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Local sub-estimates of solutions to double phase parabolic equations via nonlinear parabolic potentials

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(Presented by I. I. Skrypnik)

Dedicated to the memory of Professor Bogdan Bojarski

Abstract. For parabolic equations with nonstandard growth conditions we prove local boundedness of weak solutions in terms of nonlinear parabolic potentials of right-hand side of the equation.

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1. Introduction

In this paper we consider a class of parabolic equations with nonstandard growth condition and singular lower order term. Let Ω be a domain in $\mathbb{R}^n, T > 0$, set $\Omega_T = \Omega \times (0, T)$. We study solution to the equation

$$u_t - div\mathbb{A}(x, t, u, \nabla u) = f(x, t), (x, t) \in \Omega_T. \tag{1.1}$$

Throughout the paper we suppose that the functions $\mathbb{A}(\cdot, \cdot, u, \xi)$ are Lebesgue measurable for all $u \in \mathbb{R}^1, \xi \in \mathbb{R}^n$, $\mathbb{A}(x, t, \cdot, \cdot)$ are continuous for almost all $(x, t) \in \Omega_T$. We also assume that the following structure conditions are satisfied

$$\mathbb{A}(x, t, u, \xi)\xi \geq c_1(|\xi|^p + a(x, t)|\xi|^q),$$
$$|\mathbb{A}(x, t, u, \xi)| \leq c_2(|\xi|^{p-1} + a(x, t)|\xi|^{q-1}), \tag{1.2}$$

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where c_1, c_2 are positive constants, $a(x,t) \geq 0, a(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\Omega_T)$ with some positive $\alpha \in (0,1]$, $f \in L^1(\Omega_T)$, and

$$\frac{2n}{n+1} < p \leq q < p + \alpha. \tag{1.3}$$

The main goal of this paper is to establish local boundedness of solutions to equation (1.1) in terms of parabolic potential of the right-hand side. This fact is basically characterized by the different types of degenerate behavior according to the size of a coefficient $a(x,t)$ that determines the "phase". Indeed, on the set $a(x,t) = 0$ equation (1.1) has growth of order p with respect to the gradient (this is the "p-phase"), and at the same time this growth is of order q when $a(x,t) > 0$ (this is the "(p,q)-phase").

Before formulating the main results, let us say a few words concerning the history of the problem. In the standard case $p = q$, the class of equations (1.1) has numerous application for several decades (see e.g. [5–7] and references therein). Starting from the seminal papers by P. Marcellini [18, 19], V. V. Zhikov [23] and G. Lieberman [14] during the last decade there has been growing interest and substantial development in the quasilinear elliptic and parabolic equations. The interest grows not only from the calculus of variations but also from a number of recent applications in modeling electrorheological fluids, image processing, theory of elasticity (see e.g. [20]). The basic prototypes of elliptic equations with nonstandard growth conditions are

$$-div\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f, \left(\frac{t}{\tau}\right)^{p-1} \leq \frac{g(t)}{g(\tau)} \leq \left(\frac{t}{\tau}\right)^{q-1}, \, t \geq \tau \geq 0, \tag{1.4}$$

$$-div(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f, \, a(x) \geq 0. \tag{1.5}$$

The qualitative theory of parabolic equations with nonstandard growth conditions has not been developed yet to the same extend. Local boundedness of the gradient of solutions to quasilinear parabolic equations of the type

$$u_t - div\left(g(| \nabla u |)\frac{\nabla u}{| \nabla u |}\right) = f, \left(\frac{s}{\tau}\right)^{p-1} \leq \frac{g(s)}{g(\tau)} \leq \left(\frac{s}{\tau}\right)^{q-1}, \; s \geq \tau > 0, \tag{1.6}$$

$$u_t - div(|\nabla u|^{p-2}\nabla u + a(x,t)|\nabla u|^{q-2}\nabla u) = f, a(x,t) \geq 0 \tag{1.7}$$

were obtained in [1, 22], Hölder continuity of solutions to equation (1.6) was proved in [8–10].

To describe our results let us remind the reader the definition of a weak solution to equation (1.1). For $\xi \in \mathbb{R}^n$ set $g_a(|\xi|) := |\xi|^{p-1} + a(x,t)|\xi|^{q-1}$ and $G_a(|\xi|) = |\xi| g_a(|\xi|)$. We will write $W^{1,G_a}(\Omega_T)$ for a class of functions which are weakly differentiable with $\iint\limits_{\Omega_T} G_a(|\nabla u|)dxdt < \infty$. We say that u is a weak solution to (1.1) if $u \in V(\Omega_T) := C(0,T; L^2(\Omega)) \cap W^{1,G_a}(\Omega_T)$ and for any interval $(t_1, t_2) \subset (0,T)$ the integral identity

$$\int\limits_{\Omega} u\varphi dx\,|_{t_1}^{t_2} + \int\limits_{t_1}^{t_2}\int\limits_{\Omega} (-u\varphi_t + \mathbb{A}(x,t,u,\nabla u)\nabla\varphi)dxdt = \int\limits_{t_1}^{t_2}\int\limits_{\Omega} \varphi f dxdt \quad (1.8)$$

holds true for any testing function $\varphi \in \overset{0}{W}{}^{1,G_a}(\Omega_T)$ with $\varphi, \varphi_t \in L^\infty(\Omega_T)$.

Note that the assumptions that the testing function φ and its derivative φ_t must be bounded guarantee the time derivative and the right-hand side of (1.8) are well defined. To formulate our first main result, we define the local parabolic potential.

Let $(x_0, t_0) \in \Omega_T$ for $\rho, \theta > 0$ and let $Q_{\rho,\theta}(x_0,t_0) := Q^-_{\rho,\theta}(x_0,t_0) \cup Q^+_{\rho,\theta}(x_0,t_0)$, $Q^-_{\rho,\theta}(x_0,t_0) := B_\rho(x_0) \times (t_0 - \theta, t_0)$, $Q^+_{\rho,\theta}(x_0,t_0) := B_\rho(x_0) \times (t_0 + \theta, t_0)$. For $m > \frac{2n}{n-1}$, $\rho > 0$ define

$$D_m(\rho; x_0, t_0) := \inf_{\tau > 0}\left\{ \frac{1}{\tau^{m-2}} + \rho^{-n} \iint\limits_{Q_{\rho,\rho^m\tau^{m-2}}(x_0,t_0)} |f|dxdt \right\}. \quad (1.9)$$

Note that the above infimum is attained at some $\tau \in (0, +\infty]$ since the function under the infimum is continuous for τ. Moreover $D_2(\rho; x_0, t_0)$ $= \iint\limits_{Q_{\rho,\rho^2}(x_0,t_0)} |f|dxdt.$

Now for $j = 0, 1, 2, ...$ set $\rho_j := 2^{-j}\rho$. Following [16] we define the parabolic potential

$$P^f_m(\rho; x_0, t_0) := \sum_{j=0}^{\infty} D_m(\rho_j; x_0, t_0). \quad (1.10)$$

Particularly, there exists $\gamma > 1$ such that

$$\frac{1}{\gamma} P^f_2(\rho; x_0, t_0) \le \int\limits_0^\rho r^{-n} \iint\limits_{Q_{\rho,\rho^2}(x_0,t_0)} |f|dxdt \frac{dr}{r} \le \gamma P^f_2(\rho; x_0, t_0).$$

So that for $m = 2$ the introduced potential is equivalent to the truncated Riesz potential used in [2, 4, 12]. Note also that for $m > 2$ and for a time-independent f the minimum in the the definition of $D_m(\rho; x_0, t_0)$ is attained at

$$\tau = (m-2)^{-\frac{1}{m-1}} \left(\rho^{m-n} \int\limits_{B_\rho(x_0)} |f| dx \right)^{\frac{1}{m-1}},$$

so

$$D_m(\rho; x_0, t_0) = (m-1)(m-2)^{\frac{1}{m-1}} \left(\rho^{m-n} \int\limits_{B_\rho(x_0)} |f| dx \right)^{\frac{1}{m-1}}$$

and $P^f_m(\rho; x_0, t_0) = W^f_{1,m}(\rho; x_0)$, where $W^f_{1,m}(\rho; x_0)$ is Wolff potential defined by the formula

$$W^f_{1,m}(\rho; x_0) = \sum_{j=0}^{\infty} \left(\rho_j^{m-n} \int\limits_{B_{\rho_j}(x_0)} f\, dx \right)^{\frac{1}{m-1}}, \ \rho_j = \frac{\rho}{2^j},\ j = 0,\ 1,\ ..$$

Remark 1.1. We can estimate P^f_m by the Lebesgue norm as follows.

Let $f \in L^r(0, T; L^s(\Omega))$ for $\frac{1}{r} + \frac{n}{ms} < 1$. Then

$$\rho^{-n} \int\limits_{Q_{\rho,\rho^m\tau^{m-2}}(x_0,t_0)} |f| dx \le \gamma \tau^{(m-2)(1-\frac{1}{r})} \rho^{m(1-\frac{1}{r}-\frac{n}{ms})} ||f||_{s,r}$$

and

$$D_m(\rho; x_0, t_0) \le \gamma (\rho^{m(1-\frac{1}{r}-\frac{n}{ms})} ||f||_{s,r})^{\frac{1}{1+(m-2)(1-\frac{1}{r})}}.$$

Hence if $\frac{1}{r} + \frac{n}{ms} < 1$, then

$$P^f_m(\rho; x_0, t_0) \le \gamma (\rho^{m(1-\frac{1}{r}-\frac{n}{ms})} ||f||_{s,r})^{\frac{1}{1+(m-2)(1-\frac{1}{r})}}$$

and $\lim_{\rho\to 0} \sup\limits_{(x_0,t_0)\in\Omega_T} P^f_m(\rho; x_0, t_0) = 0$.

The main result of the paper is the local boundedness of the solutions. As it has already mentioned before the behavior of the solution in a neighborhood of a point (x_0, t_0) depends on the value of the function $a(x_0, t_0)$. In what follows we will distinguish two cases: $\sup\limits_{Q_{\rho,\rho^2}(x_0,t_0)} a(x, t) \ge 2[a]_\alpha \rho^\alpha$

(so called (p,q)-phase) and $\sup\limits_{Q_{\rho,\rho^2}(x_0,t_0)} a(x,t) \le 2[a]_\alpha \rho^\alpha$(so called p-phase), here $[a]_\alpha := \sup\limits_{\substack{(x,t),(y,\tau)\in\Omega_T\\(x,t)\neq(y,\tau)}} \frac{|a(x,t)-a(y,\tau)|}{(|x-y|+|t-\tau|)^\alpha}$.

Theorem 1.1. *(Local boundedness of solution in the (p,q)-phase). Let u be a solution of equation* (1.1) *and assumptions* (1.2), (1.3) *be fulfilled, $q \neq 2$. Fix a point $(x_0,t_0) \in \Omega_T$ such that $a_0 := a(x_0,t_0) > 0$. Let $R := (\frac{a_0}{2[a]_\alpha})^{\frac{1}{\alpha}}$ and $Q_{\rho,\theta}(x_0,t_0) \subset Q_{R,R^2}(x_0,t_0) \subset Q_{8R,(8R)^2}(x_0,t_0) \subset \Omega_T$. Then for any $0 < \lambda < \frac{p}{nq}$ the following estimate*

$$|u(x_0,t_0)| \le \gamma \left(\frac{\rho^q}{a_0\theta}\right)^{\frac{1}{q-2}}$$

$$+\gamma\left(\frac{a_0}{\rho^{n+q}} \iint_{Q_{\rho,\theta}(x_0,t_0)} |u|^{q-1+\lambda(q-1)} dxdt\right)^{\frac{1}{1+\lambda(q-1)}}$$

$$+\gamma\left(\frac{1}{\rho^{n+p}} \iint_{Q_{\rho,\theta}(x_0,t_0)} |u|^{p-1+\lambda(q-1)} dxdt\right)^{\frac{1}{1+\lambda(q-1)}}$$

$$+\gamma(1 + a_0^{-\frac{1}{q-2}}) P_q^f(2\rho; x_0; t_0) \tag{1.11}$$

holds true with a constant $\gamma > 0$ depending only on n, p, q, c_1, c_2, $[a]_\alpha$ and λ.

Theorem 1.2. *(Local boundedness of solution in the p-phase). Let u be a solution of equation* (1.1) *and assumptions* (1.2), (1.3) *be fulfilled, and assume also that $q < p\frac{n+1}{n}$. Fix a point $(x_0,t_0) \in \Omega_T$ such that $a_0 = a(x_0,t_0) = 0$. Then for any $0 < \lambda < \frac{p-n(q-p)}{nq}$ the following estimate*

$$|u(x_0,t_0)| \le \gamma \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}} + \gamma\left(\frac{1}{\rho^{n+p}} \iint\limits_{Q_{\rho,\theta}(x_0,t_0)} |u|^{p-1+\lambda(q-1)} dxdt\right)^{\frac{1}{1+\lambda(q-1)}}$$

$$+\gamma\left(\frac{1}{\rho^{n+p}} \iint\limits_{Q_{\rho,\theta}(x_0,t_0)} |u|^{(q-1)(1+\lambda)} dxdt\right)^{\frac{p}{p-n(q-p)+\lambda p(q-1)}} + \gamma P_p^f(2\rho; x_0, t_0) \tag{1.12}$$

hold true with a constant γ depending only on $n, p, q, c_1, c_2, [a]_\alpha$ and λ.

The proof of Theorems 1.1, 1.2 is based on the adaption of the Kilpeläinen–Malý technique [11] to the parabolic equations using ideas from [16].

2. Local boundedness of solutions. Proof of Theorems 1.1, 1.2

2.1. Integral estimates of the solutions

For $0<\lambda<\min(1,m-1), m>1$, set $W_m(s):=\int_0^s(1+z)^{-\frac{1+\lambda}{m}}dz=\frac{m}{m-1-\lambda}((1+s)^{\frac{m-1-\lambda}{m}}-1)$ for any $\varepsilon\in(0,1)$ evidently we have

$$W_m(s)\le\frac{m}{m-1-\lambda}s^{\frac{m-1-\lambda}{m}},\, s\le\varepsilon+\gamma(\varepsilon)W^{\frac{m}{m-1-\lambda}}(s) \tag{2.1}$$

with a constant $\gamma(\varepsilon)$ depending only on ε, m, λ. In what follows we shall also need the following simple inequality.

$$s\le\varepsilon+\gamma(\varepsilon)\int_0^s(1-(1+z)^{-\lambda})dz,\ \varepsilon,\ \lambda\in(0,1) \tag{2.2}$$

with a constant $\gamma(\varepsilon)$ depending only on ε, λ.

The next two lemmas are Cacciopolli type estimates adapted to the Kilpeläinen–Maly technique.

Lemma 2.1. *(p,q-phase). Let the conditions of Theorem 1.1 be fulfilled. Then there exists $\gamma>0$ depending only on the data such that for any $\lambda\in(0,1), k>q, l,\delta>0$, any cylinder $Q_r^{(\delta)}:=Q_{r,\frac{r^q}{a_0}\delta^{2-q}}\subset Q_{\rho,\theta}(x_0,t_0)\subset Q_{R,R^2}(x_0,t_0)$ and any $\zeta\in C_0^\infty(Q_r^{(\delta)})$, such that $0\le\zeta\le 1, |\nabla\zeta|\le\gamma r^{-1}$, $|\zeta_t|\le\gamma a_0 r^{-q}\delta^{q-2}$ one has*

$$\begin{aligned}
&\sup_{0<t<T}\delta^{-1}\int\limits_{L(t)}\int_l^u\left(1-\left(1+\frac{z-l}{\delta}\right)^{-\lambda}\right)dz\zeta^k dx\\
&+\ \delta^{p-2}\iint\limits_L\left|\nabla W_p\left(\frac{u-l}{\delta}\right)\right|^p\zeta^k dxdt\\
&+\ \delta^{q-2}a_0\iint\limits_L\left|\nabla W_q\left(\frac{u-l}{\delta}\right)\right|^q\zeta^k dxdt\\
&\le\ \gamma a_0\frac{\delta^{q-2}}{r^q}\iint\limits_L\left(1+\frac{u-l}{\delta}\right)^{q-1+\lambda(q-1)}\zeta^{k-q}dxdt\\
&+\ \gamma\frac{\delta^{p-2}}{r^p}\iint\limits_L\left(1+\frac{u-l}{\delta}\right)^{p-1+\lambda(q-1)}\zeta^{k-q}dxdt\\
&+\ \gamma\delta^{-1}\iint_{Q_r^{(\delta)}}|f|dxdt,
\end{aligned}\tag{2.3}$$

where $L:=Q_r^{(\delta)}\cap\{u>l\}, L(t):=L\cap\{\tau=t\}$.

Proof. First note that by our choice of R we have $\frac{a_0}{2} = a_0 - [a]_\alpha R^\alpha \leq a(x,t) \leq a_0 + [a]_\alpha R^\alpha = \frac{3}{2} a_0$ for any $(x,t) \in Q_r^{(\delta)} \subset Q_{R,R^2}(x_0,t_0)$. Testing identify (1.8) by $\varphi = (1-(1+(\frac{u-l}{\delta})_+)^{-\lambda})\zeta^k$, using conditions (1.2) we obtain

$$\sup_{0<t<T} \int\limits_{L(t)} \int\limits_{l}^{u} \left(1 - \left(1 + \frac{z-l}{\delta}\right)^{-\lambda}\right) dz \zeta^k dx$$

$$+\delta^{-1} \iint\limits_{L} \left(1 + \frac{u-l}{\delta}\right)^{-1-\lambda} |\nabla u|^p \zeta^k dxdt$$

$$\delta^{-1} a_0 \iint\limits_{L} \left(1 + \frac{u-l}{\delta}\right)^{-1-\lambda} |\nabla u|^q \zeta^k dxdt \leq \gamma a_0 \frac{\delta^{q-1}}{r^q}$$

$$\iint\limits_{L} \frac{u-l}{\delta} \zeta^{k-1} dxdt + \gamma r^{-1} \iint\limits_{L} |\nabla u|^{p-1} \zeta^{k-1} dxdt$$

$$+\gamma a_0 r^{-1} \iint\limits_{L} |\nabla u|^{q-1} \zeta^{k-1} dxdt + \gamma \iint\limits_{Q_r^{(\delta)}} |f| dxdt.$$

From this using the Young inequality and by our choice of $W_p(\frac{u-l}{\delta})$, $W_q(\frac{u-l}{\delta})$ we arrive at the required (2.3). □

Lemma 2.2. *(p-phase). Let the conditions of Theorem 1.2 be fulfilled. Then there exists $\gamma > 0$ depending only on the data such that for any $\lambda \in (0,1), k \geq q, l > 0, \delta \geq r^{\sigma_1}$, any cylinder $Q_r^{(\delta)} := Q_{r\delta^{\frac{p-2}{p}}, r^p \delta^{2-p}}(x_0,t_0) \subset Q_{\rho,\theta}(x_0,t_0)$ and any $\zeta \in C_0^\infty(Q_r^{(\delta)})$, such that $0 \leq \zeta \leq 1$, $|\nabla \zeta| \leq \gamma r^{-1}$, $|\zeta_t| \leq \gamma r^{-p} \delta^{p-2}$ one has*

$$\sup_{0<t<T} \int\limits_{L(t)} \int_{l}^{u} \left(1 - \left(1 + \frac{z-l}{\delta}\right)^{-\lambda}\right) dz \zeta^k dx$$

$$+\delta^{p-2} \iint\limits_{L} \left|\nabla W_p \left(\frac{u-l}{\delta}\right)\right|^p \zeta^k dxdt$$

$$\leq \gamma \delta^{p-2} r^{-p} \iint\limits_{L} \left(1 + \frac{u-l}{\delta}\right)^{p-1+\lambda(q-1)} \zeta^{k-q} dxdt$$

$$+\gamma \delta^{q-2} r^{-p} \iint\limits_{L} \left(1 + \frac{u-l}{\delta}\right)^{q-1+\lambda(q-1)} \zeta^{k-q} dxdt + \gamma \delta^{-1} \iint\limits_{Q_r^{(\delta)}} |f| dxdt. \tag{2.4}$$

Proof. Note that by our choice of δ we have an inclusion $Q_r^{(\delta)} \subset Q_{r,r^2}(x_0,t_0)$. Therefore for any $(x,t) \in Q_r^{(\delta)}$ we have $a(x,t) \le [a]_\alpha r^\alpha \le [a]_\alpha r^{q-p}$ (we have p, $q > 2$).

Testing (1.8) by $\varphi = (1-(1+(\frac{u-l}{\delta})_+)^{-\lambda})\zeta^k$, using condition (1.2) we obtain

$$\sup_{0<t<T} \int\limits_{L(t)} \int_l^u \left(1-\left(1+\frac{z-l}{\delta}\right)^{-\lambda}\right) dz\zeta^k dx$$

$$+\delta^{-1}\iint_L a(x,t)\left(\frac{u-l}{\delta}\right)^{-1-\lambda}|\nabla u|^q\zeta^k dxdt \le \gamma\frac{\delta^{p-1}}{r^p}\iint\limits_L \frac{u-l}{\delta}\zeta^{k-1}dxdt$$

$$+\gamma r^{-1}\iint\limits_L |\nabla u|^{p-1}\zeta^{k-1}dxdt + \gamma r^{-1}\iint\limits_L a(x,t)|\nabla u|^{q-1}\zeta^{k-1}dxdt$$

$$+\gamma\iint\limits_{Q_r^{(\delta)}} |f|dxdt.$$

Using the Young inequality we arrive at the required (2.4). □

2.2. Proof of Theorem 1.1

Fix a number $æ \in (0,1)$ depending only on the data and λ, which will be specified later. For $j = 0,1,2,...$ positive numbers l_j and δ_j are defined inductively as follows.

$$\delta_{-1} := \left(\frac{\rho^q}{a_0\theta}\right)^{\frac{1}{q-2}} + \left(\frac{a_0}{æ\rho^{n+q}}\iint\limits_{Q_{\rho,\theta}(x_0,t_0)} u^{q-1+\lambda(q-1)}dxdt\right)^{\frac{1}{1+\lambda(q-1)}}$$

$$+\left(\frac{1}{æ\rho^{n+p}}\iint\limits_{Q_{\rho,\theta}(x_0,t_0)} u^{p-1+\lambda(q-1)}dxdt\right)^{\frac{1}{1+\lambda(q-1)}} \qquad (2.5)$$

and $l_0 = 0$. For $j = 0,1,2,...$, given δ_{j-1} and l_j we define δ_j and l_{j+1} as follows. We denote $r_j := \rho 2^{-j}$ and $\tau_j := \sup\{\tau : \frac{1}{\tau}+r_j^{-n}\iint\limits_{Q_{r_j,r_j^q\tau^{q-2}}(x_0,t_0)} |f|dxdt = D_q(r_j;x_0,t_0)\}$, where $D_q(r_j;x_0,t_0)$ is as in (1.9). For $\delta \ge \frac{1}{2}\delta_{j-1}$ we define $B_j := B_{r_j}(x_0)$, $Q_j^{(\delta)} := Q_{r_j,\frac{r_j^p}{a_0}\delta^{2-q}}(x_0,t_0)$. Let $\zeta_j \in C_0^\infty(Q_j^{(\delta)})$ be such that

$0 \le \zeta_j \le 1$, $\zeta_j = 1$ in $\frac{1}{4}Q_j^{(\delta)}$ and $|\nabla\zeta_j| \le \gamma r_j^{-1}$, $|\frac{\partial\zeta_j}{\partial t}| \le \gamma a_0 r_j^{-q}\delta^{q-2}$. Set

$$A_j(\delta) := a_0\frac{\delta^{q-2}}{r_j^{n+q}}\iint\limits_{L_j^{(\delta)}}\left(\frac{u-l_j}{\delta}\right)^{q-1+\lambda(q-1)}\zeta_j^q\,dxdt + \frac{\delta^{p-2}}{r_j^{n+p}}\iint\limits_{L_j^{(\delta)}}\left(\frac{u-l_j}{\delta}\right)^{p-1+\lambda(q-1)}\zeta_j^q\,dxdt, \tag{2.6}$$

here $L_j^{(\delta)} := Q_j^{(\delta)} \cap \{u > l_j\}$.

If $A_j(\frac{1}{2}\delta_{j-1}) \le æ$, we set $\delta_j = \frac{1}{2}\delta_{j-1}$ and $\delta_j = l_{j+1} - l_j$. Since $A_j(\delta)$ is continuous and decreasing as a function of δ, then if $A_j(\frac{1}{2}\delta_{j-1}) > æ$ there exists $\hat\delta > \frac{1}{2}\delta_{j-1}$ such that $A_j(\hat\delta) = æ$. In this case we set $\delta_j = \hat\delta$ and $l_{j+1} = l_j + \delta_j$. Further we set $Q_j = Q_j^{(\delta_j)}$, $L_j = L_j^{(\delta_j)}$. By our choice of δ_{-1} and $\delta_j, j = 0, 1, 2, ...$ we have an inclusion $Q_j \subset Q_{j-1} \subset Q_0 \subset Q_{\rho,\theta}(x_0, t_0)$ for $j = 1, 2, ...$ and in particular $\zeta_{j-1} \equiv 1$ on Q_j, $j = 1, 2, ...$, and moreover

$$A_j(\delta_j) \le æ, j = 1, 2, ... \tag{2.7}$$

Claim. Set $B = 2^{n+q}$, then for any $j = 0, 1, 2, ...$

$$\delta_j \le B\delta_{j-1}. \tag{2.8}$$

We establish the claim by induction. By our choice of δ_{-1} we have for $j = 0$

$$A_0(B\delta_{-1}) = \frac{a_0\delta_{-1}^{-1-\lambda(q-1)}}{\rho^{n+q}B^{1+\lambda(q-1)}}\iint\limits_{Q_0} u^{q-1+\lambda(q-1)}\zeta_0^q\,dxdt + \frac{\delta_{-1}^{-1-\lambda(q-1)}}{\rho^{n+p}B^{1+\lambda(q-1)}}\iint\limits_{Q_0} u^{p-1+\lambda(q-1)}\zeta_0^q\,dxdt$$

$$\le B^{-1-\lambda(q-1)}\left\{\frac{a_0\delta_{-1}^{-1-\lambda(q-1)}}{\rho^{n+q}}\iint\limits_{Q_{\rho,\theta}(x_0,t_0)} u^{q-1+\lambda(q-1)}\,dxdt + \frac{\delta_{-1}^{-p-1-\lambda(q-1)}}{\rho^{n+p}}\iint\limits_{Q_{\rho,\theta}(x_0,t_0)} u^{p-1+\lambda(q-1)}\,dxdt\right\}$$

$$\le B^{-1}æ < æ.$$

If $\delta_0 = \frac{1}{2}\delta_{-1} \le B\delta_{-1}$, and if $A_0(\delta_0) = æ > A_0(B\delta_{-1})$, and since $A_0\delta$ is decreasing, then $\delta_0 \le B\delta_{-1}$, and in both cases we obtain $\delta_0 \le B\delta_{-1}$. Assume that (2.8) holds for $i = 1, 2, ..., j-1$, then

$$A_j(B\delta_{j-1}) = a_0 \left(\frac{2}{r_{j-1}}\right)^{n+q} \frac{\delta_{j-1}^{q-2}}{B^{1+\lambda(q-1)}} \iint\limits_{L_j} \left(\frac{u-l_j}{\delta_{j-1}}\right)^{q-1+\lambda(q-1)} \zeta_j^q dxdt$$

$$+ \left(\frac{2}{r_{j-1}}\right)^{n+p} \frac{\delta_{j-1}^{p-2}}{B^{1+\lambda(q-1)}} \iint\limits_{L_j} \left(\frac{u-l_j}{\delta_{j-1}}\right)^{p-1+\lambda(q-1)} \zeta_j^q dxdt$$

$$\le 2^{n+q} B^{-1} \left(a_0 \frac{\delta_{j-1}^{q-2}}{r_{j-1}^{n+q}} \iint\limits_{L_j} \left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)^{q-1+\lambda(q-1)} \zeta_{j-1}^q dxdt \right.$$

$$\left. + \frac{\delta_{j-1}^{p-2}}{r_{j-1}^{n+p}} \iint\limits_{L_j} \left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)^{p-1+\lambda(q-1)} \zeta_j^q dxdt \right)$$

$$\le 2^{n+q} B^{-1} A_{j-1}(\delta_{j-1}) \le æ 2^{n+q} B^{-1} \le æ.$$

If $\delta_j = \frac{1}{2}\delta_{j-1} \le B\delta_{j-1}$, $A_j(\delta_j) = æ \ge A_{j-1}(B\delta_{j-1})$, and since $A_j(\delta)$ is decreasing, then $\delta_j \le B\delta_{j-1}$, and in both cases we obtain $\delta_j \le B\delta_{j-1}$, which proves the claim.

The following lemma is a key in the Kilpeläinen–Malý technique.

Lemma 2.3. *Let the conditions of Theorem 1.1 be fulfilled. Then for any $j \ge 1$ there exists $\gamma > 0$ depending only on the data and λ, such that*

$$\delta_j \le \frac{1}{2}\delta_{j-1} + \gamma(1 + a_0^{-\frac{1}{q-2}}) D_q(r_j; x_0, t_0). \tag{2.9}$$

Proof. We shall assume later that

$$\delta_j > \frac{1}{2}\delta_{j-1},\ \delta_j > a_0^{-\frac{1}{q-2}} \frac{1}{\tau_j}, \tag{2.10}$$

since otherwise (2.9) is evident. The first inequality in (2.10) guarantees that $A_j(\delta_j) = æ$. First note the inequality

$$\frac{\delta_j^{q-2}}{r_j^{n+q}} |L_j| + \frac{\delta_{j-1}^{p-2}}{r_{j-1}^{n+p}} |L_j| \le \gamma æ, j = 1, 2, ... \tag{2.11}$$

Indeed, by (2.7) and (2.8) we have

$$a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} |L_j| + \frac{\delta_{j-1}^{p-2}}{r_{j-1}^{n+p}} |L_j|$$

$$= a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} \iint\limits_{L_j} \left(\frac{l_j - l_{j-1}}{\delta_{j-1}} \right)^{q-1+\lambda(q-1)} \zeta_{j-1}^q dxdt$$

$$+ \frac{\delta_{j-1}^{p-2}}{r_{j-1}^{n+p}} \iint\limits_{L_j} \left(\frac{l_j - l_{j-1}}{\delta_{j-1}} \right)^{p-1+\lambda(q-1)} \zeta_{j-1}^q dxdt$$

$$\leq \gamma(B) \left(a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} \iint\limits_{L_{j-1}} \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right)^{q-1+\lambda(q-1)} \zeta_{j-1}^q dxdt \right.$$

$$\left. + \frac{\delta_{j-1}^{p-2}}{r_{j-1}^{n+p}} \iint\limits_{L_{j-1}} \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right)^{p-1+\lambda(q-1)} \zeta_{j-1}^q \right) dxdt$$

$$\leq \gamma(B) A_{j-1}(\delta_{j-1}) \leq \gamma(B)æ.$$

By (2.1) and (2.11) we have for any $\varepsilon \in (0,1)$

$$æ = a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} \iint\limits_{L_j} \left(\frac{u - l_j}{\delta_j} \right)^{q-1+\lambda(q-1)} \zeta_j^q dxdt + \frac{\delta_j^{p-2}}{r_{j-1}^{n+p}}$$

$$\iint\limits_{L_j} \left(\frac{u - l_j}{\delta_j} \right)^{p-1+\lambda(q-1)} \zeta_j^q dxdt \leq a_0 \gamma \varepsilon^{q-1+\lambda(q-1)} \delta_j^{q-2} r_j^{-n-q} |L_j|$$

$$+\gamma \varepsilon^{p-1+\lambda(q-1)} \delta_j^{\frac{p-2}{r_j^{-n-q}}} |L_j| + \gamma(\varepsilon) J_1 \leq \varepsilon \gamma æ + \gamma(\varepsilon) J_1, \qquad (2.12)$$

where

$$J_1 = a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} \iint\limits_{L_j} W_q^q \left(\frac{u - l_j}{\delta_j} \right) \left(\frac{u - l_j}{\delta_j} \right)^{\lambda q} \zeta_j^q dxdt$$

$$+ \frac{\delta_j^{p-2}}{r_j^{n+q}} \iint\limits_{L_j} W_p^p \left(\frac{u - l_j}{\delta_j} \right) \left(\frac{u - l_j}{\delta_j} \right)^{\lambda q} \zeta_j^q dxdt.$$

Further we shall assume that λ satisfies the condition $0 < \lambda < \frac{p}{nq}$. By the Sobolev embedding theorem and our choice of λ we obtain

$$J_1 \leq a_0 \gamma \frac{\delta_j^{q-2}}{r_j^{n+q}} \left(\sup_{0<t<T} \int\limits_{L_j(t)} \frac{u - l_j}{\delta_j} \zeta_j^q dx \right)^{\frac{q}{n}} \iint\limits_{L_j} \left| \nabla \left(W_q \left(\frac{u - l_j}{\delta_j} \right) \zeta_j \right) \right|^q dxdt$$

$$+\gamma \frac{\delta_j^{p-2}}{r_j^{n+q}} \left(\sup_{0<t<T} \int\limits_{L_j(t)} \frac{u - l_j}{\delta_j} \zeta_j^q dx \right)^{\frac{p}{n}} \iint\limits_{L_j} \left| \nabla \left(W_p \left(\frac{u - l_j}{\delta_j} \right) \zeta_j \right) \right|^p dxdt. \quad (2.13)$$

By (2.2) and Lemma 2.1 we obtain for every $\varepsilon_1 \in (0,1)$

$$\begin{aligned}
&\sup_{0<t<T} \int\limits_{L_j(t)} \frac{u-l_j}{\delta_j}\zeta_j^q dx \le \varepsilon_1|B_j| \\
+ &\ \gamma(\varepsilon_1)\delta_j^{-1} \sup_{0<t<T} \int_{L_j(t)} \int_{l_j}^{u} \left(1-\left(1+\frac{z-l_j}{\delta_j}\right)^{-\lambda}\right) dz\zeta_j^q dx \\
\le &\ |B_j| \left(\varepsilon_1 + \gamma(\varepsilon_1) a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} \iint\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{q-1+\lambda(q-1)} dxdt\right) \\
+ &\ \gamma(\varepsilon_1)\frac{\delta_j^{p-2}}{r_j^{n+q}} \iint\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{p-1+\lambda(q-1)} dxdt \\
+ &\ \gamma(\varepsilon_1)\delta_j^{-1} r_j^{-n} \iint\limits_{Q_j} |f| dxdt. \qquad (2.14)
\end{aligned}$$

Further by (2.7), (2.8), (2.10), (2.11) and our choice of ζ_j we obtain

$$\begin{aligned}
&a_0 \frac{\delta_j^{q-2}}{r_j^{n+q}} \iint\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{q-1+\lambda(q-1)} dxdt \\
+&\frac{\delta_j^{p-2}}{r_j^{n+q}} \iint\limits_{L_j} \left(1+\frac{u-l_j}{\delta_j}\right)^{p-1+\lambda(q-1)} dxdt \le \gamma A_{j-1}(\delta_{j-1}) \le \gamma æ. \qquad (2.15)
\end{aligned}$$

Therefore, inequalities (2.13)–(2.15) and Lemma 2.1 imply

$$\begin{aligned}
æ \le \varepsilon\gamma æ + \gamma(\varepsilon) &\left(æ + \delta_j^{-1} r_j^{-n} \iint\limits_{Q_j} |f| dxdt\right) \\
\times &\left\{ \left(\varepsilon_1 + \gamma(\varepsilon_1)æ + \delta_j^{-1} r_j^{-n} \iint\limits_{Q_j} |f| dxdt\right)^{\frac{q}{n}} \right. \\
+ &\left. \left(\varepsilon_1 + \gamma(\varepsilon_1)æ + \delta_j^{-1} r_j^{-n} \iint\limits_{Q_j} |f| dxdt\right)^{\frac{p}{n}} \right\}. \qquad (2.16)
\end{aligned}$$

Now choose $\varepsilon = \frac{1}{16\gamma}$, $\varepsilon_1 = \frac{1}{16\gamma(\varepsilon)}$ and æ such that $\gamma(\varepsilon,\varepsilon_1)æ^{\frac{p}{n}} + \gamma(\varepsilon,\varepsilon_1)æ^{\frac{q}{n}} = \frac{1}{16}$. From (2.16) it follows that there exists $\gamma > 0$ such that $\delta_j^{-1} r_j^{-n} \iint_{Q_j} |f| dxdt \ge$

$\gamma æ$, hence $\delta_j \le \gamma r_j^{-n} \iint\limits_{Q_j} |f| dxdt$. By the second inequality in (2.10) we have an inclusion $Q_j \subset Q_{r_j, r_j^q \tau_j^{q-2}}(x_0,t_0)$, so

$$\delta_j \le \gamma r_j^{-n} \iint\limits_{Q_{r_j, r_j^q \tau_j^{q-2}}(x_0,t_0)} |f| dxdt \le \gamma D_q(r_j; x_0, t_0).$$

Such a way inequality (2.9) is proved, which completes the proof of Lemma 2.3. □

Summing up inequality (2.9) for $j = 1, 2, ..., J-1$ by (2.8) we obtain

$$\begin{aligned} l_J &\le \gamma\delta_0 + \gamma(1 + a_0^{-\frac{1}{q-2}}) \sum_{j=1}^{\infty} D_q(r_j; x_0, t_0) \\ &\le \gamma\delta_{-1} + \gamma(1 + a_0^{-\frac{1}{q-2}}) P_q^f(2\rho; x_0, t_0). \end{aligned} \tag{2.17}$$

Hence we can pass to the limit $J \to \infty$ in (2.17). Let $\bar{l} = \lim_{j\to\infty} l_j$, from (2.6), (2.7) we conclude that $r_j^{-n-q} \iint\limits_{Q_j} (u - \bar{l})^{q-1+\lambda(q-1)} dxdt \le \gamma \delta_j^{1+\lambda(q-1)} \to 0$, $j \to \infty$. Choosing (x_0, t_0) as a Lebesgue point of the function $(u - \bar{l})^{q-1+\lambda(q-1)}$ we conclude that $u(x_0,t_0) \le \bar{l}$ and hence $u(x_0,t_0)$ is estimated from above by the righthand side of (2.17). This completes the proof of Theorem 1.1.

2.3. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to that of Theorem 1.1. We note only the differences arising here.

Fix a number $æ \in (0,1)$ depending only on the data and λ, which will be specified later. For $j = 0, 1, 2, ...$ positive numbers l_j and δ_j are defined inductively as follows.

$$\begin{aligned} \delta_{-1} := \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}} &+ \left(\frac{1}{æ\rho^{n+p}} \iint\limits_{Q_{\rho,\theta}(x_0,t_0)} u^{p-1+\lambda(q-1)} dxdt\right)^{\frac{1}{1+\lambda(q-1)}} \\ &+ \left(\frac{1}{æ\rho^{n+p}} \iint_{Q_{\rho,\theta}(x_0,t_0)} u^{q-1+\lambda(q-1)} dxdt\right)^{\frac{p}{p-n(q-p)+\lambda p(q-1)}}, \end{aligned} \tag{2.18}$$

and $l_0 = 0$. We denote $r_j := \rho 2^{-j}$ and

$$\tau_j := \sup\left\{\tau : \frac{1}{\tau} + r_j^{-n} \iint\limits_{Q_{r_j, r_j^p \tau^{p-2}}(x_0,t_0)} |f| dxdt\right\} = D_p(r_j; x_0, t_0), \quad (2.19)$$

where $D_p(r_j; x_0, t_0)$ is defined by (1.9). For $\delta \geq \frac{1}{2}\delta_{j-1}$ we define $B_j := B_{r_j}(x_0)$, $Q_j^{(\delta)} := Q_{r_j, r_j^p \delta^{2-p}}(x_0, t_0)$ and let $\zeta_j \in C_0^\infty(Q_j^{(\delta)})$ be such that $0 \leq \zeta_j \leq 1, \zeta_j = 1$ in $\frac{1}{4}Q_j^{(\delta)}$ and $|\nabla \zeta_j| \leq \gamma r_j^{-1}$, $|\frac{\partial \zeta_j}{\partial t}| \leq \gamma r_j^{-p}\delta^{p-2}$. Set

$$A_j(\delta) := \frac{\delta^{p-2}}{r_j^{n+p}} \iint\limits_{L_j^{(\delta)}} \left(\frac{u - l_j}{\delta}\right)^{p-1+\lambda(q-1)} \zeta_j^q dxdt$$
$$+ \frac{\delta^{q-2}}{r_j^{n+p}} \iint\limits_{L_j^{(\delta)}} \left(\frac{u - l_j}{\delta}\right)^{q-1+\lambda(q-1)} \zeta_j^q dxdt, \quad (2.20)$$

where $L_j^{(\delta)} := Q_j^{(\delta)} \cap \{u > l_j\}$.

If $A_j(\frac{1}{2}\delta_{j-1}) \leq$ æ, we set $\delta_j = \frac{1}{2}\delta_{j-1}$ and $\delta_j = l_{j+1} - l_j$. Since $A_j(\delta)$ is continuous and decreasing as a function of δ, then $A_j(\frac{1}{2}\delta_{j-1}) >$ æ and there exists $\widehat{\delta} > \frac{1}{2}\delta_{j-1}$ such that $A_j(\widehat{\delta}) =$ æ. In this case we set $\delta_j = \widehat{\delta}$. Further we set $Q_j = Q_j^{(\delta_j)}$ and $L_j = L_j^{(\delta_j)}$. By our choice of $\delta_j, j = 0, 1, 2, ...$ we have an inclusion $Q_j \subset Q_{j-1} \subset Q_0 \subset Q_{\rho,\theta}(x_0, t_0)$ for $j = 1, 2, ...$, in particular, $\zeta_{j-1} \equiv 0$ on $Q_j, j = 1, 2, ...$ and

$$A_j(\delta_j) \leq æ, j = 1, 2, ... \quad (2.21)$$

Similarly to (2.8) we prove

$$\delta_j \leq B\delta_{j-1}, j = 0, 1, 2, ... \quad (2.22)$$

where $B = 2^{\sigma_3}, \sigma_3 = \frac{(n+p)p}{p-n(q-p)}$.

The next Lemma is a key in the Kilpeläinen–Malý technique in the p-phase.

Lemma 2.4. *Let the conditions of Theorem 1.2 be fulfilled. Then for any $j \geq 1$ there exists $\gamma > 0$ depending only on the data and λ such that*

$$\delta_j \leq \frac{1}{2}\delta_{j-1} + \gamma D_p(r_j; x_0, t_0). \quad (2.23)$$

Proof. We will assume that

$$\delta_j > \frac{1}{2}\delta_{j-1},\ \delta_j > \frac{1}{\tau_j},$$

since otherwise inequality (2.23) is evident. First, similarly to (2.11) we obtain

$$(\delta_j^{p-2} + \delta_j^{q-2}) r_j^{-n-p} |L_j| \le \gamma æ, j = 1, 2, ... \tag{2.24}$$

By (2.1) and (2.24) we have for any $\varepsilon \in (0,1)$

$$æ = \frac{\delta_j^{p-2}}{r_j^{n+p}} \iint\limits_{L_j} \left(\frac{u - l_j}{\delta_j}\right)^{p-1+\lambda(q-1)} \zeta_j^q dxdt$$

$$+\frac{\delta_j^{q-2}}{r_j^{n+p}} \iint\limits_{L_j} \left(\frac{u - l_j}{\delta_j}\right)^{q-1+\lambda(q-1)} \zeta_j^q dxdt \le \varepsilon æ + \gamma(\varepsilon) J_2, \tag{2.25}$$

where

$$J_2 = \frac{\delta_j^{p-2}}{r_j^{n+p}} \iint\limits_{L_j} W_p^p(\frac{u - l_j}{\delta_j})(\frac{u - l_j}{\delta_j})^{\lambda q} \zeta_j^q dxdt$$

$$+\frac{\delta_j^{q-2}}{r_j^{n+p}} \iint\limits_{L_j} W_p^p(\frac{u - l_j}{\delta_j})(\frac{u - l_j}{\delta_j})^{q-p+\lambda q} \zeta_j^q dxdt.$$

Assuming that λ satisfies the condition $0 < \lambda < \frac{p-n(q-p)}{nq}$ and using the Sobolev embedding theorem we obtain

$$\begin{aligned} J_2 \ &\le\ \gamma \left(\delta_j^{p-2} + \delta_j^{q-2+\frac{n}{p}(q-p)}\right) r_j^{-n-p} \\ &\times \left(\sup_{0<t<T} \int\limits_{L_j(t)} \frac{u - l_j}{\delta_j} \zeta_j^q dx\right)^{\frac{p}{n}} \iint\limits_{L_j} \left|\nabla \left(W_p\left(\frac{u - l_j}{\delta_j}\right)\zeta_j\right)\right|^p dxdt \\ &=\ \gamma \left(\delta_j^{p-2} + \delta_j^{q-2+\frac{n}{p}(q-p)}\right) r_j^{-n-p} J_3. \end{aligned} \tag{2.26}$$

By (2.2) and Lemma 2.2 we obtain for every $\varepsilon, \varepsilon_1 \in (0,1)$

$$\begin{aligned}
&\gamma(\varepsilon)\delta_j^{q-2+\frac{n}{p}(q-p)} r_j^{-n-p} J_3 \\
\leq\ & \gamma(\varepsilon)\left(\varepsilon_1 + \gamma(\varepsilon_1)æ + \delta_j^{-\frac{p-n(q-p)}{p}} r_j^{-n} \iint\limits_{Q_j} |f|dxdt\right)^{\frac{p}{n}} \\
\times\ & \left(æ + \delta_j^{-\frac{p-n(q-p)}{p}} r_j^{-n} \iint\limits_{Q_j} |f|dxdt\right).
\end{aligned} \tag{2.27}$$

Similarly, by (2.2) and Lemma 2.2 we have for any $\varepsilon, \varepsilon_1 \in (0,1)$

$$\begin{aligned}
&\gamma(\varepsilon)\delta_j^{-\frac{p-n(q-p)}{p}} r_j^{-n-p} J_3 \\
\leq\ & \gamma(\varepsilon)\left(\varepsilon_1 + \gamma(\varepsilon_1)æ + \delta_j^{-\frac{p-n(q-p)}{p}} r_j^{-n} \iint\limits_{Q_j} |f|dxdt\right)^{\frac{p}{n}} \\
\times\ & \left(æ + \delta_j^{-\frac{p-n(q-p)}{p}} r_j^{-n} \iint\limits_{Q_j} |f|dxdt\right).
\end{aligned} \tag{2.28}$$

Choose $\varepsilon = \frac{1}{16\gamma}, \varepsilon_1 = \frac{1}{16\gamma(\varepsilon)}$ and æ such that $\gamma(\varepsilon, \varepsilon_1)æ^{\frac{p}{n}} = \frac{1}{16}$. From (2.25)–(2.28) it follows

$$\delta_j \leq \gamma \left(r_j^{-n} \iint\limits_{Q_j} |f|dxdt\right) + \gamma \left(r_j^{-n} \iint\limits_{Q_j} |f|dxdt\right)^{\frac{p}{p-n(q-p)}}.$$

Since $\delta_j > \frac{1}{\tau_j}$ we have an inclusion $Q_j \subset Q_{r_j, r_j^p \tau_j^{p-2}}(x_0, t_0)$. From the previous we obtain

$$\delta_j \leq \gamma r_j^{-n} \iint\limits_{Q_{r_j, r_j^p \tau_j^{p-2}}(x_0,t_0)} |f|dxdt \leq \gamma D_j(r_j; x_0, t_0),$$

which proves the lemma. □

Summing inequalities (2.23) for $j = 1, 2, ..., J-1$, using (2.22) and passing to the limit $J \to \infty$, we arrive at (1.12). Here (x_0, t_0) is a Lebesgue point of the function $(u - \bar{l})^{p-1+\lambda(q-1)}$, where $\bar{l} = \lim\limits_{j\to\infty} l_j$. This completes the proof of Theorem 1.2.

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