a) $\max\limits_{i}\min\limits_{z\in C_i}|F(z)|\leq p_2\min\limits_{i}\min\limits_{z\in C_i}|F(z)|$, *or*
b) $\max\limits_{i}\max\limits_{z\in C_i}|F(z)|\leq p_2\min\limits_{i}\max\limits_{z\in C_i}|F(z)|$, *or*
c) $|F(z^*)|=\max_i\max_{z\in C_i}|F(z)|$, $|F(z^{**})|=\min_i\min_{z\in C_i}|F(z)|$, *and* z^*, z^{**} *belong to the same set* C_{i_0}

3) for every $R\in\mathbb{R}^n_+$, $|R|\leq\beta$, *there exists* $\Theta\in\mathbb{R}^n_+$, $0<\theta_j<\frac{2r_j}{2+3\lambda_{2,j}(\boldsymbol{\beta})}$, *such that for all* $z\in\mathbb{B}^n$

$$\int_{\mathbb{D}^n[z,R/\mathbf{L}(z)]}\Delta\ln|F|dV_{2n}\leq 2\pi\left(\frac{2}{3}\right)^n\prod_{j=1}^{n}\frac{\theta_j(r_j-\theta_j)\lambda_{1,j}^2(R)}{r_j^2\lambda_{2,j}(\boldsymbol{\beta})}.$$

then F *has bounded* $\mathbf{L}$*-index in joint variables.*

Proof. L. I. Ronkin [28, p. 230] deduced the following formula for entire function:

$$\int_{\mathbb{D}^n[\mathbf{0},R^*]}\Delta\ln|F|dV_{2n}=2\pi\int_{Z_F\cap\mathbb{D}^n[\mathbf{0},R^*]}\gamma_F(z)dV_{2n-2},$$

where $\gamma_F(z)$ is a multiplicity of zero point of the function F at point z, $R^*\in\mathbb{R}^n_+$ is arbitrary radius. Let $\chi_F(z)$ be a characteristic function of zero set of F. Then $\chi_F(z)\leq\gamma_F(z)$. Hence,

$$\begin{aligned}
&V_{2n}\Big(G_{R'}(F)\bigcap\mathbb{D}^n(z,R/L(z))\Big)\\
&\leq\int\limits_{z^0\in Z_F\cap\mathbb{D}^n(z,R/L(z))}V_{2n}\Big(\mathbb{D}^n(z^0,R/L(z^0))\Big)dV_{2n-2}\\
&\leq\max\Big\{V_{2n}\Big(\mathbb{D}^n(z^0,R/L(z^0))\Big):z^0\in Z_F\cap\mathbb{D}^n(z,R/L(z))\Big\}\\
&\qquad\times\int\limits_{z^0\in Z_F\cap\mathbb{D}^n(z,R/L(z))}\gamma_F(z)dV_{2n-2}\\
&=\max\Big\{V_{2n}\Big(\mathbb{D}^n(z^0,R/L(z^0))\Big):z^0\in Z_F\cap\mathbb{D}^n(z,R/L(z))\Big\}\\
&\qquad\times\frac{1}{2\pi}\int_{\mathbb{D}^n[z,\frac{R}{\mathbf{L}(z)}]}\Delta\ln|F|dV_{2n}
\end{aligned}$$

$$\leq \max\Big\{\pi^n \prod_{j=1}^n \frac{r_j^2}{l_j^2(z_0)} : z^0 \in Z_F \cap \mathbb{D}^n(z, R/L(z))\Big\}$$
$$\times \left(\frac{2}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)\lambda_{1,j}^2(R)}{r_j^2\lambda_{2,j}(\boldsymbol{\beta})}$$
$$\leq \pi^n \prod_{j=1}^n \frac{r_j^2}{\lambda_{1,j}^2(R) l_j^2(z)} \left(\frac{2}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)\lambda_{1,j}^2(R)}{r_j^2\lambda_{2,j}(\boldsymbol{\beta})}$$
$$= \left(\frac{2\pi}{3}\right)^n \prod_{j=1}^n \frac{\theta_j(r_j-\theta_j)}{\lambda_{2,j}(\boldsymbol{\beta}) l_j^2(z)}$$

i.e. inequality (4.2) holds. □

For $n = 1$ Theorem 4.1 implies the following corollary.

Corollary 4.1. *Let $l \in Q(\mathbb{D})$, $f : \mathbb{D} \to \mathbb{C}$ be an analytic function, $n(r, z^0, f)$ be a number of zeros of the f in the disc $|z - z_0| \leq \frac{r}{l(z^0)}$. If the function f satisfies the following conditions*

1) for every $r \in (0, \beta)$ there exists $p_1 = p_1(r) > 0$ such that for all $z \in \mathbb{D} \setminus G_r(f)$
$$\left|\frac{f'(z)}{f(z)}\right| \leq p_1 l(z),$$

2) for every $r \in (0, \beta)$ and $r' \geq 0$ exists $p_2 = p_2(r, r') \geq 1$ that for all $z^0 \in \mathbb{D}$ such that $\{z \in \mathbb{D}\colon |z - z^0| = \frac{r}{l(z^0)}\} \setminus G_{r'}(f) = \bigcup_i C_i \neq \emptyset$, where the sets C_i are connected disjoint sets, and either a) $\max\limits_i \min\limits_{z\in C_i} |f(z)| \leq p_2 \min\limits_i \min\limits_{z\in C_i} |f(z)|$, or b) $\max\limits_i \max\limits_{z\in C_i} |f(z)| \leq p_2 \min\limits_i \max\limits_{z\in C_i} |f(z)|$, or
c) $|f(z^*)| = \max_i \max_{z\in C_i} |f(z)|$, $|f(z^{**})| = \min_i \min_{z\in C_i} |f(z)|$, and z^*, z^{**} belong to the same set C_{i_0}

3) for every $r \in (0, \beta)$ there exist $\theta \in (0, \frac{2r}{2+3\lambda_2(\boldsymbol{\beta})})$, $r' > 0$ such that for all $z \in \mathbb{D}$
$$n(r, z, f) < \frac{2}{3}\frac{\theta(r-\theta)}{\lambda_2(\beta) l^2(z) r'^2}$$

then f has bounded l-index.

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Contact information

Andriy Ivanovych Bandura
Ivano-Frankivsk National Technical University of Oil and Gas,
Ivano-Frankivsk, Ukraine
E-Mail: andriykopanytsia@gmail.com

Oleh Bohdanovych Skaskiv
Ivan Franko National University of Lviv,
Lviv, Ukraine
E-Mail: olskask@gmail.com