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Schwarz boundary value problems for solutions of a generalized Cauchy–Riemann system with a singular line

Sergiy A. Plaksa

(Presented by V. Gutlyanskiĭ)

Dedicated to memory of Professor Bogdan Bojarski

Abstract. We consider a generalized Cauchy–Riemann system with a rectilinear singular interval of the real axis. Schwarz boundary value problems for generalized analytic functions which satisfy the mentioned system are reduced to the Fredholm integral equations of the second kind under natural assumptions relating to the boundary of a domain and the given boundary functions.

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1. Introduction

1.1. Degenerated elliptic equations associated with potential fields

It is well-known that a spatial potential solenoid field symmetric with respect to the axis Ox is described in a meridian plane xOy in terms of the axisymmetric potential φ and the Stokes flow function ψ satisfying the following system of equations:

$$y\,\frac{\partial\varphi(x,y)}{\partial x}=\frac{\partial\psi(x,y)}{\partial y}\,,\qquad y\,\frac{\partial\varphi(x,y)}{\partial y}=-\frac{\partial\psi(x,y)}{\partial x}\,. \tag{1.1}$$

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Under the condition that there exist continuous second-order partial derivatives of the functions $\varphi(x,y)$ and $\psi(x,y)$, system (1.1) implies the following equations for the axisymmetric potential and the Stokes flow function:

$$y\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi(x,y)+\frac{\partial\varphi(x,y)}{\partial y}=0\,, \tag{1.2}$$

$$y\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y)-\frac{\partial\psi(x,y)}{\partial y}=0\,, \tag{1.3}$$

that are degenerated on the axis Ox, as well as equations (1.1).

In the plane xOy, we consider a bounded simply connected domain D symmetric with respect to the axis Ox. By D_z we denote the domain in the complex plane $\mathbb{C}$ congruent to the domain D under the correspondence $z=x+iy, (x,y)\in D$, where i is the imaginary complex unit. Let ∂D and ∂D_z denote the boundaries of domains D and D_z, respectively. By b_1 and b_2 we denote the points at which the boundary ∂D_z crosses the real axis $\mathbb{R}$. We assume that $b_1<b_2$.

In what follows, $z=x+iy$, and we shall consider the solutions φ and ψ of system (1.1) given in the domain D. In this case, the complex potential $H(z):=\varphi(x,y)+i\psi(x,y)$ satisfies the following Beltrami equation:

$$\frac{\partial H(z)}{\partial\bar z}=\frac{1-\operatorname{Im}z}{1+\operatorname{Im}z}\,\frac{\partial H(z)}{\partial z}\,,\qquad z\in D_z:\operatorname{Im}z>0\,, \tag{1.4}$$

where $\frac{\partial}{\partial\bar z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$.

At the same time, the function $W(z)=\varphi(x,y)+iv(x,y)$, where $v(x,y):=\frac{\psi(x,y)}{y}$, satisfies the equation

$$2\,\frac{\partial W(z)}{\partial\bar z}-\frac{1}{z-\bar z}\,\left(W(z)-\overline{W(z)}\right)=0\,,\quad z\in D_z:\operatorname{Im}z>0\,, \tag{1.5}$$

that is the complex form of a generalized Cauchy–Riemann system with a rectilinear singular interval (b_1,b_2) of the real axis.

After substituting the function $\psi(x,y)=y\,v(x,y)$ into equation (1.3) we obtain the following equation for the function v:

$$y^2\,\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v(x,y)+y\,\frac{\partial v(x,y)}{\partial y}-v(x,y)=0\,. \tag{1.6}$$

The theory of mappings being solutions of elliptic equations is developed in the papers by M. A. Lavrentiev [1], I. N. Vekua [2], B. Bojarski [3, 4], L. I. Volkovyski [5], L. G. Mikhailov [6] and many other authors. Let us note that the theory which is developed in the mentioned papers describes properties of solutions of equations (1.4) and (1.5), as well as equations (1.1)–(1.3) and (1.6), only outside of neighbourhoods of the real axis.

1.2. Some special methods for solving boundary value problems for elliptic equations with a degeneration on the real axis

Boundary value problems for elliptic equations are considered by I. N. Vekua [2], B. Bojarski [7], A. V. Bitsadze [8], V. N. Monakhov [9], B. Bojarski, V. Gutlyanski and V. Ryazanov [10] et al.

M. V. Keldysh [11] describes some correct statements of boundary value problems for an elliptic equation with a degeneration on a straight line. It shows that there are essential differences with boundary value problems for elliptic equations without degeneration. Some special methods for researching boundary value problems for elliptic equations degenerating along the line are developed by I. I. Daniliuk [12], S. A. Tersenov [13], R. P. Gilbert [14], L. G. Mikhailov and N. Radzhabov [15], A. Yanushauskas [16], S. Rutkauskas [17, 18].

One of the ways for solving boundary value problems for axisymmetric potential solenoid fields is based on integral expressions of axisymmetric potentials via analytic functions of a complex variable (cf. E. T. Whittaker and G. N. Watson [19], H. Bateman [20], P. Henrici [21], A. G. Mackie [22], Yu. P. Krivenkov [23], N. R. Radzhabov [24], G. N. Polozhii [25], G. N. Polozhii and A.F. Ulitko [26], A.A. Kapshivyi [27], A. Ya. Aleksandrov and Yu. P. Soloviev [28], I. P. Mel'nichenko and S. A. Plaksa [29]).

In particular, in the paper [29], we developed a method for the reduction of the Dirichlet problems for the axisymmetric potential and the Stokes flow function to the Fredholm integral equations of the second kind. It is made in the case where the boundary of a simply connected domain belongs to a class being wider than the class of Lyapunov curves. These results are used below for solving the Schwarz boundary value problems for generalized analytic functions that are solutions of equation (1.5).

2. Schwarz boundary value problems for generalized analytic functions satisfying equation (1.5)

2.1. The statement of the problems

Let $\mathrm{GA}_{\mathrm{sym}}(D_z)$ be the class of generalized analytic functions $W\colon D_z \longrightarrow \mathbb{C}$ satisfying the following conditions:

- $W(x+iy)$ has the continuous partial derivatives of the first order with respect to x and y in the domain $\{z \in D_z : \operatorname{Im} z > 0\}$ and satisfies equation (1.5);

- W submits to Schwarz reflection principle, i.e,

$$W(\bar{z}) = \overline{W(z)} \qquad \forall\, z \in D_z\,. \tag{2.1}$$

Thus, for a function $W \in \mathrm{GA}_{\mathrm{sym}}(D_z)$, the function $\varphi(x,y) = \operatorname{Re} W(z)$ is even with respect to the variable y, and the function $v(x,y) = \operatorname{Im} W(z)$ is odd with respect to the variable y.

Consider two *Schwarz boundary value problems* for solutions of equation (1.5):

Schwarz BVP–I: to find a function $W \in \mathrm{GA}_{\mathrm{sym}}(D_z)$ which is continuously extended onto the boundary ∂D_z, when values of its real part are given on ∂D_z, i.e.,

$$\operatorname{Re} W(z) = u_1(x,y) \quad \forall\, z \in \partial D_z\,, \tag{2.2}$$

where $u_1\colon \partial D \longrightarrow \mathbb{R}$ is a given continuous function even with respect to the variable y;

Schwarz BVP–II: to find a function $W \in \mathrm{GA}_{\mathrm{sym}}(D_z)$ which is continuously extended onto the set $\partial D_z \setminus \{b_1, b_2\}$, when values of its imaginary part are given on $\partial D_z \setminus \{b_1, b_2\}$, i.e.,

$$\operatorname{Im} W(z) = u_2(x,y) \quad \forall\, z \in \partial D_z \setminus \{b_1, b_2\}\,, \tag{2.3}$$

where $u_2\colon \big(\partial D \setminus \{(b_1,0),(b_2,0)\}\big) \longrightarrow \mathbb{R}$ is a given continuous function odd with respect to the variable y. In addition, it is required that the function W should satisfy the estimate

$$|W(z)| \le c\left(|z-b_1|^{-\beta_W} + |z-b_2|^{-\beta_W}\right) \quad \forall\, z \in D_z\,, \tag{2.4}$$

where the constant c does not depend on z and the constant $\beta_W \in (0;1)$ is dependent only of the function W.

In contradistinction to the classical Schwarz boundary value problem for analytic functions of a complex variable, Schwarz BVP–II is not reduced to Schwarz BVP–I because, at least, the real and imaginary parts of a generalized analytic function $W \in \mathrm{GA}_{\mathrm{sym}}(D_z)$ satisfy the different equations, viz., the function $\varphi(x,y) = \operatorname{Re} W(z)$ satisfies equation (1.2) while the function $v(x,y) = \operatorname{Im} W(z)$ satisfies equation (1.6).

2.2. Preliminary notes

For every $z \in D_z$ with $\operatorname{Im} z \neq 0$, we fix an arbitrary Jordan rectifiable curve $\Gamma_{z\bar{z}}$ in the domain D_z which is symmetric with respect to the real axis $\mathbb{R}$ and connects the points z and $\bar{z}$. For every $z \in \partial D_z$ with $\operatorname{Im} z \neq 0$ by $\Gamma_{z\bar{z}}$ we denote the Jordan subarc of the boundary ∂D_z with the end points z and $\bar{z}$ which contains the point b_1.

Denote by $\overline{D_z}$ the closure of domain D_z.

For $z \in \overline{D_z}$, $\operatorname{Im} z \neq 0$, let $\sqrt{(t-z)(t-\bar{z})}$ be that continuous branch of the function $G(t) = \sqrt{(t-z)(t-\bar{z})}$ analytic outside of the cut along $\Gamma_{z\bar{z}}$ for which $G(b_2) > 0$. We define $\sqrt{(t-z)(t-\bar{z})} := t - z$ for each $z \in \overline{D_z}$ with $\operatorname{Im} z = 0$.

It is well-known that any function $W(x+iy)$ which belongs to the class $\mathrm{GA}_{\mathrm{sym}}(D_z)$ is also analytic with respect to the variable x and y outside of any neighbourhood of the real axis. Therefore, as a consequence of integral expressions for the axisymmetric potential and the Stokes flow function obtained in Theorem 1 in [30] and Theorem 1 in [31] (cf. also Theorems 3.4 and 3.5 in [29]), one can conclude that there exists a unique analytic function $F\colon D_z \longrightarrow \mathbb{C}$ which submits to Schwarz reflection principle of the type (2.1) and such that the equality

$$W(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{F(t)}{\sqrt{(t-z)(t-\bar{z})}} \left(1 - i\,\frac{(t-x)}{y}\right) dt \tag{2.5}$$

is fulfilled for all $z \in D_z$ with $\operatorname{Im} z \neq 0$, where γ is an arbitrary closed Jordan rectifiable curve in D_z which surrounds $\Gamma_{z\bar{z}}$.

Let us note that if the function F is continuously extended onto the set $\partial D_z \setminus \{b_1, b_2\}$ and satisfies an estimate of the type (2.4), then the formula (2.5) can be transformed to the form

$$W(z) = \frac{1}{2\pi i} \int\limits_{\partial D_z} \frac{F(t)}{\sqrt{(t-z)(t-\bar{z})}} \left(1 - i\,\frac{(t-x)}{y}\right) dt \tag{2.6}$$

for all $z \in D_z$ with $\operatorname{Im} z \neq 0$. For all $z \in D_z$ with $\operatorname{Im} z = 0$, equality (2.6) is transformed by continuity into the equality $W(z) = F(z)$.

Our immediate purpose is to use integral expression (2.6) of any function $W \in \mathrm{GA}_{\mathrm{sym}}(D_z)$ for solving Schwarz BVP–I and Schwarz BVP–II.

We assume that the given function u_1 belongs to the set $\widetilde{\mathcal{H}}_\alpha(\partial D)$ of functions $u : \partial D \to \mathbb{R}$ satisfying the following condition

$$|u(x_1, y_1) - u(x_2, y_2)|$$

$$\leq c\,(\max\{|z_1 - b_1||z_1 - b_2|, |z_2 - b_1||z_2 - b_2|\})^{-\nu}\, |z_1 - z_2|^\alpha$$

$$\forall\,(x_1, y_1), (x_2, y_2) \in \partial D,$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $\alpha \in (1/2; 1]$, $\nu \in [0; \alpha)$ and the constant c does not depend on x_1, y_1, x_2, y_2.

We assume also that the given function u_2 is of the form $u_2(x, y) \equiv \widetilde{u}_2(x, y)/y$, where $\widetilde{u}_2 \in \widetilde{\mathcal{H}}_\alpha(\partial D)$ and $\widetilde{u}_2(b_1, 0) = \widetilde{u}_2(b_2, 0) = 0$.

We shall formulate conditions on the boundary ∂D_z in terms of the conformal mapping $\sigma(Z)$ which maps the unit disk $\{Z \in \mathbb{C} : |Z| < 1\}$ onto the domain D_z in such a way that $\sigma(-1) = b_1$, $\sigma(1) = b_2$ and $\mathrm{Im}\, \sigma(i) > 0$. Moreover,

$$\sigma(\bar{Z}) = \overline{\sigma(Z)} \qquad \forall\, Z \in \{Z \in \mathbb{C} : |Z| \leq 1\}.$$

We assume that the conformal mapping σ has the nonvanishing continuous contour derivative on the unit circle and its modulus of continuity

$$\omega(\sigma', \varepsilon) := \sup_{|Z_1 - Z_2| \leq \varepsilon} |\sigma'(Z_1) - \sigma'(Z_2)|$$

satisfies the following condition:

$$\int_0^1 \frac{\omega(\sigma', \eta)}{\eta}\, \ln^3 \frac{2}{\eta}\, d\eta < \infty. \tag{2.7}$$

One can observe that if the boundary ∂D is the Lyapunov curve, then the condition (2.7) is satisfied. It follows from the Kellog theorem. The condition (2.7) is also satisfied in a more general case where the boundary ∂D is a smooth curve and, furthermore, the tangent angle $\theta(s)$ as function of the arc length s has the modulus of continuity $\omega(\theta, \varepsilon)$ satisfying the following condition

$$\int_0^1 \frac{\omega(\theta, \eta)}{\eta}\, \ln^4 \frac{2}{\eta}\, d\eta < \infty.$$

The last statement follows from an estimate for the modulus of continuity of conformal mapping of the unit disk given in the papers [32, 33].

2.3. Reduction of Schwarz BVP–I to the Fredholm integral equation

Preparatory to formulate a result on the reduction of Schwarz BVP–I to the Fredholm integral equation, let us introduce some denotations.

We introduce the function

$$M(Z,T) := \sqrt{\frac{(T-Z)(T-\bar{Z})}{(\sigma(T)-\sigma(Z))(\sigma(T)-\sigma(\bar{Z}))}}\ .$$

For each $Z \neq -1$ it is understood as a continuous branch of the function analytic with respect to the variable T in the unit disk and satisfying the condition $M(Z,-1) > 0$.

Now, consider the conformal mapping $Z = \frac{\xi - i}{\xi + i}$ of the complex plane. This mapping assigns the points Z and $\bar{Z}$ of the unit circle to the points ξ and $-\xi$ of the real axis, respectively. Let us introduce the function

$$m(\xi,\tau) := M(Z,T)$$

of real variables ξ and τ, where $T = \frac{\tau - i}{\tau + i}$. Denote

$$A(\xi,\tau) := 2\ \mathrm{Re}\, m(\xi,\tau), \qquad B(\xi,\tau) := 2\ \mathrm{Im}\, m(\xi,\tau).$$

Consider the functions

$$\widetilde{m}(\xi,\tau) := \begin{cases} \frac{2\xi}{\pi} \int\limits_{\tau}^{\xi} \frac{s(m(s,\tau)-m(\xi,\tau))}{(\xi^2-s^2)^{3/2}\sqrt{s^2-\tau^2}}\, ds\,, & \text{when } \xi\tau > 0\,, |\tau| < |\xi|, \\ 0, & \text{when } \xi\tau < 0 \text{ or } \xi\tau > 0\,, |\tau| > |\xi|\,, \end{cases}$$

$$\widetilde{A}(\xi,\tau) := 2\ \mathrm{Re}\, \tilde{m}(\xi,\tau), \qquad \widetilde{B}(\xi,\tau) := 2\ \mathrm{Im}\, \tilde{m}(\xi,\tau),$$

$$k_p(\xi,\tau) := -\frac{\xi}{|\xi|}\ \widetilde{A}(\xi,\tau) + \frac{1}{\pi} \int\limits_{0}^{\xi} \frac{\widetilde{B}(\xi,\eta)}{\eta - \tau}\, d\eta\,,$$

$$P(\xi) := \sqrt{\sigma'\left(\frac{\xi - i}{\xi + i}\right)} - \sqrt{\sigma'\left(\frac{\xi + i}{\xi - i}\right)}$$

and the integral operators

$$(k_p f)(\xi) := \int\limits_{-\infty}^{\infty} \frac{k_p(\xi,\tau)}{\sqrt{\tau^2+1}}\ f(\tau)\, d\tau,$$

$$(Rf)(\xi) := \sqrt{\xi^2+1}\left(\frac{A(\xi,\xi)}{(A(\xi,\xi))^2+(B(\xi,\xi))^2} f(|\xi|)\right.$$
$$\left. + \frac{P(\xi)}{4\pi i}\int\limits_{-\infty}^{\infty} f(|\tau|)\sqrt{\frac{(\tau^2+1)\,\left|\operatorname{Im}\sigma(\frac{\tau-i}{\tau+i})\right|}{2\,|\tau|}}\,\frac{d\tau}{\tau-\xi}\right).$$

For a given function $u_1\colon \partial D \longrightarrow \mathbb{R}$, we define the function

$$f^*(\xi) := u_1^*(\xi) - \xi\int\limits_0^{\xi}\frac{s(u_1^*(s)-u_1^*(\xi))}{(\xi^2-s^2)^{3/2}}\,ds\,,$$

where the function u_1^* is expressed via the given function u_1 in the following form:

$$u_1^*(\xi) := \frac{1}{\sqrt{\xi^2+1}}\big(u_1(x,y)-u_1(b_2,0)\big), \qquad x+iy = \sigma\left(\frac{\xi-i}{\xi+i}\right).$$

Let $C(\mathbb{R})$ be the Banach space of continuous functions $u^* : (\mathbb{R}\cup\{\infty\}) \to \mathbb{C}$ with the norm $\|u^*\|_{C(\mathbb{R})} := \sup\limits_{\tau\in\mathbb{R}}|u^*(\tau)|$ and $C_e(\mathbb{R})$ be the subspace of $C(\mathbb{R})$ containing all even continuous functions.

Denote by $\mathcal{D}(\mathbb{R})$ the set of functions $u^*\in C(\mathbb{R})$ for which the modulus of continuity

$$\omega_{\mathbb{R}}(u^*,\varepsilon) := \sup_{\tau_1,\tau_2\in\mathbb{R},|\tau_1-\tau_2|\le\varepsilon}|u^*(\tau_1)-u^*(\tau_2)|$$

and the local centralized (with respect to the infinitely remote point) modulus of continuity

$$\omega_{\mathbb{R},\infty}(u^*,\varepsilon) := \sup_{\tau\in\mathbb{R},|\tau|\ge 1/\varepsilon}|u^*(\tau)-u^*(\infty)|$$

satisfy the Dini conditions

$$\int\limits_0^1\frac{\omega_{\mathbb{R}}(u^*,\eta)}{\eta}\,d\eta<\infty, \qquad \int\limits_0^1\frac{\omega_{\mathbb{R},\infty}(u^*,\eta)}{\eta}\,d\eta<\infty\,.$$

We denote also by $\mathcal{D}_e(\mathbb{R})$ the set of even functions from $\mathcal{D}(\mathbb{R})$.

The following theorem establishes sufficient conditions for the reduction of Schwarz BVP–I to the Fredholm integral equation.

Theorem 2.1. *Suppose that* $u_1 \in \widetilde{\mathcal{H}}_\alpha(\partial D)$, *and the conformal mapping* $\sigma(Z)$ *has the nonvanishing continuous contour derivative on the unit circle, and its modulus of continuity satisfies condition* (2.7). *Then the solution of the Schwarz BVP–I is given by formula* (2.6), *in which*

$$F(z) = u_1(b_2, 0) - \frac{2(\xi + i)}{\pi\, \sigma'(\frac{\xi - i}{\xi + i})} \int_{-\infty}^{\infty} \frac{U_0(\tau)}{\sqrt{\tau^2 + 1}\,(\tau - \xi)}\, d\tau,$$
$$z = \sigma\left(\frac{\xi + i}{\xi - i}\right) \quad \forall\, \xi \in \mathbb{C} : \operatorname{Im} \xi > 0\,, \tag{2.8}$$

where U_0 *is a solution of the Fredholm integral equation*

$$U_0(\xi) + (R(k_p U_0))(\xi) = (Rf^*)(\xi) \quad \forall\, \xi \in \mathbb{R} \tag{2.9}$$

in the space $C_e(\mathbb{R})$. *Moreover, the operator* $R\, k_p$ *is compact in the space* $C_e(\mathbb{R})$, *and equation* (2.9) *has a unique solution* $U_0 \in C_e(\mathbb{R})$ *which belongs necessarily to the set* $\mathcal{D}_e(\mathbb{R})$.

Proof. Let us use Theorem 3.16 in [29] on reduction of Dirichlet boundary value problem for the axisymmetric potential $\varphi(x, y) = \operatorname{Re} W(z)$ to the Fredholm integral equation (2.9). It follows from Theorem 3.16 in [29] that

$$\varphi(x, y) = \operatorname{Re} W(z) = \frac{1}{2\pi i} \int_{\partial D_z} \frac{F(t)}{\sqrt{(t - z)(t - \bar{z})}}\, dt \quad \forall\, z \in D_z\,, \tag{2.10}$$

where F is defined by equality (2.8), and the function (2.10) is continuously extended onto ∂D_z, and the boundary condition (2.2) is satisfied.

Further, it follows from Theorem 3.10 in [29] (cf. also Theorem 6 in [34]) that the function

$$v(x, y) = \operatorname{Im} W(z) = -\frac{1}{2\pi i y} \int_{\partial D_z} \frac{F(t)(t - x)}{\sqrt{(t - z)(t - \bar{z})}}\, dt \tag{2.11}$$

is continuously extended from D_z onto the set $\partial D_z \setminus \{b_1, b_2\}$.

To complete the proof, it remains to prove that $\operatorname{Im} W(z) \to 0$ when $z \in D_z$, $\operatorname{Im} z \neq 0$ and $z \to b_j$, $j = 1, 2$.

Since ∂D_z is a smooth curve, there exists $\delta_0 > 0$ such that each of the circles $\{t \in \mathbb{C} : |t - b_j| = \delta\}$, $j = 1, 2$, crosses ∂D_z only at two points for all $\delta < \delta_0$. Then for all $z \in D_z$ with $\operatorname{Im} z \neq 0$ and $|z - b_j| = \delta$, $j = 1, 2$, we can replace the cut $\Gamma_{z\bar{z}}$ by another cut $\Gamma'_{z\bar{z}}$ of the plane $\mathbb{C}$ without changing the values of the function $\sqrt{(t-z)(t-\bar{z})}$ for all $t \in \partial D_z$. Moreover, we can take $\Gamma'_{z\bar{z}}$ as the arc of the circle $\{t \in \mathbb{C} : |t-b_j| = \delta\}$ with the end points z and $\bar{z}$ that is located in D_z.

Now, for $z \in D_z$, $\operatorname{Im} z \neq 0$ and $|z-b_j| = \delta$, $j = 1, 2$, using the Cauchy integral theorem, we obtain the equalities

$$\operatorname{Im} W(z) = -\frac{1}{2\pi i y} \int\limits_{\partial D_z} \frac{\big(F(t)-F(b_j)\big)(t-x)}{\sqrt{(t-z)(t-\bar{z})}}\, dt =$$

$$= -\frac{1}{\pi i y} \int\limits_{\Gamma'_{z\bar{z}}} \frac{\big(F(t)-F(b_j)\big)(t-x)}{\big(\sqrt{(t-z)(t-\bar{z})}\,\big)^{+}}\, dt\,,$$

where $\big(\sqrt{(t-z)(t-\bar{z})}\,\big)^{+}$ denotes the values of the function $\sqrt{(t-z)(t-\bar{z})}$ for $t \in \Gamma'_{z\bar{z}}$ that are taken on the right side of the cut $\Gamma'_{z\bar{z}}$.

Since ∂D_z is a smooth curve, there exists a constant $c_0 \in (0;1)$ such that in the case $z \in D_z : |z-b_j| < \delta_0$ and $|\operatorname{Im} z| \le c_0\,|z-b_j|$, $j = 1, 2$, the two-sided inequality $b_1 < \operatorname{Re} z < b_2$ is fulfilled. Therefore, in this case $|t-x|/|y| \le 1$ for all $t \in \Gamma'_{z\bar{z}}$. In the case $z \in D_z : |z-b_j| < \delta_0$ and $|\operatorname{Im} z| > c_0\,|z-b_j|$, $j = 1, 2$, one has the relations $|t-x|/|y| \le 2\,|z-b_j|/(c_0\,|z-b_j|) = 2/c_0$ for all $t \in \Gamma'_{z\bar{z}}$. Thus, the quotient $|t-x|/|y|$ is bounded for all $t \in \Gamma'_{z\bar{z}}$.

Therefore, we obtain the relations

$$|\operatorname{Im} W(z)| \le c_1 \max_{t \in \Gamma'_{z\bar{z}}} \big|F(t)-F(b_j)\big| \int\limits_{\Gamma'_{z\bar{z}}} \frac{|dt|}{\sqrt{|t-z||t-\bar{z}|}} \le$$

$$\le c_2 \max_{t \in \Gamma'_{z\bar{z}}} \big|F(t)-F(b_j)\big| \to 0\,, \quad z \to b_j\,,\ j = 1, 2\,,$$

where the constants c_1, c_2 are independent of z. □

2.4. Reduction of Schwarz BVP–II to the Fredholm integral equation

Preparatory to formulate a result on the reduction of Schwarz BVP–II to the Fredholm integral equation, let us introduce some denotations.

Consider the function $\Pi_*(\xi) := |\xi|^{\beta_0}\,(|\xi|+1)^{\beta_\infty-\beta_0}$, where the numbers β_0, β_∞ satisfy the inequalities $1-\alpha+\nu < \beta_0 < 1$ and $0 < \beta_\infty < \min\{\alpha-\nu, 1/2\}$, in which α and ν are the same numbers as in the definition of the set $\mathcal{H}_\alpha(\partial D)$.

Let us introduce the function

$$n(\xi,\tau) := M(Z,T)\ \left(\sigma(T) - \operatorname{Re}\sigma(Z)\right)$$

of real variables ξ and τ, where $Z = \frac{\xi-i}{\xi+i}$ and $T = \frac{\tau-i}{\tau+i}$.

Consider also the functions

$$\widetilde{n}(\xi,\tau) := \begin{cases} \frac{2\xi}{\pi} \int\limits_{\tau}^{\xi} \frac{s(n(s,\tau)-n(\xi,\tau))}{(\xi^2-s^2)^{3/2}\sqrt{s^2-\tau^2}}\, ds\,, & \text{when } \xi\tau > 0\,, |\tau| < |\xi|\,, \\ 0\,, & \text{when } \xi\tau < 0 \text{ or } \xi\tau > 0\,, |\tau| > |\xi|\,, \end{cases}$$

$$\widetilde{C}(\xi,\tau) := 2\ \operatorname{Re}\tilde{n}(\xi,\tau), \qquad \widetilde{D}(\xi,\tau) := 2\ \operatorname{Im}\tilde{n}(\xi,\tau),$$

$$k_f(\xi,\tau) := -\frac{\xi}{|\xi|}\ \widetilde{D}(\xi,\tau) - \frac{1}{\pi}\int\limits_0^{\xi} \frac{\widetilde{C}(\xi,\eta)}{\eta-\tau}\, d\eta$$

and the integral operators

$$(k_f g)(\xi) := \int\limits_{-\infty}^{\infty} \frac{k_f(\xi,\tau)}{\Pi_*(\tau)}\ g(\tau)\, d\tau\,,$$

$$(Qg)(\xi) := \Pi_*(\xi)\ \Bigg(\frac{A(\xi,\xi)}{(A(\xi,\xi))^2+(B(\xi,\xi))^2}\ \frac{g(|\xi|)}{\operatorname{Im}\sigma(\frac{\xi-i}{\xi+i})}$$

$$+\frac{P(\xi)}{4\pi i}\int\limits_{-\infty}^{\infty} \frac{g(|\tau|)}{\operatorname{Im}\sigma(\frac{\tau-i}{\tau+i})}\ \sqrt{\frac{(\tau^2+1)\ \left|\operatorname{Im}\sigma(\frac{\tau-i}{\tau+i})\right|}{2\ |\tau|}}\ \frac{d\tau}{\tau-\xi}\Bigg).$$

For a given function $u_2\colon \big(\partial D \setminus \{(b_1,0),(b_2,0)\}\big) \longrightarrow \mathbb{R}$, we define the function

$$g_*(\xi) := u_2^*(\xi) - \xi \int\limits_0^{\xi} \frac{s(u_2^*(s)-u_2^*(\xi))}{(\xi^2-s^2)^{3/2}}\, ds\,,$$

where the function u_2^* is expressed via the given function u_2 in the following form:

$$u_2^*(\xi) := \frac{y\, u_2(x,y)}{\sqrt{\xi^2+1}}\,, \qquad x+iy = \sigma\left(\frac{\xi-i}{\xi+i}\right).$$

Denote by $C_u(\mathbb{R})$ the subspace of $C(\mathbb{R})$ including all odd continuous functions. Denote also by $\mathcal{D}_u(\mathbb{R})$ the set of odd functions from $\mathcal{D}(\mathbb{R})$.

The following theorem establishes sufficient conditions for the reduction of Schwarz BVP–II to the Fredholm integral equation.

Theorem 2.2. *Suppose that the given function u_2 is of the form $u_2(x,y) \equiv \widetilde{u}_2(x,y)/y$, where $\widetilde{u}_2 \in \widetilde{\mathcal{H}}_\alpha(\partial D)$ and $\widetilde{u}_2(b_1,0) = \widetilde{u}_2(b_2,0) = 0$. Suppose also that the conformal mapping $\sigma(Z)$ has the nonvanishing continuous contour derivative on the unit circle, and its modulus of continuity satisfies condition* (2.7)*. Then the solution of the Schwarz BVP–II is given by formula* (2.6)*, in which*

$$F(z) = \frac{2(\xi+i)}{\pi i\,\sigma'(\frac{\xi-i}{\xi+i})} \int\limits_{-\infty}^{\infty} \frac{V_0(\xi)}{\Pi_*(\xi)(\tau-\xi)}\,d\tau + C\,,$$

$$z = \sigma\left(\frac{\xi+i}{\xi-i}\right) \quad \forall\, \xi \in \mathbb{C} : \operatorname{Im}\xi > 0\,, \quad (2.12)$$

where C is a real constant, V_0 is a solution of the Fredholm integral equation

$$V_0(\xi) + (Q(k_f V_0))(\xi) = (Qg_*)(\xi) \quad \forall\, \xi \in \mathbb{R} \qquad (2.13)$$

in the space $C_u(\mathbb{R})$. Moreover, the operator $Q\,k_f$ is compact in the space $C_u(\mathbb{R})$, and equation (2.13) *has a unique solution $V_0 \in C_u(\mathbb{R})$ which belongs necessarily to the set $\mathcal{D}_u(\mathbb{R})$ and satisfies the equality*

$$V_0(0) = V_0(\infty) = 0\,.$$

Proof. Let us use Theorem 3.27 in [29] on reduction of Dirichlet boundary value problem for the Stokes flow function $\psi(x,y) = y \operatorname{Im} W(z)$ with the given boundary function $\widetilde{u}_2(x,y)$ to the Fredholm integral equation (2.13). As a result, we obtain that the function (2.11), where F is defined by equality (2.12), is continuously extended from D_z onto the set $\partial D_z \setminus \{b_1, b_2\}$, and the boundary condition (2.3) is satisfied. Moreover, for the function $\operatorname{Im} W(z)$ as well as for the function F an estimate of the type (2.4) is fulfilled. Finally, it is easy to establish that the function (2.10) satisfies an estimate of the type (2.4) if the function F satisfies such an estimate. $\square$

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CONTACT INFORMATION

Sergiy A. Plaksa Department of Complex Analysis and Potential Theory, Institute of Mathematics of the National Academy of Science of Ukraine, Kyiv, Ukraine
E-Mail: plaksa62@gmail.com