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Finite spaces pretangent to metric spaces at infinity

Viktoriia Bilet, Oleksiy Dovgoshey

Abstract. Let X be an unbounded metric space and $\tilde{r}$ be a sequence of positive real numbers tending to infinity. We define the pretangent space $\Omega^X_{\infty,\tilde{r}}$ to X at infinity as a metric space whose points are equivalence classes of sequences $\tilde{x} \subset X$ which tend to infinity with the speed of $\tilde{r}$. It is proved that all pretangent spaces are complete. We also prove that for every finite metric space Y there is an unbounded metric space X such that Y and $\Omega^X_{\infty,\tilde{r}}$ are isometric for some $\tilde{r}$. The finiteness conditions of $\Omega^X_{\infty,\tilde{r}}$ are completely described.

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1. Introduction

The pretangent spaces to an unbounded metric space (X, d) at infinity are by definition some limits of rescaling metric spaces $\left(X, \frac{1}{r_n}d\right)$ with r_n tending to infinity. The Gromov–Hausdorff convergence and the asymptotic cones are most often used for construction of such limits. Both of these constructions are based on high–order logic abstractions (see, for example, [23]), which makes them very powerful, but it does away the constructiveness. In this paper we consider a more constructive approach to building an asymptotic structure of unbounded metric spaces at infinity.

Let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers tending to infinity, $\lim\limits_{n\to\infty} r_n = \infty$. In what follows $\tilde{r}$ will be called a *scaling sequence*

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and, moreover, any formula of the form $(x_n)_{n\in\mathbb{N}} \subset A$ will mean that all elements of the sequence $(x_n)_{n\in\mathbb{N}}$ belong to the set A.

Let (X, d) be an unbounded metric space.

Definition 1.1. *Two sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ and $\tilde{y} = (y_n)_{n\in\mathbb{N}} \subset X$ are* mutually stable *with respect to the scaling sequence $\tilde{r}$ if there is a finite limit*

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{1.1}$$

Let $p \in X$. Denote by $Seq(X, \tilde{r})$ the set of all sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ for which $\lim\limits_{n\to\infty} d(x_n, p) = \infty$ and there are finite limits

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n} := \tilde{\tilde{d}}_{\tilde{r}}(\tilde{x}). \tag{1.2}$$

Definition 1.2. *A subset F of $Seq(X, \tilde{r})$ is* self-stable *if any two $\tilde{x}, \tilde{y} \in F$ are mutually stable. F is* maximal self-stable *if it is self-stable and, for arbitrary $\tilde{t} \in Seq(X, \tilde{r})$, we have either $\tilde{t} \in F$ or there is $\tilde{x} \in F$ such that $\tilde{x}$ and $\tilde{t}$ are not mutually stable,*

$$\liminf_{n\to\infty} \frac{d(x_n, t_n)}{r_n} < \limsup_{n\to\infty} \frac{d(x_n, t_n)}{r_n}.$$

The maximal self-stable subsets of $Seq(X, \tilde{r})$ will be denoted by $\tilde{X}_{\infty,\tilde{r}}$.

Remark 1.1. If $\tilde{x} \in Seq(X, \tilde{r})$ and p, $b \in X$, then, using the triangle inequality, we obtain

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n} = \lim_{n\to\infty} \frac{d(x_n, b)}{r_n}. \tag{1.3}$$

In particular, the set $Seq(X, \tilde{r})$ itself and its self-stable and maximal self-stable subsets are invariant under the choice of the point $p \in X$.

Let $\tilde{X}_{\infty,\tilde{r}}$ be a maximal self-stable subset of $Seq(X, \tilde{r})$. Consider the function $\tilde{d}\colon \tilde{X}_{\infty,\tilde{r}} \times \tilde{X}_{\infty,\tilde{r}} \to \mathbb{R}$ defined by (1.1). Obviously, $\tilde{d}$ is symmetric and nonnegative and $\tilde{d}(\tilde{x}, \tilde{x}) = 0$ holds for every $\tilde{x} \in \tilde{X}_{\infty,\tilde{r}}$. Moreover, the triangle inequality for d gives us the triangle inequality for $\tilde{d}$,

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y}).$$

Hence $(\tilde{X}_{\infty,\tilde{r}}, \tilde{d})$ is a pseudometric space.

Now we are ready to define the main object of our research.

Definition 1.3. *Let (X, d) be an unbounded metric space, let $\tilde{r}$ be a scaling sequence and let $\tilde{X}_{\infty,\tilde{r}}$ be a maximal self-stable subset of $Seq(X, \tilde{r})$. The* pretangent space *to (X, d) (at infinity, with respect to $\tilde{r}$) is the* metric identification *of the pseudometric space $(\tilde{X}_{\infty,\tilde{r}}, \tilde{d})$.*

Since the notion of pretangent space is basic for the paper, we recall the metric identification construction. Define a relation $\equiv$ on $Seq(X, \tilde{r})$ as

$$(\tilde{x} \equiv \tilde{y}) \Leftrightarrow \left(\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\right). \tag{1.4}$$

The reflexivity and the symmetry of $\equiv$ are evident. Let $\tilde{x}, \tilde{y}, \tilde{z} \in Seq(X, \tilde{r})$ and $\tilde{x} \equiv \tilde{y}$ and $\tilde{y} \equiv \tilde{z}$. Then the inequality

$$\limsup_{n\to\infty} \frac{d(x_n, z_n)}{r_n} \leq \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} + \lim_{n\to\infty} \frac{d(y_n, z_n)}{r_n}$$

implies $\tilde{x} \equiv \tilde{z}$. Thus $\equiv$ is an equivalence relation.

Write $\Omega^X_{\infty,\tilde{r}}$ for the set of equivalence classes generated by the restriction of $\equiv$ on the set $\tilde{X}_{\infty,\tilde{r}}$. Using general properties of pseudometric spaces we can prove (see, for example, [18]) that the function $\rho \colon \Omega^X_{\infty,\tilde{r}} \times \Omega^X_{\infty,\tilde{r}} \to \mathbb{R}$ with

$$\rho(\alpha, \beta) := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}), \quad \tilde{x} \in \alpha \in \Omega^X_{\infty,\tilde{r}}, \quad \tilde{y} \in \beta \in \Omega^X_{\infty,\tilde{r}}, \tag{1.5}$$

is a well-defined metric on $\Omega^X_{\infty,\tilde{r}}$. The metric identification of $(\tilde{X}_{\infty,\tilde{r}}, \tilde{d})$ is the metric space $(\Omega^X_{\infty,\tilde{r}}, \rho)$.

Let $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ be a strictly increasing sequence. Denote by $\tilde{r}'$ the subsequence $(r_{n_k})_{k\in\mathbb{N}}$ of the scaling sequence $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ and, for every $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$, write $\tilde{x}' := (x_{n_k})_{k\in\mathbb{N}}$. It is clear that we have

$$\{\tilde{x}' : \tilde{x} \in Seq(X, \tilde{r})\} \subseteq Seq(X, \tilde{r}')$$

and that $\tilde{\tilde{d}}_{\tilde{r}'}(\tilde{x}') = \tilde{\tilde{d}}_{\tilde{r}}(\tilde{x})$ holds for every $\tilde{x} \in Seq(X, \tilde{r})$. Furthermore, if some sequences $\tilde{x}$, $\tilde{y} \in Seq(X, \tilde{r})$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}'). \tag{1.6}$$

Consequently $\{\tilde{x}' \colon \tilde{x} \in \tilde{X}_{\infty,\tilde{r}}\}$ is a self-stable subset of $Seq(X, \tilde{r})$ for every $\tilde{X}_{\infty,\tilde{r}}$. By Zorn's lemma there is $\tilde{X}_{\infty,\tilde{r}'} \subseteq Seq(X, \tilde{r}')$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{\infty,\tilde{r}}\} \subseteq \tilde{X}_{\infty,\tilde{r}'}. \tag{1.7}$$

Denote by $\varphi_{\tilde{r}'}$ the mapping from $\tilde{X}_{\infty,\tilde{r}}$ to $\tilde{X}_{\infty,\tilde{r}'}$ with $\varphi_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$. It follows from (1.6) that, after corresponding metric identifications, the mapping $\varphi_{\tilde{r}'}$ passes to an isometric embedding $em'\colon \Omega^X_{\infty,\tilde{r}} \to \Omega^X_{\infty,\tilde{r}'}$ such that the diagram

$$\begin{array}{ccc} \tilde{X}_{\infty,\tilde{r}} & \xrightarrow{\varphi_{\tilde{r}'}} & \tilde{X}_{\infty,\tilde{r}'} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ \Omega^X_{\infty,\tilde{r}} & \xrightarrow{em'} & \Omega^X_{\infty,\tilde{r}'} \end{array} \tag{1.8}$$

is commutative. Here π and π' are the natural projections,

$$\pi(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{\infty,\tilde{r}}\colon \tilde{d}_{\tilde{r}}(\tilde{x},\tilde{y}) = 0\},\ \pi'(\tilde{t}) \quad := \{\tilde{y} \in \tilde{X}_{\infty,\tilde{r}'}\colon \tilde{d}_{\tilde{r}'}(\tilde{t},\tilde{y}) = 0\} \tag{1.9}$$

for $\tilde{x} \in \tilde{X}_{\infty,\tilde{r}}$ and $\tilde{t} \in \tilde{X}_{\infty,\tilde{r}'}$.

Definition 1.4. *Let (X,d) be an unbounded metric space and let $\tilde{r}$ be a scaling sequence. A pretangent $\Omega^X_{\infty,\tilde{r}}$ is* tangent *if $em'\colon \Omega^X_{\infty,\tilde{r}} \to \Omega^X_{\infty,\tilde{r}'}$ is surjective for every $\Omega^X_{\infty,\tilde{r}'}$.*

It is can be proved that the following statements are equivalent.

- The metric space $\Omega^X_{\infty,\tilde{r}}$ is tangent.
- The mapping $em' : \Omega^X_{\infty,\tilde{r}} \to \Omega^X_{\infty,\tilde{r}'}$ is an isometry for every $\Omega^X_{\infty,\tilde{r}'}$.
- The set $\{\tilde{x}' : \tilde{x} \in \tilde{X}_{\infty,\tilde{r}}\}$ is a maximal self-stable subset of the set $Seq(X,\tilde{r}')$ for every $\tilde{r}'$.
- The mapping $\varphi_{\tilde{r}'}\colon \tilde{X}_{\infty,\tilde{r}} \to \tilde{X}_{\infty,\tilde{r}'}$ is onto whenever $\tilde{X}_{\infty,\tilde{r}'} \supseteq \{\tilde{x}'\colon \tilde{x} \in \tilde{X}_{\infty,\tilde{r}}\}$.

In conclusion of this brief introduction we note that there exist other techniques which allow us to investigate the asymptotic properties of metric spaces at infinity. As examples, we mention only the Gromov product which can be used to define a metric structure on the boundaries of hyperbolic spaces [12, 24], the balleans theory [22] and the Wijsman convergence [20, 27, 28]. Moreover, it should be noted that some infinitesimal analogues of pretangent and tangent spaces were studied in [1–3, 7–9, 11, 13–16].

2. Distinguished point of pretangent spaces

Let (X,d) be an unbounded metric space and let $p \in X$. Denote by $\tilde{X}_\infty$ the set of sequences $\tilde{x} \subset X$ which satisfy the condition

$\lim_{n\to\infty} d(x_n, p) = \infty$. For every scaling sequence $\tilde{r}$ define the subset $\tilde{X}^0_{\infty,\tilde{r}}$ of $\tilde{X}_\infty$ as:

$$\left((z_n)_{n\in\mathbb{N}} \in \tilde{X}^0_{\infty,\tilde{r}}\right) \Leftrightarrow \left((z_n)_{n\in\mathbb{N}} \in \tilde{X}_\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{d(z_n, p)}{r_n} = 0\right). \tag{2.1}$$

It is clear that the set $\tilde{X}^0_{\infty,\tilde{r}}$ is invariant under replacing of $p \in X$ by an arbitrary point of X.

Proposition 2.1. [10] *Let (X, d) be an unbounded metric space, $p \in X$ and let $\tilde{r}$ be a scaling sequence. Then the following statements hold.*

1. *The set $\tilde{X}^0_{\infty,\tilde{r}}$ is nonempty.*
2. *If $\tilde{z} \in \tilde{X}^0_{\infty,\tilde{r}}$ and $\tilde{y} \in \tilde{X}_\infty$ and $\tilde{d}_{\tilde{r}}(\tilde{z}, \tilde{y}) = 0$, then $\tilde{y} \in \tilde{X}^0_{\infty,\tilde{r}}$ holds.*
3. *If $F \subseteq Seq(X, \tilde{r})$ is self-stable, then $\tilde{X}^0_{\infty,\tilde{r}} \cup F$ is also self-stable. In particular $\tilde{X}^0_{\infty,\tilde{r}}$ is self-stable.*
4. *The inclusion $\tilde{X}^0_{\infty,\tilde{r}} \subseteq \tilde{X}_{\infty,\tilde{r}}$ holds for every $\tilde{X}_{\infty,\tilde{r}}$.*
5. *Let $\tilde{z} \in \tilde{X}^0_{\infty,\tilde{r}}$ and $\tilde{y} \in Seq(X, \tilde{r})$ and $\tilde{x} \in \tilde{X}_\infty$. Then*

   $$\tilde{\tilde{d}}_{\tilde{r}}(\tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{z})$$

   holds and we have $\tilde{x} \in Seq(X, \tilde{r})$ if and only if $\tilde{x}$ and $\tilde{z}$ are mutually stable.
6. *Denote by $\mathbf{\Omega}^{\mathbf{X}}_{\infty,\tilde{\mathbf{r}}}$ the set of all pretangent to X at infinity (with respect to $\tilde{r}$) spaces. Then the membership relation*

   $$\tilde{X}^0_{\infty,\tilde{r}} \in \bigcap_{\Omega^X_{\infty,\tilde{r}} \in \mathbf{\Omega}^{\mathbf{X}}_{\infty,\tilde{\mathbf{r}}}} \Omega^X_{\infty,\tilde{r}}$$

   holds.

The set $\tilde{X}^0_{\infty,\tilde{r}}$ is a common point of all pretangent spaces $\Omega^X_{\infty,\tilde{r}}$ (with given scaling sequence $\tilde{r}$). We may consider the pretangent spaces to (X, d) at infinity as the triples $(\Omega^X_{\infty,\tilde{r}}, \rho, \nu_0)$, where ρ is defined by (1.5) and $\nu_0 := \tilde{X}^0_{\infty,\tilde{r}}$. The point ν_0 can be informally described as follows. The points of the pretangent space $\Omega^X_{\infty,\tilde{r}}$ are infinitely removed from the initial space (X, d), but $\Omega^X_{\infty,\tilde{r}}$ contains a unique point ν_0 which is close to (X, d) as much as possible.

Example 2.1. Let $X = [0, \infty)$ and let d be the standard metric on X, $d(x, y) = |x - y|$. Then, for every scaling sequence $\tilde{r}$, there is the unique maximal self-stable family $\tilde{X}_{\infty,\tilde{r}}$. A sequence $\tilde{x} \in \tilde{X}_{\infty}$ belongs to $\tilde{X}_{\infty,\tilde{r}}$ if and only if there is a finite limit

$$\tilde{d}_{\tilde{r}}(\tilde{x}) = \lim_{n\to\infty} \frac{x_n}{r_n}.$$

In particular, $\tilde{x} \in \tilde{X}^0_{\infty,\tilde{r}}$ if and only if $x_n = o(r_n)$, where "o" is the corresponding Landau symbol. Note also that the pretangent space $\Omega^X_{\infty,\tilde{r}}$ is tangent and isometric to $[0, \infty)$. A proof of these simple statements can be obtained by appropriate modification of the proofs of Proposition 2.4 and Proposition 2.6 from [4].

3. Completeness of pretangent spaces

It is well know that the Gromov–Hausdorff limits and the asymptotic cones of metric spaces are always complete. The quasi-metrics on the boundaries of hyperbolic spaces are also complete (see, for example, Proposition 6.1 in [24]). The goal of this section is to show that every pretangent space is complete. For the proof of this fact we shall use several lemmas.

The following lemma can be found in [10].

Lemma 3.1. *Let (X, d) be an unbounded metric space, $p \in X$ and $\tilde{y} \in \tilde{X}_{\infty}$, let $\tilde{r}$ be a scaling sequence and let $\tilde{X}_{\infty,\tilde{r}}$ be a maximal self-stable set. If $\tilde{y}$ and $\tilde{x}$ are mutually stable for every $\tilde{x} \in \tilde{X}_{\infty,\tilde{r}}$, then $\tilde{y} \in \tilde{X}_{\infty,\tilde{r}}$.*

Lemma 3.2. *Let (X, d) be an unbounded metric space, $\tilde{r}$ be a scaling sequence, $\tilde{X}_{\infty,\tilde{r}}$ be maximal self-stable, $\tilde{x} \in \tilde{X}_{\infty}$ and let $(\tilde{\gamma}^m)_{m\in\mathbb{N}} \subset \tilde{X}_{\infty,\tilde{r}}$ such that $\tilde{\gamma}^m$ and $\tilde{x}$ are mutually stable for every $m \in \mathbb{N}$ and let*

$$\lim_{m\to\infty} \tilde{d}(\tilde{x}, \tilde{\gamma}^m) = 0. \tag{3.1}$$

Then $\tilde{x}$ belongs to $\tilde{X}_{\infty,\tilde{r}}$.

Proof. By Lemma 3.1 $\tilde{x} \in \tilde{X}_{\infty,\tilde{r}}$ if and only if for every $\tilde{y} \in \tilde{X}_{\infty,\tilde{r}}$ there is a finite limit

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}. \tag{3.2}$$

Let $\tilde{y} \in \tilde{X}_{\infty,\tilde{r}}$. It follows from the triangle inequality for $\tilde{d}$ that

$$|\tilde{d}(\tilde{y}, \tilde{\gamma}^{m_1}) - \tilde{d}(\tilde{y}, \tilde{\gamma}^{m_2})| \leq \tilde{d}(\tilde{\gamma}^{m_1}, \tilde{\gamma}^{m_2}) \leq \tilde{d}(\tilde{x}, \tilde{\gamma}^{m_1}) + \tilde{d}(\tilde{x}, \tilde{\gamma}^{m_2}) \tag{3.3}$$

for all $m_1, m_2 \in \mathbb{N}$. Now (3.1) and (3.3) imply that $(\tilde{d}(\tilde{y}, \tilde{\gamma}^m))_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Consequently, there is a finite limit $\lim\limits_{m\to\infty} \tilde{d}(\tilde{y}, \tilde{\gamma}^m)$. We claim that limit (3.2) exists and

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = \lim_{m\to\infty} \tilde{d}(\tilde{y}, \tilde{\gamma}^m). \tag{3.4}$$

This statement holds if and only if

$$\limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = \lim_{m\to\infty} \tilde{d}(\tilde{y}, \tilde{\gamma}^m) \tag{3.5}$$

and

$$\liminf_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = \lim_{m\to\infty} \tilde{d}(\tilde{y}, \tilde{\gamma}^m). \tag{3.6}$$

Equality (3.5) holds if and only if

$$\lim_{m\to\infty} \left| \tilde{d}(\tilde{y}, \tilde{\gamma}^m) - \limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \right| = 0.$$

It is clear that

$$\lim_{m\to\infty} \left| \tilde{d}(\tilde{y}, \tilde{\gamma}^m) - \limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n}) \right|$$

$$= \lim_{m\to\infty} \limsup_{n\to\infty} \left| \frac{d(y_n, \gamma_n^m)}{r_n} - \frac{d(x_n, y_n)}{r_n} \right|$$

$$\leq \lim_{m\to\infty} \limsup_{n\to\infty} \frac{d(\gamma_n^m, x_n)}{r_n} = \lim_{m\to\infty} \tilde{d}(\tilde{\gamma}^m, \tilde{x}) = 0,$$

where $(\gamma_n^m)_{n\in\mathbb{N}} = \tilde{\gamma}^m$. Equality (3.5) follows. Equality (3.6) can be proved similarly. ☐

Lemma 3.3. *Let (X, d) be an unbounded metric space and let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence. Then for every $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ there is $\tilde{y} = (y_n)_{n\in\mathbb{N}} \in \tilde{X}_\infty$ such that $\lim\limits_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = 0$.*

Proof. Let $\tilde{z} = (z_n)_{n\in\mathbb{N}} \in \tilde{X}^0_{\infty,\tilde{r}}$ and let $p \in X$. For every $\tilde{x} \subset X$ define a sequence $\tilde{y} = (y_n)_{n\in\mathbb{N}} \subset X$ by the rule

$$y_n := \begin{cases} x_n, & \text{if } d(x_n, p) \geq d(z_n, p) \\ z_n, & \text{if } d(x_n, p) < d(z_n, p). \end{cases} \tag{3.7}$$

It follows from (3.7) that the inequality

$$d(y_n, p) \geq d(z_n, p) \tag{3.8}$$

holds for every $n \in \mathbb{N}$. Since we have $\tilde{z} \in \tilde{X}^0_{\infty,\tilde{r}} \subset \tilde{X}_\infty$, inequality (3.8) implies $\tilde{y} = (y_n)_{n\in\mathbb{N}} \in \tilde{X}_\infty$. Moreover, from (3.8) we also have

$$0 \le d(x_n, y_n) \le 2d(z_n, p). \tag{3.9}$$

The equality

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = 0$$

follows from (3.9) and $\lim\limits_{n\to\infty} \frac{d(z_n,p)}{r_n} = 0$. □

Theorem 3.1. *Let (X, d) be an unbounded metric space. Then all pretngent spaces to (X, d) at infinity are complete.*

Proof. Let $p \in X$, let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence and let $\tilde{X}_{\infty,\tilde{r}}$ be a maximal self-stable set with metric identification $(\Omega^X_{\infty,\tilde{r}}, \rho)$. The metric space $(\Omega^X_{\infty,\tilde{r}}, \rho)$ is complete if and only if the pseudometric space $(\tilde{X}_{\infty,\tilde{r}}, \tilde{d})$ is complete, i.e., for every Cauchy sequence $(\tilde{\gamma}^m)_{m\in\mathbb{N}} \subset \tilde{X}_{\infty,\tilde{r}}$ there is $\tilde{x} = (x_n)_{n\in\mathbb{N}} \in \tilde{X}_{\infty,\tilde{r}}$ such that

$$\lim_{m\to\infty} \tilde{d}(\tilde{x}, \tilde{\gamma}^m) = 0. \tag{3.10}$$

By Lemma 3.2 if (3.10) holds with some $\tilde{x} \in \tilde{X}_\infty$, then $\tilde{x} \in \tilde{X}_{\infty,\tilde{r}}$. Let $(\tilde{\gamma}^m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(\tilde{X}_{\infty,\tilde{r}}, \tilde{d})$. We first find $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ for which (3.10) holds. Then, using Lemma 3.3, we obtain $\tilde{x} \in \tilde{X}_\infty$ satisfying (3.10). Let $(\varepsilon)_{k\in\mathbb{N}} \subset (0, \infty)$ be a decreasing sequence such that

$$\sum_{k=1}^{\infty} \varepsilon_k < \infty. \tag{3.11}$$

There is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ for which

$$\tilde{d}(\tilde{\gamma}^m, \tilde{\gamma}^{m_k}) \le \varepsilon_k$$

holds whenever $m \ge m_k$. Now we construct $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ such that $\tilde{x}$ and $\tilde{\gamma}^{m_k}$ are mutually stable for every $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \tilde{d}(\tilde{\gamma}^{m_k}, \tilde{x}) = 0. \tag{3.12}$$

For every $m \in \mathbb{N}$, we set $\tilde{\gamma}^m := (\gamma^m_n)_{n\in\mathbb{N}}$. Let $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $\tilde{\beta}^k = (\beta^k_n)_{n\in\mathbb{N}} \in \tilde{X}_\infty$ be inductively defined by the rule: if $k = 1$, then $N_1 = 1$

and $(\beta_n^1)_{n\in\mathbb{N}} := (\gamma_n^{m_1})_{n\in\mathbb{N}}$; if $k \geq 2$, then N_k is the smallest $l \in \mathbb{N}$ which satisfies the inequality $l > N_{k-1}$ and

$$\beta_n^k := \begin{cases} \beta_n^1 & \text{if } N_1 \leq n < N_2, \\ \beta_n^2 & \text{if } N_2 \leq n < N_1, \\ \cdots & \cdots\cdots\cdots\cdots, \\ \beta_n^{k-1} & \text{if } N_{k-1} \leq n < N_k, \\ \gamma_n^{m_k} & \text{if } n \geq N_k. \end{cases} \tag{3.13}$$

Define $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ as

$$x_n := \beta_n^k \quad \text{for} \quad n \in [N_k, N_{k+1}), k = 1, 2, 3, \ldots. \tag{3.14}$$

It follows from (3.12) and (3.13) that

$$\lim_{k\to\infty} \frac{d(\beta_n^k, \gamma_n^{m_k})}{r_n} = 0 \tag{3.15}$$

for every $k \in \mathbb{N}$, and

$$\frac{d(\beta_n^k, \beta_n^{k-1})}{r_n} < 2\varepsilon_k \tag{3.16}$$

for all $n, k \in \mathbb{N}$. Limit relation (3.15) implies that (3.12) holds if and only if

$$\lim_{k\to\infty} \limsup_{n\to\infty} \left(\frac{d(x_n, \beta_n^k)}{r_n} \right) = 0. \tag{3.17}$$

Using (3.14), we see that for every $n \in \mathbb{N}$ there is $K(n) \in \mathbb{N}$ such that

$$\frac{d(x_n, \beta_n^k)}{r_n} = \frac{d(\beta_n^{K(n)}, \beta_n^k)}{r_n}.$$

If k is given, then, for sufficiently large n, the inequality $K(n) > k$ holds. Consequently by (3.16) we have

$$\frac{d(\beta_n^{K(n)}, \beta_n^k)}{r_n} \leq \sum_{i=0}^{K(n)-k} \frac{d(\beta_n^{k+i+1}, \beta_n^{k+i})}{r_n} \leq 2\sum_{i=k}^{\infty} \varepsilon_i. \tag{3.18}$$

Inequalities (3.11) and (3.18) imply (3.17). □

4. When are the pretangent spaces finite?

The main goals of this section are to find conditions under which all pretangent spaces to a given metric space are finite and to show that any finite metric space is isometric to a pretangent space.

Lemma 4.1. *Let (X,d) be an unbounded metric space. Then there exists a pretangent space $\Omega^X_{\infty,\tilde r}$ such that $\left|\Omega^X_{\infty,\tilde r}\right| \geq 2$.*

Proof. Let $\tilde x = (x_n)_{n\in\mathbb{N}} \in \tilde X_\infty$ and let $p \in X$. Define a sequence $\tilde r = (r_n)_{n\in\mathbb{N}}$ as:

$$r_n := \begin{cases} d(x_n,p) & \text{if } x_n \neq p \\ 1 & \text{if } x_n = p. \end{cases}$$

From $\tilde x \in \tilde X_\infty$ it follows that $\lim_{n\to\infty} r_n = \infty$. Hence $\tilde r$ is a scaling sequence. It is clear that $\tilde{\tilde d}_{\tilde r}(\tilde x) = 1$. Consequently, $Seq(X,\tilde r)$ contains $\tilde x$. There is $\tilde X_{\infty,\tilde r} \subseteq Seq(X,\tilde r)$ such that $\tilde x \in \tilde X_{\infty,\tilde r}$. Let $\Omega^X_{\infty,\tilde r}$ be the metric identification of $\tilde X_{\infty,\tilde r}$. Then the inequality $|\Omega^X_{\infty,\tilde r}| \geq 2$ holds. □

The following lemma is an analog of Lemma 5 from [1].

Lemma 4.2. *Let (X,d) be an unbounded metric space, $p \in X$, let $\mathfrak{B}$ be a countable subset of $\tilde X_\infty$ and let $\tilde r = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence. Suppose that*

$$\limsup_{n\to\infty} \frac{d(b_n,p)}{r_n} < \infty \tag{4.1}$$

holds for every $\tilde b = (b_n)_{n\in\mathbb{N}} \in \mathfrak{B}$. Then there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the family

$$\mathfrak{B}' := \{\tilde b' = (b_{n_k})_{k\in\mathbb{N}} : \tilde b \in \mathfrak{B}\}$$

is self-stable at infinity with respect to $\tilde r' = (r_{n_k})_{k\in\mathbb{N}}$.

Proof. It is sufficient to consider the case when $\mathfrak{B}$ is countably infinite. Then the set of all ordered pairs $(\tilde b, \tilde x) \in \mathfrak{B}^2$ can be enumerated as $(\tilde b^1, \tilde x^1), (\tilde b^2, \tilde x^2), \ldots$. The triangle inequality and (4.1) imply

$$\sup_{n\in\mathbb{N}} \frac{d(b^j_n, x^j_n)}{r_n} < \infty$$

for each pair $(\tilde b^j, \tilde x^j) \in \mathfrak{B}^2$, $\tilde b^j = (b^j_n)_{n\in\mathbb{N}}$ and $\tilde x^j = (x^j_n)_{n\in\mathbb{N}}$. In particular, we have

$$\sup_{n\in\mathbb{N}} \frac{d(b^1_n, x^1_n)}{r_n} < \infty.$$

Since every bounded, infinite sequence contains a convergent subsequence, there is a strictly increasing sequence $\tilde n^1 = (n^1_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{d(b^1_{n^1_k}, x^1_{n^1_k})}{r_{n^1_k}}, \quad \lim_{k\to\infty} \frac{d(x^1_{n^1_k}, p)}{r_{n^1_k}} \quad \text{and} \quad \lim_{k\to\infty} \frac{d(b^1_{n^1_k}, p)}{r_{n^1_k}}$$

are finite. Hence, the sequences $(b^1_{n^1_k})_{k\in\mathbb{N}}$ and $(x^1_{n^1_k})_{k\in\mathbb{N}}$ are mutually stable with respect to $(r_{n^1_k})_{k\in\mathbb{N}}$. Analogously, by induction, we can prove that for every integer $i \geq 2$ there is a subsequence $\tilde{n}_i = (n^i_k)_{k\in\mathbb{N}}$ of sequence $\tilde{n}_{i-1}$ such that $(b^i_{n^i_k})_{k\in\mathbb{N}}$ and $(x^i_{n^i_k})_{k\in\mathbb{N}}$ are mutually stable with respect to $(r_{n^i_k})_{k\in\mathbb{N}}$. Using Cantor's diagonal construction, write $\tilde{r}' := (r_{n^k_k})_{k\in\mathbb{N}}$ and, for every $\tilde{b} = (b_n)_{n\in\mathbb{N}} \in \mathfrak{B}$, define $\tilde{b}'$ as $\tilde{b}' := (b_{n^k_k})_{k\in\mathbb{N}}$. Then the family $\mathfrak{B}' := \{\tilde{b}' : \tilde{b} \in \mathfrak{B}\}$ is self-stable at infinity with respect to $\tilde{r}'$. □

Let (X, d) be an unbounded metric space and let p be a point of X. Denote by X^n the set of all n-tuples $x = (x_1, \dots, x_n)$ with $x_k \in X$ for $k = 1, \dots, n$, $n \geq 2$ and define the function $F_n : X^n \to \mathbb{R}$ as

$$F_n(x_1,\dots,x_n) := \begin{cases} 0 & \text{if } (x_1,\dots,x_n) = (p,\dots,p) \\ \dfrac{\min\limits_{1\le k\le n} d(x_k,p) \prod\limits_{1\le k<l\le n} d(x_k,x_l)}{\left(\max\limits_{1\le k\le n} d(x_k,p)\right)^{\frac{n(n-1)}{2}+1}} & \text{otherwise.} \end{cases} \tag{4.2}$$

Theorem 4.1. *Let (X, d) be an unbounded metric space and let $n \geq 2$ be an integer number. Then the inequality*

$$\left|\Omega^X_{\infty,\tilde{r}}\right| \le n \tag{4.3}$$

holds for every $\Omega^X_{\infty,\tilde{r}}$ if and only if

$$\lim_{x_1,\dots,x_n\to\infty} F_n(x_1,\dots,x_n) = 0. \tag{4.4}$$

Proof. Let (4.3) hold for all pretangent spaces $\Omega^X_{\infty,\tilde{r}}$. Suppose $\tilde{x}^i = (x^i_m)_{m\in\mathbb{N}} \in \tilde{X}_\infty$, $i = 1, \dots, n$ such that

$$\lim_{m\to\infty} F_n(x^1_m,\dots,x^n_m) = \limsup_{x_1,\dots,x_n\to\infty} F_n(x_1,\dots,x_n) > 0. \tag{4.5}$$

If $\tilde{r} = (r_m)_{m\in\mathbb{N}}$ is a scaling sequence with

$$r_m = \max\{1, d(x^1_m,p),\dots,d(x^n_m,p)\}$$

for every $m \in \mathbb{N}$, where p is a point of X in definition (4.2), then the inequality

$$\limsup_{m\to\infty} \frac{d(x^k_m,p)}{r_m} \le 1$$

holds for every $k \in \{1, \dots, n\}$. Using Lemma 4.2 we may suppose that the family $\{\tilde{x}^1, \dots, \tilde{x}^n\}$ is self-stable with respect to $\tilde{r}$. Now (4.5) and (4.2) imply

$$\tilde{\tilde{d}}_{\tilde{r}}(\tilde{x}^k) > 0 \quad \text{and} \quad \tilde{d}_{\tilde{r}}(\tilde{x}^k, \tilde{x}^j) > 0$$

for all distinct $k, j \in \{1, \dots, n\}$. Adding $\tilde{z} \in \tilde{X}^0_{\infty,\tilde{r}}$ to the family $\{\tilde{x}^1, \dots, \tilde{x}^n\}$ we see that the family $\{\tilde{z}, \tilde{x}^1, \dots, \tilde{x}^n\}$ is self-stable by statement (iii) of Proposition 2.1. Consequently there is $\Omega^X_{\infty,\tilde{r}}$ with $\left|\Omega^X_{\infty,\tilde{r}}\right| \geq n+1$, contrary to (4.3). Equality (4.4) follows.

To prove the converse statement it suffices to consider some different $n+1$ points $\nu_0, \nu_1, \dots, \nu_n \in \Omega^X_{\infty,\tilde{r}}$ such that $\nu_0 = \tilde{X}^0_{\infty,\tilde{r}}$, (see (1.8)). Then, for the sequences $\tilde{x}^1, \dots, \tilde{x}^n$ with

$$\pi(\tilde{x}^k) = \nu_k, \ \tilde{x}^k = (x^k_m)_{m \in \mathbb{N}}, \ k \in \{1, \dots, n\},$$

we obtain

$$\lim_{m \to \infty} F_n(x^1_m, \dots, x^n_m) = \frac{\min\limits_{1 \leq k \leq n} \rho(\nu_0, \nu_k) \prod\limits_{1 \leq k < l \leq n} \rho(\nu_k, \nu_l)}{\left(\max\limits_{1 \leq k \leq n} \rho(\nu_k, \nu_0)\right)^{\frac{n(n-1)}{2}+1}} \neq 0. \qquad (4.6)$$

□

Corollary 4.1. *Let (X, d) be an unbounded metric space and let $n \geq 2$ be an integer number. Suppose $\lim_{x_1, \dots, x_n \to \infty} F_n(x_1, \dots, x_n) = 0$ holds. Then every pretangent space $\Omega^X_{\infty,\tilde{r}}$ with $|\Omega^X_{\infty,\tilde{r}}| = n$ is tangent.*

In what follows $\mathbb{C}$ is the complex plane.

Example 4.1. Let $\tilde{r} = (r_m)_{m \in \mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that

$$\lim_{m \to \infty} \frac{r_{m+1}}{r_m} = \infty,$$

let $n \geq 2$ be an integer number, let $R_i = \{z \in \mathbb{C} \colon \arg z = \theta_i\}$ be the ray starting at origin with the angle of $\theta_i = \frac{\pi i}{2n}$ with the positive real axis, $i = 0, \dots, n-1$ and let

$$C_m = \{z \in \mathbb{C} \colon |z| = r_m\}$$

be the circle in $\mathbb{C}$ with the radius r_m, $m \in \mathbb{N}$, and the center 0. Write

$$X_n := \left(\bigcup_{i=0}^{n-1} R_i\right) \bigcap \left(\bigcup_{m=1}^{\infty} C_m\right)$$

and define the distance function d on X_n as

$$d(z, w) = |z - w|$$

(see Figure 1 for X_n with $n = 3$).

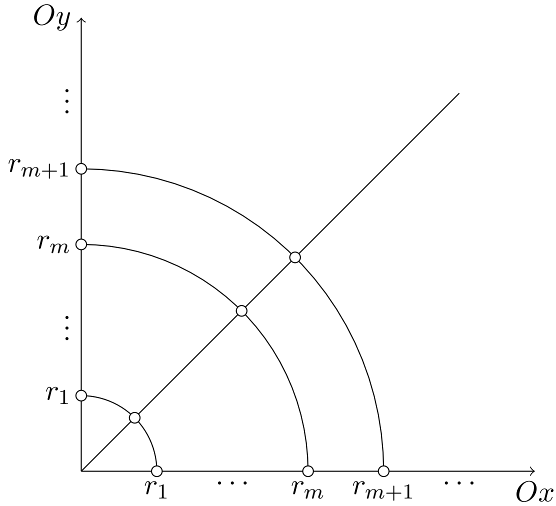

Figure 1: The graphical representation of X_3. The points of X_3 are depicted here as small circles.

Then we obtain

$$\lim_{\substack{x_1,\dots,x_{n+1}\to\infty \\ x_1,\dots,x_{n+1}\in X_n}} F_{n+1}(x_1, \dots, x_{n+1}) = 0 < \limsup_{\substack{x_1,\dots,x_n\to\infty \\ x_1,\dots,x_n\in X_n}} F_n(x_1, \dots, x_n).$$

In particular, for $n = 3$, the equality

$$\limsup_{\substack{x,y,z\to\infty \\ x,y,z\in X_3}} F_3(x, y, z) = 2\sqrt{2} - 2$$

holds. (See Figure 2 for all possible pretangent spaces with $n = 3$.)

The following theorem directly follows from Theorem 4.7 [10].

Theorem 4.2. *For every finite nonempty metric space (Y, δ) with $|Y| \geq 2$ and every $y^* \in Y$ there are an unbounded metric space (X, d), and a scaling sequence $\tilde{r}$ and an isometry $f\colon Y \to \Omega^X_{\infty,\tilde{r}}$ such that $f(y^*) = \nu_0$ holds and $\Omega^X_{\infty,\tilde{r}}$ is tangent, and $\nu_0 = \tilde{X}^0_{\infty,\tilde{r}}$.*

This theorem remains valid if $|Y| = 1$ that follows from Theorem 5.1 of the next section of the paper. Theorem 4.1 has no direct generalizations to the case of infinite Y even if (Y, δ) is complete and countable. (See Example 6.1 in the last section of the paper.)

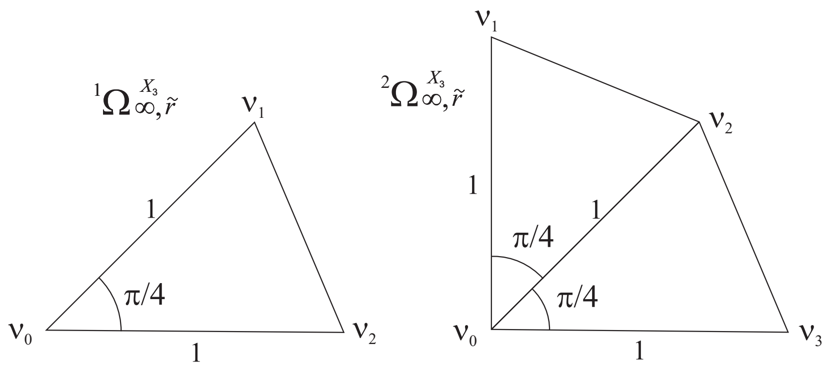

Figure 2: Every pretangent space $\Omega^{X_3}_{\infty,\tilde r}$ (with respect to $\tilde r$ given above) is isometric either $^1\Omega^{X_3}_{\infty,\tilde r}$ or $^2\Omega^{X_3}_{\infty,\tilde r}$.

5. Finite tangent spaces and strong porosity at infinity.

Theorem 4.1 gives, in particular, a condition guaranteeing the finiteness of all pretangent spaces. The goal of present section is to obtain the *existence conditions* for finite tangent spaces.

Let (X, d) be an unbounded metric space and let $p \in X$. The finiteness of $\Omega^X_{\infty,\tilde r}$ is closely connected with a porosity of the set

$$Sp(X) := \{d(x, p) : x \in X\}$$

at infinity.

Definition 5.1. *Let $E \subseteq \mathbb{R}^+$. The* porosity *of E at infinity is the quantity*

$$p^+(E, \infty) := \limsup_{h\to\infty} \frac{l(\infty, h, E)}{h}, \tag{5.1}$$

where $l(\infty, h, E)$ is the length of the longest interval in the set $[0, h] \setminus E$. The set E is strongly porous at infinity *if $p^+(E, \infty) = 1$ and, respectively, E is* nonporous at infinity *if $p^+(E, \infty) = 0$.*

The standard definition of porosity at a finite point can be found in [26]. See [4, 6, 8, 9] for some applications of porosity to studies of infinitesimal properties of metric spaces.

Lemma 5.1. *Let $E \subseteq \mathbb{R}^+$ and let $p_1 \in (0, 1)$. If the double inequality*

$$p^+(E, \infty) < p_1 < 1 \tag{5.2}$$

holds, then for every infinite, strictly increasing sequence of real numbers r_n with $\lim_{n\to\infty} r_n = \infty$ there is a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ such that for every

$k \in \mathbb{N}$ *there are points* $x_1^{(k)}, \ldots, x_k^{(k)} \in E$ *which satisfy the inequalities*

$$\begin{aligned} r_{n_k} < \frac{r_{n_k}}{1-p_1} \leq x_1^{(k)} \leq \frac{r_{n_k}}{(1-p_1)^2} \\ < \frac{r_{n_k}}{(1-p_1)^3} \leq x_2^{(k)} \leq \frac{r_{n_k}}{(1-p_1)^4} \\ \ldots\ldots\ldots\ldots\ldots \\ < \frac{r_{n_k}}{(1-p_1)^{2k-1}} \leq x_k^{(k)} \leq \frac{r_{n_k}}{(1-p_1)^{2k}} < r_{n_{k+1}}. \end{aligned} \tag{5.3}$$

Proof. Suppose that (5.2) holds. Let n_1 be the first natural number such that $l(\infty, h, E) < p_1 h$ for all $h \in (r_{n_1}, \infty)$. If $r_{n_1}, \ldots, r_{n_k}$ are defined, then write n_{k+1} for the first m with

$$r_m > (1-p_1)^{-2k} r_{n_k}.$$

It is easy to show that the equality

$$\frac{r_{n_k}}{(1-p_1)^m} - \frac{r_{n_k}}{(1-p_1)^{m-1}} = p_1 \cdot \frac{r_{n_k}}{(1-p_1)^m}$$

holds for all $m \in \mathbb{N}$. Using the last equality, Definition 5.1 and inequality (5.2) we obtain

$$E \cap \left[\frac{r_{n_k}}{(1-p_1)^m}, \frac{r_{n_k}}{(1-p_1)^{m+1}}\right] \neq \varnothing$$

for all $m \in \mathbb{N}$. Hence, there are points $x_1^{(k)}, \ldots, x_k^{(k)} \in E$ which satisfy (5.3). □

Theorem 5.1. *Let (X, d) be an unbounded metric space, $p \in X$. The following statements are equivalent:*

1. *The set $Sp(X)$ is strongly porous at infinity;*
2. *There is a single-point, tangent space $\Omega^X_{\infty,\tilde{r}}$;*
3. *There is a finite tangent space $\Omega^X_{\infty,\tilde{r}}$;*
4. *There is a compact tangent space $\Omega^X_{\infty,\tilde{r}}$;*
5. *There is a bounded, separable tangent space $\Omega^X_{\infty,\tilde{r}}$.*

Proof. $1 \Rightarrow 2$. Suppose the equality

$$p^+(Sp(X), \infty) = 1$$

holds. Let $\tilde{h} = (h_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers such that

$$\lim_{n\to\infty} h_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{l(\infty, h_n, Sp(X))}{h_n} = 1.$$

Let us consider a sequence of intervals $(c_n, d_n) \subseteq [0, h_n] \backslash Sp(X)$ for which

$$\lim_{n\to\infty} \frac{|d_n - c_n|}{h_n} = 1. \tag{5.4}$$

Passing, if it is necessary, to a subsequence we suppose that

$$0 < c_n < d_n \leq h_n \tag{5.5}$$

holds for every $n \in \mathbb{N}$. A pretangent $\Omega^X_{\infty,\tilde{r}}$ is single-point if and only if $\tilde{X}_{\infty,\tilde{r}} = \tilde{X}^0_{\infty,\tilde{r}}$ holds for the corresponding $\tilde{X}_{\infty,\tilde{r}}$. Hence, it suffices to prove that

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n} = 0 \tag{5.6}$$

holds for every $x \in \tilde{X}_{\infty,\tilde{r}}$. Write

$$r_n := \sqrt{d_n c_n} \tag{5.7}$$

for every $n \in \mathbb{N}$ and define $\tilde{r} := (r_n)_{n\in\mathbb{N}}$.

Let us prove equality (5.6). It is evident that (5.4) and (5.5) imply the limit relations

$$\lim_{n\to\infty} \frac{c_n}{h_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{d_n}{h_n} = 1. \tag{5.8}$$

Consequently, we obtain

$$\lim_{n\to\infty} \frac{c_n}{d_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{d_n}{c_n} = \infty. \tag{5.9}$$

Since $(c_n, d_n) \subseteq [0, h_n] \setminus Sp(X)$, we have either $d(x_n, p) \leq c_n$ or $d(x_n p) \geq d_n$ for all $n \in \mathbb{N}$. Thus, either

$$\frac{d(x_n, p)}{r_n} \leq \sqrt{\frac{c_n}{d_n}} \tag{5.10}$$

or

$$\frac{d(x_n, p)}{r_n} \geq \sqrt{\frac{d_n}{c_n}} \tag{5.11}$$

holds for every $n \in \mathbb{N}$. The second relation in (5.9) implies that (5.11) cannot be valid for sufficient large n because $\tilde{\tilde{d}}_{\tilde{r}}(\tilde{x})$ is finite. Now, (5.6) follows from (5.10).

It is proved that if $\tilde{r}$ is defined by (5.7), then there is a unique pretangent space $\Omega^X_{\infty,\tilde{r}}$ and this space is single-point. Note also that $\Omega^X_{\infty,\tilde{r}}$ is tangent. To prove it we can consider the subsequences $\tilde{x}' = (x_{n_k})_{k\in\mathbb{N}}$, $\tilde{z}' = (z_{n_k})_{k\in\mathbb{N}}$ and $\tilde{r}' = (r_{n_k})_{k\in\mathbb{N}}$ of $\tilde{x}$, $\tilde{z}$ and $\tilde{r}$, and repeat the proof of equality (5.6) substituting $d_{n_k}, c_{n_k}, h_{n_k}$ and r_{n_k} instead of d_n, c_n, h_n and r_n, respectively. The implication $1 \Rightarrow 2$ follows.

$2 \Rightarrow 3$, $3 \Rightarrow 4$, $4 \Rightarrow 5$. These implications are evident.

$5 \Rightarrow 1$. Let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence and let $\Omega^X_{\infty,\tilde{r}}$ be bounded, separable and tangent. Suppose there is $p_1 \in (0,1)$ such that $p^+(Sp(X),\infty) < p_1$. Applying Lemma 5.1 with $E = Sp(X)$ we can find a subsequence $\tilde{r}' = (r_{n_k})_{k\in\mathbb{N}}$ of the sequence $\tilde{r}$ such that for every $k \in \mathbb{N}$ there are points $x_1^{(k)}, \ldots, x_k^{(k)} \in X$ for which

$$\begin{gathered}\frac{1}{1-p_1} \le \frac{d(x_1^{(k)},p)}{r_{n_k}} \le \frac{1}{(1-p_1)^2},\\ \frac{1}{(1-p_1)^3} \le \frac{d(x_2^{(k)},p)}{r_{n_k}} \le \frac{1}{(1-p_1)^4},\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \frac{1}{(1-p_1)^{2k-1}} \le \frac{d(x_k^{(k)},p)}{r_{n_k}} \le \frac{1}{(1-p_1)^{2k}}.\end{gathered} \tag{5.12}$$

Let $\tilde{z} = (z_k)_{k\in\mathbb{N}} \in \tilde{\tilde{X}}_{\infty,\tilde{q}}$ and $\tilde{q} = (q_k)_{k\in\mathbb{N}}$. Write $\tilde{x}_j$ for the j-th column of the following infinite matrix

$$\begin{pmatrix} x_1^{(1)} & z_1 & z_1 & z_1 & z_1 & \cdots\\ x_1^{(2)} & x_2^{(2)} & z_2 & z_2 & z_2 & \cdots\\ x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & z_3 & z_3 & \cdots\\ x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & z_4 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It follows from (5.12) that the inequalities

$$\frac{1}{(1-p_1)^{2j-1}} \le \liminf_{k\to\infty} \frac{d(x_j^{(k)},p)}{r_{n_k}} \le \limsup_{k\to\infty} \frac{d(x_j^{(k)},p)}{r_{n_k}} \le \frac{1}{(1-p_1)^{2j}} \tag{5.13}$$

hold for all $j \in \mathbb{N}$. Let $\tilde{X}_{\infty,\tilde{r}}$ be the maximal self-stable family with the metric identification $\Omega^X_{\infty,\tilde{r}}$ and let $\tilde{X}'_{\infty,\tilde{r}} = \{(x_{n_k})_{k\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in \tilde{X}_{\infty,\tilde{r}}\}$.

The family $\mathfrak{B} := \{\tilde{x}_1, \tilde{x}_2, \ldots\}$ satisfies the conditions of Lemma 4.2. Hence there is a subsequence $\tilde{r}''$ of $\tilde{r}'$ such that $\tilde{X}_{\infty,\tilde{r}''} \supseteq \mathfrak{B}'$. The first inequality in (5.12) implies that $\Omega^X_{\infty,r''}$ is unbounded, contrary to 5. $\square$

6. Kuratowski limits and pretangent spaces.

Let (Y, δ) be a metric space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty sets $A_n \subseteq Y$. The *Kuratowski limit inferior* of $(A_n)_{n\in\mathbb{N}}$ is the subset $\underset{n\to\infty}{Li} A_n$ of Y defined by the rule:

$$\left(y \in \underset{n\to\infty}{Li} A_n\right) \Leftrightarrow (\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \colon B(y, \varepsilon) \cap A_n \neq \varnothing), \quad (6.1)$$

where $B(y, \varepsilon)$ is the open ball of radius $\varepsilon > 0$ centered at the point $y \in Y$,

$$B(y, \varepsilon) = \{x \in Y : \delta(x, y) < \varepsilon\}.$$

Similarly, the *Kuratowski limit superior* of $(A_n)_{n\in\mathbb{N}}$ can be defined as the subset $\underset{n\to\infty}{Ls} A_n$ of Y for which

$$\left(y \in \underset{n\to\infty}{Ls} A_n\right) \Leftrightarrow (\forall \varepsilon > 0\ \forall n \in \mathbb{N}\ \exists n_0 \geq n : B(y, \varepsilon) \cap A_{n_0} \neq \varnothing). \quad (6.2)$$

Remark 6.1. The Kuratowski limit inferior and limit superior are basic concepts of set-valued analysis in metric spaces and have numerous applications (see, for example, [5]).

We denote $tA := \{tx : x \in A\}$ for any nonempty set $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, and write, as above,

$$\nu_0 := \tilde{X}^0_{\infty,\tilde{r}}$$

for any scaling sequence $\tilde{r}$ and an unbounded metric space (X, d). Moreover, we denote by $\mathbf{\Omega^X_{\infty,\tilde{r}}}$ the set of all pretangent at infinity spaces to (X, d) with respect to $\tilde{r}$ and write

$$Sp\left(\Omega^X_{\infty,\tilde{r}}\right) := \{\rho(\nu_0, \nu) \colon \nu \in \Omega^X_{\infty,\tilde{r}}\} \text{ and } Sp(X) := \{d(p, x) \colon x \in X\}. \quad (6.3)$$

Proposition 6.1. *Let (X, d) be an unbounded metric space, $p \in X$, $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence and let $\tilde{\mathbf{R}}$ be the set of all infinite subsequences of $\tilde{r}$. Then the equalities*

$$\bigcup_{\Omega^X_{\infty,\tilde{r}} \in \mathbf{\Omega^X_{\infty,\tilde{r}}}} Sp\left(\Omega^X_{\infty,\tilde{r}}\right) = \underset{n\to\infty}{Li} \left(\frac{1}{r_n} Sp(X)\right) \quad (6.4)$$

and

$$\bigcup_{\Omega^X_{\infty,\tilde{r}'}\in\mathbf{\Omega}^X_{\infty,\tilde{r}'},\,\tilde{r}'\in\tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde{r}'}) = \mathop{Ls}_{n\to\infty}\left(\frac{1}{r_n}(Sp(X))\right) \tag{6.5}$$

hold.

Proof. Let us prove the inclusion

$$\bigcup_{\Omega^X_{\infty,\tilde{r}}\in\mathbf{\Omega^X_{\infty,\tilde{r}}}} Sp\left(\Omega^X_{\infty,\tilde{r}}\right) \subseteq \mathop{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right). \tag{6.6}$$

Let $\Omega^X_{\infty,\tilde{r}} \in \mathbf{\Omega^X_{\infty,\tilde{r}}}$ and $\nu \in \Omega^X_{\infty,\tilde{r}}$ be arbitrary. Let $\tilde{X}_{\infty,\tilde{r}}$ be a maximal self-stable family with the metric identification $\Omega^X_{\infty,\tilde{r}}$, and let $\tilde{x} = (x_n)_{n\in\mathbb{N}} \in \tilde{X}_{\infty,\tilde{r}}$, $\tilde{z} = (z_n)_{n\in\mathbb{N}} \in \tilde{X}^0_{\infty,\tilde{r}}$ such that

$$\pi(\tilde{x}) = \nu \quad \text{and} \quad \pi(\tilde{z}) = \nu_0.$$

Then, by the definition of pretangent spaces, we have

$$\lim_{n\to\infty}\frac{d(x_n,z_n)}{r_n} = \rho(\nu_0,\nu).$$

Consequently, for every $\varepsilon > 0$ the inequality

$$\left|\frac{1}{r_n}d(x_n,p) - \rho(\nu_0,\nu)\right| < \varepsilon$$

holds for all sufficiently large n. Since $\Omega^X_{\infty,\tilde{r}}$ is an arbitrary element of $\mathbf{\Omega^X_{\infty,\tilde{r}}}$ and ν is an arbitrary point of $\Omega^X_{\infty,\tilde{r}}$ and $\frac{1}{r_n}d(x_n,p) \in \frac{1}{r_n}Sp(X)$, inclusion (6.6) follows.

To obtain the converse inclusion, we consider an arbitrary

$$t \in \mathop{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right). \tag{6.7}$$

It is evident that $0 \in Sp\left(\Omega^X_{\infty,\tilde{r}}\right)$ holds for every $\Omega^X_{\infty,\tilde{r}}$. Suppose $t > 0$ and write

$$\operatorname{dist}\left(t, \frac{1}{r_n}Sp(X)\right) := \inf\left\{|t-s| \colon s \in \frac{1}{r_n}Sp(X)\right\}.$$

Using (6.1), we see that (6.7) holds if and only if

$$\lim_{n\to\infty}\operatorname{dist}\left(t,\frac{1}{r_n}Sp(X)\right) = 0. \tag{6.8}$$

Consequently, there is a sequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} |\tau_n - t| = 0 \tag{6.9}$$

and $\tau_n \in \frac{1}{r_n}Sp(X)$ for every $n \in \mathbb{N}$. Using the definition of $\frac{1}{r_n}Sp(X)$, we may rewrite the last statement as: "There is a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$\lim_{n\to\infty} \left| \frac{d(x_n,p)}{r_n} - t \right| = 0$$

holds". Thus, we have

$$\lim_{n\to\infty} \frac{d(x_n,p)}{r_n} = t. \tag{6.10}$$

The inequality $t > 0$ implies that $(x_n)_{n\in\mathbb{N}} \in \tilde{X}_{\infty,\tilde{r}}$. Let $\tilde{X}_{\infty,\tilde{r}}$ be a maximal self-stable family for which $(x_n)_{n\in\mathbb{N}} \in \tilde{X}_{\infty,\tilde{r}}$ and let $\Omega^X_{\infty,\tilde{r}}$ be the metric identification of $\tilde{X}_{\infty,\tilde{r}}$. Limit relation (6.10) implies $t \in Sp\left(\Omega^X_{\infty,\tilde{r}}\right)$. Since t is an arbitrary positive number from $\underset{n\to\infty}{Li}\left(\frac{1}{r_n}Sp(X)\right)$, we obtain

$$\bigcup_{\Omega^X_{\infty,\tilde{r}} \in \mathbf{\Omega^X_{\infty,\tilde{r}}}} Sp\left(\Omega^X_{\infty,\tilde{r}}\right) \supseteq \underset{n\to\infty}{Li}\left(\frac{1}{r_n}Sp(X)\right).$$

Equality (6.2) follows.

Equality (6.5) follows from (6.2) because, for every $t \geq 0$, we have $t \in \underset{n\to\infty}{Ls}\left(\frac{1}{r_n}Sp(X)\right)$ if and only if $t \in \underset{n\to\infty}{Li}\left(\frac{1}{r_{n_k}}Sp(X)\right)$ holds for some $(r_{n_k})_{k\in\mathbb{N}} \in \tilde{\mathbf{R}}$. □

Corollary 6.1. *Let (X,d) be an unbounded metric space, $\tilde{r}$ be a scaling sequence and let ${}^1\Omega^X_{\infty,\tilde{r}}$ be tangent and separable. Then we have*

$$\underset{n\to\infty}{Li}\left(\frac{1}{r_n}Sp(X)\right) = \underset{n\to\infty}{Ls}\left(\frac{1}{r_n}Sp(X)\right) = Sp\left({}^1\Omega^X_{\infty,\tilde{r}}\right). \tag{6.11}$$

Proof. Using Lemma 4.2, for every $\tilde{r}' \in \tilde{\mathbf{R}}$ and every

$$s \in \bigcup_{\Omega^X_{\infty,\tilde{r}'} \in \mathbf{\Omega}^X_{\infty,\tilde{r}'}} \Omega^X_{\infty,\tilde{r}'},$$

we can find $\nu \in {}^1\Omega^X_{\infty,\tilde{r}}$ such that $\rho(\nu_0,\nu) = s$. Consequently,

$$Sp\left({}^1\Omega^X_{\infty,\tilde{r}}\right) \supseteq \bigcup_{\Omega^X_{\infty,\tilde{r}'} \in \mathbf{\Omega}^X_{\infty,\tilde{r}'},\, \tilde{r}' \in \tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde{r}'})$$

holds. It is evident that

$$\bigcup_{\Omega^X_{\infty,\tilde r'}\in\mathbf{\Omega}^X_{\infty,\tilde r'},\,\tilde r'\in\tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde r'}) \supseteq \bigcup_{\Omega^X_{\infty,\tilde r}\in\mathbf{\Omega}^X_{\infty,\tilde r}} Sp(\Omega^X_{\infty,\tilde r}) \supseteq Sp({}^1\Omega^X_{\infty,\tilde r}).$$

Hence we have the equalities

$$\bigcup_{\Omega^X_{\infty,\tilde r'}\in\mathbf{\Omega}^X_{\infty,\tilde r'},\,\tilde r'\in\tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde r'}) = \bigcup_{\Omega^X_{\infty,\tilde r}\in\mathbf{\Omega}^X_{\infty,\tilde r}} Sp\left(\Omega^X_{\infty,\tilde r}\right) = Sp\left({}^1\Omega^X_{\infty,\tilde r}\right).$$

The last statement, (6.2) and (6.5) imply (6.11). ☐

Since the Kuratowski limit inferior and limit superior are closed (see, for example, [5, p. 18]), we obtain the following corollary of Proposition 6.1.

Corollary 6.2. *Let (X,d) be an unbounded metric space and let $\tilde r$ be a scaling sequence. Then the sets*

$$\bigcup_{\Omega^X_{\infty,\tilde r}\in\mathbf{\Omega}^{\mathbf{X}}_{\infty,\tilde{\mathbf{r}}}} Sp(\Omega^X_{\infty,\tilde r}) \quad \text{and} \quad \bigcup_{\Omega^X_{\infty,\tilde r'}\in\mathbf{\Omega}^{\mathbf{X}}_{\infty,\tilde{\mathbf{r}}'},\,\tilde r'\in\tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde r'})$$

are closed in $[0,\infty)$.

Let us consider a metric space (Y,δ) such that:

1. (Y,δ) is strongly rigid, i.e.,

   $$\delta(x,y)=\delta(t,z)>0$$

   implies $\{x,y\}=\{t,z\}$ for all x, y, z, $t\in Y$;

2. $\delta(x,y)<2$ for all x, $y\in Y$;

3. $\operatorname{diam} Y=2$;

4. The cardinality of the open ball

   $$B(y^*,r)=\{y\in Y:\delta(y,y^*)<r\}$$

   is finite for every $r\in(0,2)$ and every $y^*\in Y$.

Corollary 6.3. *Let (X,d) be an unbounded metric space, $\tilde r$ be a scaling sequence, $\Omega^X_{\infty,\tilde r}$ be tangent and let (Y,δ) be a metric space satisfying conditions (i_1)–(i_4). If $f\colon \Omega^X_{\infty,\tilde r}\to Y$ is an isometric embedding, then $\Omega^X_{\infty,\tilde r}$ is finite.*

Proof. Let $f\colon \Omega^X_{\infty,\tilde{r}} \to Y$ be an isometric embedding and let

$$y^* := f^{-1}(\nu_0), \quad \nu_0 := \tilde{X}^0_{\infty,\tilde{r}} \quad \text{and} \quad Y_1 := f(\Omega^X_{\infty,\tilde{r}}).$$

Conditions (i_3) and (i_4) imply that Y is countable. Consequently, $\Omega^X_{\infty,\tilde{r}}$ is also countable. Using Corollary 6.2, Corollary 6.1 and (i_3) we obtain that $Sp\left(\Omega^X_{\infty,\tilde{r}}\right)$ is a closed subset of $[0,2]$. Since

$$Sp\left(\Omega^X_{\infty,\tilde{r}}\right) = \{\delta(y,y^*)\colon y \in Y_1\}$$

holds, the set $\{\delta(y,y^*)\colon y \in Y_1\}$ is closed. Suppose $\Omega^X_{\infty,\tilde{r}}$ is infinite. Then Y_1 is infinite. Since Y is strongly rigid, we have

$$\lim_{n\to\infty} \delta(y^*, y_n) = 2$$

for every sequence $(y_n)_{n\in\mathbb{N}}$ of distinct points $y_n \in Y_1$. Hence

$$2 \in \{d(y,y^*) : y \in Y_1\}$$

holds, contrary to (i_2). □

Example 6.1. Let (Y,δ) be a metric space with $Y = \mathbb{N}$ and the metric δ defined such that:

$$\delta(1,2) = 1 + \frac{1}{2};$$

$$\delta(1,3) = 1 + \frac{2}{3}, \quad \delta(2,3) = 1 + \frac{3}{4};$$

$$\delta(1,4) = 1 + \frac{4}{5}, \quad \delta(2,4) = 1 + \frac{5}{6}, \quad \delta(3,4) = 1 + \frac{6}{7};$$

$$\delta(1,5) = 1 + \frac{7}{8}, \quad \delta(2,5) = 1 + \frac{8}{9}, \quad \delta(3,5) = 1 + \frac{9}{10}, \quad \delta(4,5) = 1 + \frac{10}{11};$$

....

Then (Y,δ) is a countable and complete metric space satisfying conditions (i_1)–(i_4). By Corollary 6.3 no tangent space $\Omega^X_{\infty,\tilde{r}}$ is isometric to (Y,δ).

Corollary 6.4. *Let (X,d) be an unbounded metric space and let $\tilde{r}$ be a scaling sequence. Then the following statements are equivalent:*

1. *There is a single-point pretangent space $\Omega^X_{\infty,\tilde{r}}$;*

2. *All $\Omega^X_{\infty,\tilde{r}}$ are single-point;*

3. The equality

$$\underset{n\to\infty}{Li}\left(\frac{1}{r_n}Sp(X)\right)=\{0\}$$

holds.

Proof. It suffices to show that the implication 1 $\Rightarrow$ 2 is valid. Suppose contrary that there exist pretangent spaces ${}^1\Omega^X_{\infty,\tilde r}$ and ${}^2\Omega^X_{\infty,\tilde r}$ such that

$$\left|{}^1\Omega^X_{\infty,\tilde r}\right|=1 \quad\text{and}\quad \left|{}^2\Omega^X_{\infty,\tilde r}\right|\geq 2.$$

Write ${}^1\tilde X_{\infty,\tilde r}$ and ${}^2\tilde X_{\infty,\tilde r}$ for the maximal self-stable sets corresponding ${}^1\Omega^X_{\infty,\tilde r}$ and ${}^2\Omega^X_{\infty,\tilde r}$ respectively. Then the equality $\left|{}^1\Omega^X_{\infty,\tilde r}\right|=1$ implies the equality

$${}^1\tilde X_{\infty,\tilde r}=\tilde X^0_{\infty,\tilde r}.$$

By statement 6 of Proposition 2.1 we have $\tilde X^0_{\infty,\tilde r}\in {}^2\Omega^X_{\infty,\tilde r}$. It follows from the inequality $\left|{}^2\Omega^X_{\infty,\tilde r}\right|\geq 2$ that

$${}^2\tilde X_{\infty,\tilde r}\setminus \tilde X^0_{\infty,\tilde r}\neq\varnothing.$$

Consequently we have

$${}^1\tilde X_{\infty,\tilde r}\subseteq {}^2\tilde X_{\infty,\tilde r} \quad\text{and}\quad {}^2\tilde X_{\infty,\tilde r}\setminus {}^1\tilde X_{\infty,\tilde r}\neq\varnothing.$$

Since ${}^2\tilde X_{\infty,\tilde r}$ is self-stable, the set ${}^1\tilde X_{\infty,\tilde r}$ is not maximal self-stable, contrary to the definition. □

Using Corollary 6.4 we can construct an unbounded metric space (X,d) such that there exist single-point pretangent spaces but these spaces are never tangent to (X,d) at infinity.

Example 6.2. Let $\mathbb{Z}$ be the set of all integer numbers, $t\in(1,\infty)$ and let X be a subset of the real line $\mathbb{R}$ (with the standard metric $d(x,y)=|x-y|$) such that $x\in X$ if and only if $x=0$ or $x=t^i$ for some $i\in\mathbb{Z}$. Let us define a scaling sequence $\tilde r=(r_n)_{n\in\mathbb{N}}$ as:

$$r_n:=t^{n/2},\quad n\in\mathbb{N} \tag{6.12}$$

and put $p=0$. Then we have

$$Sp(X)=\{|x-0|\colon x\in X\}=X$$

and

$$\frac{1}{r_n}Sp(X)=\begin{cases}X & \text{if } n \text{ is even}\\ \sqrt{t}X & \text{if } n \text{ is odd.}\end{cases} \tag{6.13}$$

It is easy to see that the inclusion

$$\operatorname*{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right) \subseteq \operatorname*{Li}_{k\to\infty}\left(\frac{1}{r_{n_k}}Sp(X)\right) \tag{6.14}$$

holds for every infinite subsequence $(r_{n_k})_{k\in\mathbb{N}}$ of $\tilde{r}$. Using (6.13) and (6.14) with $n_k = 2k$, $k \in \mathbb{N}$ and with $n_k = 2k+1$ we obtain

$$\operatorname*{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right) \subseteq X$$

and, respectively,

$$\operatorname*{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right) \subseteq \sqrt{t}X.$$

Consequently, we have

$$\operatorname*{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right) \subseteq (\sqrt{t}X)\cap X = \{0\}.$$

It is clear that

$$0 \in \operatorname*{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right).$$

Thus we obtain the equality

$$\operatorname*{Li}_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right) = \{0\}.$$

Now Corollary 6.4 implies that, for $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ defined by (6.12), there is a unique pretangent space $\Omega^X_{\infty,\tilde{r}}$ and this space is single-point. A simple calculation shows that the equality

$$p^+(Sp(X),\infty) = \frac{t-1}{t} \tag{6.15}$$

holds. Consequently, by Theorem 5.1 the metric space (X,d) does not have any single-point *tangent* spaces at infinity.

Letting t to 1 we obtain the following.

Proposition 6.2. *For every $\varepsilon > 0$ there are an unbounded metric space (X,d) and a scaling sequence $\tilde{r}$ such that $p^+\big(Sp(X),\infty\big) < \varepsilon$ and all $\Omega^X_{\infty,\tilde{r}}$ are single-point.*

In the previous proposition, we considered the metric spaces having an arbitrary small positive porosity at infinity. What happens if this porosity becomes zero?

Proposition 6.3. *Let (X, d) be an unbounded metric space, $p \in X$. If $Sp(X)$ is a nonporous at infinity set, then the inequality*

$$|\Omega^X_{\infty,\tilde{r}}| \geq 2 \tag{6.16}$$

holds for every pretangent space $\Omega^X_{\infty,\tilde{r}}$.

Proof. Suppose $Sp(X)$ is nonporous at infinity,

$$p^+(Sp(X), \infty) = 0. \tag{6.17}$$

Let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence. By Corollary 6.4 it suffices to show that there is a pretangent space $\Omega^X_{\infty,\tilde{r}}$ satisfying (6.16). From Definition 5.1 and (6.17) it follows that

$$\lim_{n\to\infty} \frac{l(\infty, r_n, Sp(X))}{r_n} = 0, \tag{6.18}$$

where $l(\infty, r_n, Sp(X))$ is the length of the longest interval in $[0, r_n) \setminus Sp(X)$. Write

$$\tau_n := \sup([0, r_n) \cap Sp(X)), \quad n \in \mathbb{N}.$$

Then (6.18) implies the equality

$$\lim_{n\to\infty} \frac{r_n - \tau_n}{r_n} = 0.$$

Thus

$$\lim_{n\to\infty} \frac{\tau_n}{r_n} = 1 \tag{6.19}$$

hold. It is easy to see that, for every $n \in \mathbb{N}$, we have

$$\tau_n = \sup\{d(p, x) \colon x \in B(p, r_n)\}, \tag{6.20}$$

where $B(p, r_n)$ is the open ball $\{x \in X \colon d(x, p) < r_n\}$. It follows from (6.19), (6.20) and the definition of $Seq(X, \tilde{r})$ that there is $\tilde{x} \in Seq(X, \tilde{r})$ such that

$$\tilde{\tilde{d}}_{\tilde{r}}(\tilde{x}) = \lim_{n\to\infty} \frac{d(x_n, p)}{r_n} = 1.$$

Consequently if $\tilde{X}_{\infty,\tilde{r}}$ is maximal self-stable subset of $Seq(X, \tilde{r})$ such that $\tilde{x} \in \tilde{X}_{\infty,\tilde{r}}$, then the inequality

$$|\Omega^X_{\infty,\tilde{r}}| \geq 2$$

holds for the metric identification $\Omega^X_{\infty,\tilde{r}}$ of $\tilde{X}_{\infty,\tilde{r}}$. ☐

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Contact information

Viktoriia Bilet
Institute of Applied Mathematics
and Mechanics of NASU,
Sloviansk, Ukraine
E-Mail: viktoriiabilet@gmail.com

Oleksiy Dovgoshey
Institute of Applied Mathematics
and Mechanics of NASU,
Sloviansk, Ukraine
E-Mail: oleksiy.dovgoshey@gmail.com