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Constructing balleans

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Abstract. A ballean is a set endowed with a coarse structure. We introduce and explore three constructions of balleans from a pregiven family of balleans: bornological products, bouquets and combs. Also we analyze the smallest and the largest coarse structures on a set X compatible with a given bornology on X.

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1. Introduction

Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X := \{(x, x) : x \in X\}$ of X;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z \ \ ((x, z) \in E,\ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$.

Elements $E \in \mathcal{E}$ of the coarse structure are called *entourages* on X.

For $x \in X$ and $E \in \mathcal{E}$ the set $E[x] := \{y \in X : (x, y) \in E\}$ is called the *ball of radius E centered at x*. Since $E = \bigcup_{x\in X}\{x\} \times E[x]$, the entourage E is uniquely determined by the family of balls $\{E[x] : x \in X\}$. A subfamily $\mathcal{B} \subset \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if each set $E \in \mathcal{E}$ is contained in some $B \in \mathcal{B}$.

The pair $(X, \mathcal{E})$ is called a *coarse space* [11] or a *ballean* [8,10]. In [8] every base of a coarse structure, defined in terms of balls, is called a *ball structure*. We prefer the name balleans not only by the authors rights but also because a coarse spaces sounds like some special type of topological

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spaces. In fact, balleans can be considered as non-topological antipodes of uniform topological spaces. Our compromise with [11] is in usage the name coarse structure in place of the ball structure.

In this paper, all balleans under consideration are supposed to be *connected*: for any $x, y \in X$, there is $E \in \mathcal{E}$ such $y \in E[x]$. A subset $Y \subseteq X$ is called *bounded* if $Y = E[x]$ for some $E \in \mathcal{E}$, and $x \in X$. The family $\mathcal{B}_X$ of all bounded subsets of X is a bornology on X. We recall that a family $\mathcal{B}$ of subsets of a set X is a *bornology* if $\mathcal{B}$ contains the family $[X]^{<\omega}$ of all finite subsets of X and $\mathcal{B}$ is closed under finite unions and taking subsets. A bornology $\mathcal{B}$ on a set X is called *unbounded* if $X \notin \mathcal{B}$.

Each subset $Y \subseteq X$ defines a *subbalean* $(Y, \mathcal{E}|_Y)$ of $(X, \mathcal{E})$, where $\mathcal{E}|_Y = \{E \cap (Y \times Y) : E \in \mathcal{E}\}$. A subbalean $(Y, \mathcal{E}|_Y)$ is called *large* if there exists $E \in \mathcal{E}$ such that $X = E[Y]$, where $E[Y] = \bigcup_{y \in Y} E[y]$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be balleans. A mapping $f : X \to X'$ is called *coarse (or macrouniform)* if for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}$ such that $f(E(x)) \subseteq E'(f(x))$ for each $x \in X$. If f is a bijection such that f and f^{-1} are coarse, then f is called an *asymorphism.* If $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ contains large asymorphic subballeans, then they are called *coarsely equivalent.*

For coarse spaces $(X_\alpha, \mathcal{E}_\alpha)$, $\alpha \in \kappa$, their product is the Cartesian product $X = \prod_{\alpha \in \alpha} X_\alpha$ endowed with the coarse structure generated by the base consisting of the entourages

$$\big\{\big((x_\alpha)_{\alpha\in\kappa}, (y_\alpha)_{\alpha\in\kappa}\big) \in X \times X : \forall \alpha \in \kappa \;\; (x_\alpha, y_\alpha) \in E_\alpha\big\},$$

where $(E_\alpha)_{\alpha\in\kappa} \in \prod_{\alpha\in\kappa} \mathcal{E}_\alpha$.

A class $\mathfrak{M}$ of balleans is called a *variety* if $\mathfrak{M}$ is closed under formation of subballeans, coarse images and Cartesian products. For characterization of all varieties of balleans, see [7].

Given a family $\mathfrak{F}$ of subsets of $X \times X$, we denote by $\mathcal{E}$ the intersection of all coarse structures, containing each $F \cup \triangle_X$, $F \in \mathfrak{F}$, and say that $\mathcal{E}$ is generated by $\mathfrak{F}$. It is easy to see that $\mathcal{E}$ has a base of subsets of the form $E_1 \circ E_1 \circ \ldots \circ E_n$, where

$$E_1, \ldots, E_n \in \{F \cup F^{-1} \cup \{(x, y)\} \cup \triangle_X : F \in \mathfrak{F}, \;\; x, y \in X\}.$$

By a *pointed ballean* we shall understand a ballean $(X, \mathcal{E})$ with a distinguished point $e_* \in X$.

2. Metrizability and normality

Every metric d on a set X defines the coarse structure $\mathcal{E}_d$ on X with the base $\{\{(x,y) : d(x,y) < n\} : n \in \mathbb{N}\}$. A ballean $(X, \mathcal{E})$ is called *metrizable* if there is a metric d on such that $\mathcal{E} = \mathcal{E}_d$.

Theorem 1 ([5])**.** *A ballean $(X, \mathcal{E})$ is metrizable if and only if $\mathcal{E}$ has a countable base.*

Let $(X, \mathcal{E})$ be a ballean. A subset $U \subseteq X$ is called an *asymptotic neighbourhood* of a subset $Y \subseteq X$ if for every $E \in \mathcal{E}$ the set $E[Y] \setminus U$ is bounded.

Two subset Y, Z of X are called *asymptotically disjoint (separated)* if for every $E \in \mathcal{E}$ the intersection $E[Y] \cap E[Z]$ is bounded (Y and Z have disjoint asymptotic neighbourhoods).

A ballean $(X, \mathcal{E})$ is called *normal* [6] if any two asymptotically disjoint subsets of X are asymptotically separated. Every ballean $(X, \mathcal{E})$ with linearly ordered base of $\mathcal{E}$ is normal. In particular, every metrizable ballean is normal, see [6].

A function $f : X \to \mathbb{R}$ is called *slowly oscillating* if for any $E \in \mathcal{E}$ and $\varepsilon > 0$, there exists a bounded subset B of X such that $\operatorname{diam} f(E[x]) < \varepsilon$ for each $x \in X \setminus B$.

Theorem 2 ([6])**.** *A ballean $(X, \mathcal{E})$ is normal if and only if for any two disjoint asymptotically disjoint subsets Y, Z of X there exists a slowly oscillating function $f : X \to [0, 1]$ such that $f(Y) \subset \{0\}$ and $f(Z) \subset \{1\}$.*

For any unbounded bornology $\mathcal{B}$ on a set X the cardinals

$$\begin{aligned}
&\operatorname{add}(\mathcal{B}) = \min\{\mathcal{A} \subset \mathcal{B} : \bigcup\mathcal{A} \notin \mathcal{B}\},\\
&\operatorname{cov}(\mathcal{B}) = \min\{|\mathcal{C}| : \mathcal{C} \subset \mathcal{B},\ \bigcup\mathcal{C} = X\} \quad \text{and}\\
&\operatorname{cof}(\mathcal{B}) = \min\{\mathcal{C} \subset \mathcal{B} : \forall B \in \mathcal{B}\ \ \exists C \in \mathcal{C}\ \ B \subset C\}
\end{aligned}$$

are called the *additivity*, the *covering number* and the *cofinality* of $\mathcal{B}$, respectively. It is well-known (and easy to see) that $\operatorname{add}(\mathcal{B}) \le \operatorname{cov}(\mathcal{B}) \le \operatorname{cof}(\mathcal{B})$.

The following theorem was proved in [10, 1.4].

Theorem 3. *If the product $X \times Y$ of balleans X, Y is normal then*

$$\operatorname{add}(\mathcal{B}_X) = \operatorname{cof}(\mathcal{B}_X) = \operatorname{cof}(\mathcal{B}_Y) = \operatorname{add}(\mathcal{B}_Y).$$

Theorem 4. *Let X be the Cartesian product of a family $\mathcal{F}$ of metrizable balleans. Then the following statements are equivalent:*

1. X is metrizable;

2. X is normal;

3. All but finitely many balleans from $\mathcal{F}$ are bounded.

Proof. We need only to show (2) $\Rightarrow$ (3). Assume the contrary. Then there exists a family $(Y_n)_{n<\omega}$ of unbounded metrizable balleans such that the Cartesian product $Y = \prod_{n\in\omega} Y_n$ is normal. On the other hand, $\mathrm{add}(\mathcal{B}_Y) \le \mathrm{add}(\mathcal{B}_{Y_0}) = \aleph_0$ and a standard diagonal argument shows that $\mathrm{cof}(\mathcal{B}_Y) > \aleph_0$, contradicting Theorem 3. □

3. Bornological products

Let $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}$ be an indexed family of pointed balleans and let $\mathcal{B}$ be a bornology on the index set A. For each $\alpha \in A$ by e_α we denote the distinguished point of the ballean X_α.

The $\mathcal{B}$-*product* of the family of pointed balleans $\{X_\alpha : \alpha \in A\}$ is the set

$$X_{\mathcal{B}} = \big\{(x_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha : \{\alpha \in A : x_\alpha \neq e_\alpha\} \in \mathcal{B}\big\},$$

endowed with the coarse structure $\mathcal{E}_{\mathcal{B}}$, generated by the base consisting of the entourages

$$\big\{\big((x_\alpha)_{\alpha\in A}, (y_\alpha)_{\alpha\in A}\big) \in X_{\mathcal{B}} \times X_{\mathcal{B}} : \forall \alpha \in B \;\; (x_\alpha, y_\alpha) \in E_\alpha\big\}$$

where $B \in \mathcal{B}$ and $(E_\alpha)_{\alpha\in B} \in \prod_{\alpha\in B} \mathcal{E}_\alpha$.

For the bornology $\mathcal{B} = \mathcal{P}_A$ consisting of all subsets of the index set A, the $\mathcal{B}$-product $X_{\mathcal{B}}$ coincides with the Cartesian product $\prod_{\alpha\in A} X_\alpha$ of the coarse spaces $(X_\alpha, \mathcal{E}_\alpha)$.

If each X_α is the doubleton $\{0, 1\}$ with distnguished point $e_\alpha = 0$, then the $\mathcal{B}$-product is called the $\mathcal{B}$-*macrocube* on A. If $|A| = \omega$ and $\mathcal{B} = [A]^{<\omega}$, then we get the well-known Cantor macrocube, whose coarse characterization was given by Banakh and Zarichnyi in [2].

For relations between macrocubes and hyperballeans, see [3], [9].

Theorem 5. *Let $\mathcal{B}$ be a bornology on a set and let $X_{\mathcal{B}}$ be the $\mathcal{B}$-product of a family of unbounded metrizable pointed balleans. Then the following statements are equivalent:*

1. $X_{\mathcal{B}}$ is metrizable;

2. $X_{\mathcal{B}}$ is normal;

3. $|A| = \omega$ *and* $\mathcal{B} = [A]^{<\omega}$.

Proof. To see that (2) $\Rightarrow$ (3), repeat the proof of Theorem 4. □

Theorem 6. *Let $\mathcal{B}$ be a bornology on a set A and let $X_{\mathcal{B}}$ be the $\mathcal{B}$-product of a family $\{X_\alpha : \alpha \in A\}$ of bounded pointed balleans which are not singletons. The coarse space $X_{\mathcal{B}}$ is metrizable if and only if the bornology $\mathcal{B}$ has a countable base.*

Proof. Apply Theorem 1. □

Let X be a macrocube on a set A and Y be a macrocube on a set B, $A \cap B = \emptyset$. Then $X \times Y$ is a macrocube on $A \cup B$ and, by Theorem 3, $X \times Y$ needs not to be normal.

Question 1. *How can one detect whether a given macrocube is normal? Is a $\mathcal{B}$-macrocube on an infinite set A normal provided that $\mathcal{B} \neq \mathcal{P}_A$ is a maximal unbounded bornology on A?*

Let $\{X_n : n < \omega\}$ be a family of finite balleans, $\mathcal{B} = [\omega]^{<\omega}$. By [10], the $\mathcal{B}$-product of the family $\{X_n : n < \omega\}$ is coarsely equivalent to the Cantor macrocube.

Question 2. *Let $\{X_\alpha : \alpha \in A\}$ be a family of finite (bounded) pointed balleans and let $\mathcal{B}$ be a bornology on A. How can one detect whether a $\mathcal{B}$-product of $\{X_\alpha : \alpha \in A\}$ is coarsely equivalent to some macrocube?*

4. Bouquets

Let $\mathcal{B}$ be a bornology on a set A and let $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}$ be a family of pointed balleans. The subballean

$$\bigvee_{\alpha \in A} X_\alpha := \big\{(x_\alpha)_{\alpha \in A} \in X_{\mathcal{B}} : |\{\alpha \in A : x_\alpha \neq e_\alpha\}| \leq 1\big\}$$

of the $\mathcal{B}$-product $X_{\mathcal{B}}$ is called the $\mathcal{B}$*-bouquet* of the family $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}$. The point $e = (e_\alpha)_{\alpha \in A}$ is the distinguished point of the ballean $\bigvee_{\alpha \in A} X_\alpha$.

For every $\alpha \in A$ we identify the ballean X_α with the subballean $\{(x_\beta)_{\beta \in A} \in X_{\mathcal{B}} : \forall \beta \in A \setminus \{\alpha\} \;\; x_\beta = e_\beta\}$ of $\bigvee_{\alpha \in A} X_\alpha$. Under such identification $\bigvee_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} X_\beta$ and $X_\alpha \cap X_\beta = \{e\} = \{e_\alpha\} = \{e_\beta\}$ for any distinct indices $\alpha, \beta \in A$.

Applying Theorem 1, we can prove the following two theorems.

Theorem 7. *Let $\mathcal{B}$ be a bornology on a set A and let $\{X_\alpha : \alpha \in A\}$ be a family of unbounded pointed metrizable balleans. The $\mathcal{B}$-bouquet $\bigvee_{\alpha\in A} X_\alpha$ is metrizable if and only if $|A| = \omega$ and $\mathcal{B} = |A|^{<\omega}$.*

Theorem 8. *Let $\mathcal{B}$ be a bornology on a set A and let $\{X_\alpha : \alpha \in A\}$ be a family of bounded pointed balleans, which are not singletons. The $\mathcal{B}$-bouquet $\bigvee_{\alpha\in A} X_\alpha$ is metrizable if and only if the bornology $\mathcal{B}$ has a countable base.*

Theorem 9. *A bornological bouquet of any family of pointed normal balleans is normal.*

Proof. Let $\mathcal{B}$ be a bornology on a non-empty set A and X be the $\mathcal{B}$-bouquet of pointed normal balleans X_α, $\alpha \in A$. Given two disjoint asymptotically disjoint sets $Y, Z \subset X$, we shall construct a slowly oscillating function $f : X \to [0,1]$ such that $f(Y) \subset \{0\}$ and $f(Z) \subset \{1\}$. The definition of the coarse structure on the $\mathcal{B}$-bouquet ensures that for every $\alpha \in A$ the subsets $Y \cap X_\alpha$ and $Z \cap X_\alpha$ are asymptotically disjoint in the coarse space X_α, which is identified with the subspace $\{(x_\beta) \in X : \forall \beta \in A\setminus\{\alpha\}\ \ x_\beta = e_\beta\}$ of the $\mathcal{B}$-bouquet X. By the normality of X_α, there exists a slowly oscillating function $f_\alpha : X_\alpha \to [0,1]$ such that $f_\alpha(Y \cap X_\alpha) \subset \{0\}$ and $f_\alpha(Z \cap X_\alpha) \subset \{1\}$. Changing the value of f_α in the distinguished point e_α of X_α, we can assume that $f_\alpha(e_\alpha) = f_\beta(e_\beta)$ for any $\alpha, \beta \in A$. Then the function $f : X \to [0,1]$, defined by $f{\restriction}X_\alpha = f_\alpha$ for $\alpha \in A$ is slowly ascillating and has the desired property: $f(Y) \subset \{0\}$ and $f(Z) \subset \{1\}$. By Theorem 2, the ballean X is normal. □

5. Combs

Let $(X, \mathcal{E})$ be a ballean and A be a subset of X. Let $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}$ be a family of pointed balleans with the marked points $e_\alpha \in X_\alpha$ for $\alpha \in A$.

The bornology $\mathcal{B}_X$ of the ballean $(X, \mathcal{E})$ induces a bornology $\mathcal{B} := \{B \in \mathcal{B}_X : B \subset A\}$ on the set A. Let $\bigvee_{\alpha\in A} X_\alpha$ be the $\mathcal{B}$-bouquet of the family of pointed balleans $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}$, and let e we denote the distinguished point of the bouquet $\bigvee_{\alpha\in A} X_\alpha$.

For for every $\alpha \in A$ we identify the ballean X_α with the subballean $\{(x_\beta)_{\beta\in A} \in \bigvee_{\alpha\in A} X_\alpha : \forall \beta \in A \setminus \{\alpha\}\ \ x_\beta = e_\beta\}$ of $\bigvee_{\alpha\in A} X_\alpha$. Then $\bigvee_{\alpha\in A} X_\alpha = \bigcup_{\alpha\in A} X_\alpha$ and $X_\alpha \cap X_\beta = \{e\} = \{e_\alpha\} = \{e_\beta\}$ for any distinct indices $\alpha, \beta \in A$.

The suballean

$$X \coprod_{\alpha\in A} X_\alpha := (X \times \{e\}) \cup \bigcup_{\alpha\in A} (\{\alpha\} \times X_\alpha)$$

of the ballean $X \times \bigvee_{\alpha\in A} X_\alpha$ is called the *comb* with handle X and spines X_α, $\alpha \in A \subset X$. We shall identify the handle X and the spines X_α with the subsets $X \times \{e\}$ and $\{\alpha\} \times X_\alpha$ in the comb $X \perp\!\!\!\perp_{\alpha\in A} X_\alpha$. It can be shown that the comb $X \perp\!\!\!\perp_{\alpha\in A} X_\alpha$ carries the smallest coarse structure such that the identity inclusions of the balleans X and X_α, $\alpha \in A$, into $X \perp\!\!\!\perp_{\alpha\in A} X_\alpha$ are macrouniform.

Theorem 10. *The comb $X \perp\!\!\!\perp_{\alpha\in A} X_\alpha$ is metrizable if the balleans X and X_α, $\alpha \in A$, are metrizable, and for each bounded set $B \subset X$ the intersection $A \cap B$ is finite.*

Proof. Applying Theorem 7, we conclude that the bouquet $\bigvee_{\alpha\in A} X_\alpha$ is metrizable. Then the comb $X \perp\!\!\!\perp_{\alpha\in A} X_\alpha$ is metrizable being a subspace of the metrizable ballean $X \times \bigvee_{\alpha\in A} X_\alpha$. □

By analogy with Theorem 9 we can prove

Theorem 11. *The comb $X \perp\!\!\!\perp_{\alpha\in A} X_\alpha$ is normal if the balleans X and X_α, $\alpha \in A$, are normal.*

6. Coarse structures, determined by bornologies

Let $\mathcal{B}$ be a bornology on a set X. We say that a coarse structure $\mathcal{E}$ on X is *compatible* with $\mathcal{B}$ if $\mathcal{B}$ coincides with the bornology $\mathcal{B}_X$ of all bounded subsets of $(X, \mathcal{E})$.

The family of all coarse structures, compatible with a given bornology $\mathcal{B}$ has the smallest and largest elements $\Downarrow\!\mathcal{B}$ and $\Uparrow\!\mathcal{B}$.

The smallest coarse structure $\Downarrow\!\mathcal{B}$ is generated by the base consisting of the entourages $(B \times B) \cup \triangle_X$, where $B \in \mathcal{B}$.

The largest coarse structure $\Uparrow\!\mathcal{B}$ consists of all entourages $E \subseteq X \times X$ such that $E^{-1}[B] \cup E[B] \in \mathcal{B}$ for every $B \in \mathcal{B}$.

An unbounded ballean $(X, \mathcal{E})$ is called

- *discrete* if $\mathcal{E} = \Downarrow\!\mathcal{B}_X$,
- *ultradiscrete* if X is discrete and its bornology $\mathcal{B}_X$ is maximal by inclusion in the family of all unbounded bornologies on X;
- *maximal* if its coarse structure is maximal by inclusion in the family of all unbounded coarse structures on X;
- *relatively maximal* if $\mathcal{E} = \Uparrow\!\mathcal{B}_X$.

It can be shown that an unbounded ballean $(X, \mathcal{E})$ is discrete if and only if for every $E \in \mathcal{E}$ there exists a bounded set $B \subset X$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. In [10, Chapter 3] discrete balleans are called pseudodiscrete.

It is clear that each maximal ballean is relatively maximal. For maximal balleans, see [10, Chapter 10]. For any regular cardinal κ the ballean $(\kappa, \Uparrow[\kappa]^{<\kappa})$ is maximal.

Each ultradiscrete ballean is both discrete and relatively maximal.

A ballean $(X, \mathcal{E})$ is called *ultranormal* if X contains no two unbounded asymptotically disjoint subsets. By [10, Theorem 10.2.1], every unbounded subset of a maximal ballean is large, which implies that each maximal ballean is ultranormal. A discrete ballean is ultranormal if and only if it is ultradiscrete.

Example 1. *For every infinite set X, there exists a bornology $\mathcal{B}$ on X such that $\Downarrow\mathcal{B} = \Uparrow\mathcal{B}$ but the ballean $(X, \Downarrow\mathcal{B}) = (X, \Uparrow\mathcal{B})$ is not ultradiscrete. Consequently, the ballean $(X, \Downarrow\mathcal{B}) = (X, \Uparrow\mathcal{B})$ is discrete and relatively maximal but not ultranormal.*

Proof. By Theorem 3.1.6 [4], there are two free ultrafilters p, q on X such that for every function $f : X \to X$ and any $P \in p$ and $Q \in q$ we have $f(P) \notin q$ and $f(Q) \notin p$. We put $\mathcal{B} = \{B \subseteq X : B \notin p,\ B \notin q\}$ and note that $\mathcal{B}$ is a bornology on X.

To show that $\Downarrow\mathcal{B} = \Uparrow\mathcal{B}$, we need to check that for any entourage $E \in \Uparrow\mathcal{B}$, the set $Y = \{x \in X : E[x] \neq \{x\}\}$ belongs to the bornology $\mathcal{B}$. To derive a contradiction, assume that $Y \notin \mathcal{B}$. For every $x \in Y$ choose a point $f(x) \in E[x] \setminus \{x\}$. By Zorn's Lemma, there exists a maximal subset $Z \subset Y$ such that $Z \cap f(Z) = \emptyset$. By the maximality of Z, for any $y \in Y \setminus Z$ we get $f(y) \in Z$ and hence $f(Y \setminus Z) \subset Z$. It follows from $Y \notin \mathcal{B}$ that $Z \notin \mathcal{B}$ or $Y \setminus Z \notin \mathcal{B}$.

First assume that $Z \notin \mathcal{B}$. Then $Z \in p$ or $Z \in q$. Without loss of generality, $Z \in p$. Then $f(Z) \notin p$ and $f(Z) \notin q$ (by the choice of p, q). Consequently, $f(Z) \in \mathcal{B}$ and $Z \subset E^{-1}[f(Z)] \in \mathcal{B}$, which is a desired contradiction.

The case $Y \setminus Z \notin \mathcal{B}$ can be considered by analogy.

Since X can be written as the union $X = P \cup Q$ of two disjoint unbounded sets $P \in p$, $Q \in q$, the ballean $(X, \Uparrow\mathcal{B})$ is not ultradiscrete and not ultranormal. $\square$

By a *bornological space* we understand a pair $(X, \mathcal{B}_X)$ consisting of a set X and a bornology $\mathcal{B}_X$ on X. A bornological space $(X, \mathcal{B}_X)$ is *unbounded* if $X \notin \mathcal{B}_X$. For two bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$

their product is the bornological space $(X \times Y, \mathcal{B})$ endowed with the bornology

$$\mathcal{B}_{X\times Y} = \{B \subset X \times Y : B \subset B_X \times B_Y \text{ for some } B_X \in \mathcal{B}_X,\ B_Y \in \mathcal{B}_Y\}.$$

The following theorem allows us to construct many examples of bornological spaces $(X, \mathcal{B})$ for which the coarse space $(X, \Uparrow\mathcal{B})$ is not normal.

Theorem 12. *Let $(X \times Y, \mathcal{B})$ be the product of two unbounded bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$. If $\mathrm{cov}(\mathcal{B}_Y) < \mathrm{add}(\mathcal{B}_X)$, then the coarse space $(X \times Y, \Uparrow\mathcal{B})$ is not normal.*

Proof. Fix any point $(x_0, y_0) \in X \times Y$. Assuming that $\mathrm{cov}(\mathcal{B}_Y) < \mathrm{add}(\mathcal{B}_X)$, we shall prove that for a coarse structure $\mathcal{E}$ on $X \times Y$ is not normal if $\mathcal{E}$ has the following three properties:

1. $\mathcal{E}$ is compatible with the bornology $\mathcal{B}$;
2. for any $B_Y \in \mathcal{B}_Y$ there exists $E \in \mathcal{E}$ such that $X \times B_Y \subset E[X \times \{y_0\}]$;
3. for any $B_X \in \mathcal{B}_X$ there exists $E \in \mathcal{E}$ such that $B_X \times Y \subset E[\{x_0\} \times Y]$.

It is easy to see that the coarse structure $\Uparrow\mathcal{B}$ has these three properties.

By the definition of the cardinal $\kappa = \mathrm{cov}(\mathcal{B}_Y)$, there there is a family $\{Y_\alpha\}_{\alpha\in\kappa} \subset \mathcal{B}_Y$ such that $\bigcup_{\alpha\in\kappa} Y_\alpha = Y$.

Assume that $\mathcal{E}$ is a coarse structure on $X \times Y$ satisfying the conditions (1)–(3). First we check that the sets $X \times \{y_0\}$ and $\{x_0\} \times Y$ are asymptotically disjoint in $(X \times Y, \mathcal{E})$. Given any entourage $E \in \mathcal{E}$, we should prove that the intersection $E[X \times \{y_0\}] \cap E[\{x_0\} \times Y]$ is bounded. By the condition (1), for every $\alpha \in \kappa$ the bounded set $E^{-1}[E[\{x_0\} \times Y_\alpha]]$ is contained in the product $B_\alpha \times Y$ for some bounded set $B_\alpha \in \mathcal{B}_X$. Since $\kappa < \mathrm{add}(\mathcal{B}_X)$, the union $B_{<\kappa} := \bigcup_{\alpha\in\kappa} B_\alpha$ belongs to the bornology $\mathcal{B}_X$. Given any point $(u, v) \in E[X \times \{y_0\}] \cap E[\{x_0\} \times Y]$, find $x \in X$ and $y \in Y$ such that $(u, v) \in E[(x, y_0)] \cap E[(x_0, y)]$. Since $Y = \bigcup_{\alpha\in\kappa} Y_\alpha$, there exists $\alpha \in \kappa$ such that $y \in Y_\alpha$. Then $(x, y_0) \in E^{-1}[E[(x_0, y)]] \subset E^{-1}[E[\{x_0\} \times Y_\alpha]] \subset B_\alpha \times Y \subset B_{<\kappa} \times Y$ and hence $(u, v) \in E[(x, y_0)] \subset E[B_{<\kappa} \times \{y_0\}]$, which implies that the intersection

$$E[X \times \{y_0\}] \cap E[\{x_0\} \times Y] \subset E[B_{<\kappa} \times \{y_0\}]$$

is bounded in $(X \times Y, \mathcal{E})$.

Assuming that the coarse space $(X \times Y, \mathcal{E})$ is normal, we can find disjoint asymptotical neighborhoods U and V of the asymptotically disjoint sets $X \times \{y_0\}$ and $Y \times \{x_0\}$. By the condition (2), for every $\alpha \in \kappa$ there exists an entourage $E_\alpha \in \mathcal{E}$ such that $X \times Y_\alpha \subset E_\alpha[X \times \{y_0\}]$. Since U is an asymptotic neighborhood of the set $X \times \{y_0\}$ in $(X \times Y, \mathcal{E})$, the set $(X \times Y_\alpha) \setminus U \subset E_\alpha[X \times \{y_0\}] \setminus U$ is bounded in $(X \times Y, \mathcal{E})$. Now the condition (1) implies that $(X \times Y_\alpha) \setminus U \subset D_\alpha \times Y$ for some bounded set $D_\alpha \in \mathcal{B}_X$.

We claim that the family $\{D_\alpha\}_{\alpha\in\kappa}$ is cofinal in $\mathcal{B}_X$. Indeed, given any bounded set $D \in \mathcal{B}_X$, use the condition (3) and find a entourage $E \in \mathcal{E}$ such that $D \times Y \subset E[\{x_0\} \times Y]$. Since V is an asymptotic neighborhood of the set $\{x_0\} \times Y$, the set $E[\{x_0\} \times Y] \setminus V$ is bounded in $(X \times Y, \mathcal{E})$ and the condition (1) ensures that it has bounded projection onto Y. Since $Y \notin \mathcal{B}_Y$, we can find a point $y \in Y$ such that $X \times \{y\}$ is disjont with $E[\{x_0\} \times Y] \setminus V$. Find $\alpha \in \kappa$ with $y \in Y_\alpha$. Then $(X \times \{y\}) \cap E[\{x_0\} \times Y] \subset V$ and hence

$$D \times \{y\} \subset (X \times y\}) \cap E[\{x_0\} \times Y] \subset (X \times Y_\alpha) \cap V \subset (X \times Y_\alpha) \setminus U \subset D_\alpha \times Y,$$

which yields the desired inclusion $D \subset D_\alpha$. Therefore,

$$\mathrm{cof}(\mathcal{B}_X) \leq |\{D_\alpha\}_{\alpha\in\kappa}| \leq \kappa = \mathrm{cov}(\mathcal{B}_Y) < \mathrm{add}(\mathcal{B}_X),$$

which contradicts the known inequality $\mathrm{add}(\mathcal{B}_X) \leq \mathrm{cof}(\mathcal{B}_X)$. □

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