About one extremal problem for open sets and partially non-overlapping domains

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Abstract. Sharp estimates of product of inner radii for pairwise disjoint domains are obtained. In particular, we solve an extremal problem in the case of arbitrary finite number of free poles on the system points on the rays.

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1. Introduction

This paper belongs to the theory of extremal problems on classes of disjoint domains, which is a separate direction in the geometric theory of functions of a complex variable. The start of these investigations is associated with the paper by M. A. Lavrent'ev [1]. He found the maximum of some functional with respect to two simply connected domains with two fixed points. We note that he applied this result to some aerodynamics problems. In 1947, G. M. Goluzin solved a similar problem for three fixed points on the complex plane [2]. Then the topic began to evolve rapidly. In this connection, we may recall papers of many authors, including Y. E. Alenitsin, M. A. Lebedev, J. Jenkins, P. M. Tamrazov, P. P. Kufarev, and others. Using the idea of P. M. Tamrazov, G. P. Bakhtina solved, for the first time, the problem with the so-called "free poles" on a unit circle (see, e.g., [3]).

An important step for the development of this topic was the papers by V. N. Dubinin. He proposed a new method of piecewise separating

transformation and first solved a number of extremal problems for any multiconnected disjoint domains (see, e.g., [4–6]). Now, this type of extremal problems is used in investigations in the holomorphic dynamics.

In the last decade, Bakhtin's method of "managing functional" is actively used. A. K. Bakhtin solved a number of extremal problems for the so-called "radial systems of points" (see, e.g., [4, 7–20]). Namely this method will be applied in what follows.

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, let $\mathbb{C}$ be the plane of complex numbers, and let $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ be the Riemannian sphere, $\mathbb{R}_+ = (0; \infty)$.

For a fixed number $n \in \mathbb{N}$ of points

$$A_n = \{a_k\}_{k=1}^n,$$

the following relations are valid:

$$0 = \arg a_1 < \arg a_2 < ... < \arg a_n < 2\pi. \tag{1.1}$$

For such systems of points, we consider the following sizes:

$$\sigma_k = \frac{1}{\pi} \left(\arg a_{k+1} - \arg a_k\right), \ k = 1, 2, ..., n, \quad a_{n+1} := a_1.$$

Consider the system of angular domains

$$M_k := \{w : \arg a_k < \arg w < \arg a_{k+1}\}, \quad k = \overline{1, n}, \quad a_{n+1} := a_1,$$

and the following "managing functional" for an arbitrary system of points A_n:

$$\mathrm{T}\left(A_n\right) = \prod_{k=1}^n \chi \left(\left| \frac{a_k}{a_{k+1}} \right|^{\frac{1}{2\sigma_k}} \right) |a_k|,$$

where $\chi(t) = \frac{1}{2}(t + \frac{1}{t})$.

Let D, $D \subset \overline{\mathbb{C}}$, be any open set, and let the point $w = a \in D$. By $D(a)$, we denote a connected component of the set D, which contains the point a. For an arbitrary system of points $A_n = \{a_k \in \mathbb{C} : k = \overline{1, n}\}$, satisfying condition (1.1) and for the open set D, $A_n \subset D$, we denote, by $D_k\left(a_s\right)$, the connected component of the set $D\left(a_s\right) \bigcap \overline{M_k}$ containing the point a_s, $k = \overline{1, n}$, $s = k, k+1$, $a_{n+1} := a_1$.

We say that the open set D, $A_n \subset D$, satisfies the condition of disjointness relative to the system of points A_n, if the relation

$$D_k(a_k) \bigcap D_k(a_{k+1}) = \varnothing, \tag{1.2}$$

$k = \overline{1,n}$, on all angles $\overline{M_k}$, holds.

System domains $\{B_k\}_{k=1}^n$, $k = \overline{1,n}$, the define system partially non-overlapping domains, if

$$D := \bigcup_{k=1}^n B_k, \tag{1.3}$$

is open set, if she meets a condition (1.2).

Let

$$g_B(z, a) = h_{B,a}(z) + \log \frac{1}{|z - a|}$$

be the generalized Green function of a domain B relative to the point $a \in B$. If $a = \infty$, then

$$g_B(z, \infty) = h_{B,\infty}(z) + \log \frac{1}{|z|}.$$

The quantity

$$r(B, a) := \exp(h_{B,a}(z))$$

stands for the internal radius of the domain $B \subset \overline{\mathbb{C}}$ relative to the point $a \in B$ (see [4–6, 16, 17, 21]).

We use the concept of a quadratic differential. We note that the main results on the theory of quadratic differentials can be found in work [22].

In what follows, we consider the following problems.

Problem 1. Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \geq 0$. To find a maximum of the functional

$$\prod_{k=1}^n \left(|a_{k+1} - a_k|^\alpha \cdot r(D, a_k)\right),$$

where $A_n = \{a_k\}_{k=1}^n$ is any ray system of points that satisfies condition (1.1), D is an open set that satisfies condition (1.2), $a_k \in D \subset \overline{\mathbb{C}}$, and to describe all extremals $(k = \overline{1,n})$.

Problem 2. Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \geq 0$. To find a maximum of the functional

$$\prod_{k=1}^n \left(|a_{k+1} - a_k|^\alpha \cdot r(B_k, a_k)\right),$$

where $A_n = \{a_k\}_{k=1}^n$ is any ray system of points that satisfies condition (1.1), $\{B_k\}_{k=1}^n$ is any collection of partially non-overlapping domains that satisfies condition (1.3), $a_k \in B_k \subset \overline{\mathbb{C}}$, and to describe all extremals $(k = \overline{1,n})$.

2.   The case of open set

Lemma 2.1. *The function*

$$P(\tau) = \ln \sin \frac{\pi\tau}{2}$$

is convex for $\tau \in (0, 2)$.

The proof of the lemma can be found in the papers [20].

Theorem 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \geq 0$. Then for all system points $A_n = \{a_k\}_{k=1}^n$, the satisfied condition (1.1) and condition*

$$(|a_k| - |a_{k+1}|)^2 = 4\sin^2 \frac{\pi\sigma_k}{2}\left(1 - |a_k||a_{k+1}|\right), \ k = \overline{1, n}, \qquad (2.1)$$

arbitrary open set D, the satisfied condition (1.2), $a_k \in D \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, be satisfied inequality

$$\prod_{k=1}^n \left(|a_{k+1} - a_k|^\alpha \cdot r\left(D, a_k\right)\right) \leq \left(\frac{2^{\alpha+2}}{n} \cdot \sin^\alpha \frac{\pi}{n}\right)^n \cdot \mathrm{T}\left(A_n\right).$$

The equality obtain in this inequality, when $D = \bigcup\limits_{k=1}^n B_k$, where points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{w^{n-2}}{(w^n - 1)^2}dw^2. \qquad (2.2)$$

Proof. At once we will note that from the condition of unapplying follows that $\mathrm{cap}\,\overline{\mathbb{C}}\backslash D > 0$ and set D possesses Green's generalized function $g_D(z, a)$, where

$$g_D(z, a) = \begin{cases} g_{D(a)}(z, a), \ z \in D(a), \\ 0, \ z \in \overline{\mathbb{C}}\backslash\overline{D(a)}, \\ \lim\limits_{\zeta \to z} g_{D(a)}(\zeta, a), \ \zeta \in D(a), z \in \partial D(a) \end{cases}$$

– Green's generalized function open set D concerning a point $a \in D$, and $g_{D(a)}(z, a)$ – Green's function domain $D(a)$ concerning a point $a \in D(a)$.

Further, we will use methods of works [4, 6, 7]. Sets we will consider $E_0 = \overline{\mathbb{C}}\backslash D$; $E(a_k, t) = \{w \in \mathbb{C} : |w - a_k| \leqslant t\}$, $k = \overline{1, n}$, $n \geqslant 2$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$. The condenser we will enter into consideration for rather small $t > 0$

$$C\left(t, D, A_n\right) = \{E_0, E_1\},$$

where

$$E_1 = \bigcup_{k=1}^{n} E(a_k, t).$$

Capacity of the condenser $C\,(t,\,D,\,A_{2n,2m-1})$ is called as ([4], [6])

$$\operatorname{cap} C\,(t,\,D,\,A_n) = \inf \int\int \left[(G'_x)^2 + (G'_y)^2 \right]\, dx dy,$$

where an infimum undertakes on all continuous and to the lipschicevym in $\overline{\mathbb{C}}$ functions $G = G(z)$, such that $G\Big|_{E_0} = 0,\, G\Big|_{E_1} = 1.$

Let is named the module of condenser C, reverse the capacity of condenser

$$|C| = [\operatorname{cap} C]^{-1}$$

From a theorem 1 [16] get

$$|C\,(t, D, A_{2n,2m-1})\,| = \frac{1}{2\pi} \cdot \frac{1}{n} \cdot \log \frac{1}{t} + M(D, A_n) + o(1),\, t \to 0, \quad (2.3)$$

where

$$M(D, A_n) = \frac{1}{2\pi} \cdot \frac{1}{n^2} \cdot \left[\sum_{k=1}^{n} \log r(D, a_k) + 2 \sum_{k \neq q} g_D(a_k, a_q) \right]. \quad (2.4)$$

Function

$$\zeta_k\,(w) = -i\left(e^{-i\arg a_k} w\right)^{\frac{1}{\sigma_k}}, \quad k = 1, 2, \dots, n \quad (2.5)$$

realizes univalent and conformal transformations of domain M_k to the right half-plane $\operatorname{Re}\zeta > 0$, for all $k = \overline{1, n}$.

From a formula (2.5) we receive the following asymptotic expressions

$$|\zeta_k\,(w) - \zeta_k\,(a_m)| \sim \frac{1}{\sigma_k} |a_m|^{\frac{1}{\sigma_k} - 1} |w - a_m|,$$

$$w \to a_m, \qquad k = 1, 2, ..., n,\, m = k, k+1. \quad (2.6)$$

It's obvious that

$$\zeta_k\,(a_k) = -i|a_k|^{\frac{1}{\sigma_k}},\, \zeta_k\,(a_{k+1}) = i|a_{k+1}|^{\frac{1}{\sigma_k}},\, k = 1, 2, ..., n,\, a_{n+1} := a_1. \quad (2.7)$$

For any domain $\Delta \in \mathbb{C}$ the define $(\Delta)^* := \left\{ w \in \overline{\mathbb{C}} : \overline{w} \in \Delta \right\}.$

Let $\boldsymbol{\Omega}_k^{(1)}$ define connected component $\zeta_k\left(D \bigcap \overline{M}_k\right) \bigcup \left(\zeta_k\left(D \bigcap \overline{M}_k\right)\right)^*$, containing a point $\zeta_k\,(a_k)$, a $\boldsymbol{\Omega}_{k-1}^{(2)}$ – connected component

$\zeta_{k-1} \left(D \bigcap \overline{M}_{k-1} \right) \bigcup \left(\zeta_{k-1} \left(D \bigcap \overline{M}_{k-1} \right) \right)^*$, containing a point $\zeta_{k-1}(a_k)$, $k = \overline{1,n}$, $\mathbf{\Omega}_0^{(2)} := \mathbf{\Omega}_n^{(2)}$. It is clear, that $\mathbf{\Omega}_k^{(s)}$ generally speaking, domains are multiconnected domains, $k = \overline{1,n}$, $s = 1,2$. Pair of domains $\mathbf{\Omega}_{k-1}^{(2)}$ and $\mathbf{\Omega}_k^{(1)}$ grows out of piece-dividing transformation open set D concerning families $\{M_{k-1}, M_k\}$, $\{\zeta_{k-1}, \zeta_k\}$ in point a_k, $k = \overline{1,n}$, $\overline{M}_0 := \overline{M}_n$, $\zeta_0 := \zeta_n$.

Let's consider condensers

$$C_k(t, D, A_n) = \left(E_0^{(k)}, E_1^{(k)} \right),$$

where

$$E_s^{(k)} = \zeta_k \left(E_s \bigcap \overline{P}_k \right) \bigcup \left[\zeta_k \left(E_s \bigcap \overline{P}_k \right) \right]^*,$$

$k = \overline{1,n}$, $s = 0,1$, $\{M_k\}_{k=1}^n$ – the system of corners corresponding to system of points A_n; operation $[A]^*$ compares to any the set $A \subset \overline{\mathbb{C}}$ a set, symmetric a set A is relative unit circle $|w| = 1$. From this it follows that to the condenser $C(t, D, A_n)$, at dividing transformation is relative $\{P_k\}_{k=1}^n$ and $\{\zeta_k\}_{k=1}^n$, there corresponds a set of condensers the system of corners corresponding to system of points A_n; operation $[A]^*$ compares to any the set $A \subset \overline{\mathbb{C}}$ a set, symmetric a set A is relative unit circle $|w| = 1$. From this it follows that to the condenser $C(t, D, A_n)$, at dividing transformation is relative $\{M_k\}_{k=1}^n$ and $\{\zeta_k\}_{k=1}^n$, there corresponds a set of condensers $\{C_k(t, D, A_n)\}_{k=1}^n$, symmetric relatively $\{z : |z| = 1\}$. According to works [4, 16], we will receive

$$\mathrm{cap}\, C(t, D, A_n) \geqslant \frac{1}{2} \sum_{k=1}^n \mathrm{cap}\, C_k(t, D, A_n). \qquad (2.8)$$

From here follows

$$|C(t, D, A_n)| \leqslant 2 \left(\sum_{k=1}^n |C_k(t, D, A_n)|^{-1} \right)^{-1}. \qquad (2.9)$$

The formula (2.3) gives a module asymptotics $C(t, D, A_n)$ at $t \to 0$, and $M(D, A_n)$ is the given module of a set D relatively A_n. Using formulas (2.6) and that fact that a set D meets the condition of unapplied in relation to the system of points A_n, for condensers we will receive similar asymptotic representations $C_k(t, D, A_n)$, $k = \overline{1,n}$

$$|C_k(t, D, A_n)| = \frac{1}{4\pi} \log \frac{1}{t} + M_k(D, A_n) + o(1),\ t \to 0, \qquad (2.10)$$

where

$$M_k(D, A_n) = \frac{1}{8\pi} \cdot \left[\log \frac{r\left(\mathbf{\Omega}_k^{(1)}, \zeta_k(a_k)\right)}{\frac{1}{\sigma_k} |a_k|^{\frac{1}{\sigma_k} - 1}} + \log \frac{r\left(\mathbf{\Omega}_{k-1}^{(2)}, \zeta_{k-1}(a_k)\right)}{\frac{1}{\sigma_{k-1}} |a_k|^{\frac{1}{\sigma_{k-1}} - 1}} \right].$$

By means of (2.10), we receive

$$|C_k(t, D, A_n)|^{-1} = \frac{4\pi}{\log \frac{1}{t}} \cdot \left(1 + \frac{4\pi}{\log \frac{1}{t}} M_k(D, A_n) + o\left(\frac{1}{\log \frac{1}{t}} \right) \right)^{-1}$$

$$= \frac{4\pi}{\log \frac{1}{t}} - \left(\frac{4\pi}{\log \frac{1}{t}} \right)^2 M_k(D, A_n) + o\left(\left(\frac{1}{\log \frac{1}{t}} \right)^2 \right), \quad t \to 0. \quad (2.11)$$

Further, from (2.11), follows that

$$\sum_{k=1}^{n} |C_k(t, D, A_n)|^{-1}$$

$$= \frac{4\pi n}{\log \frac{1}{t}} - \left(\frac{4\pi}{\log \frac{1}{t}} \right)^2 \cdot \sum_{k=1}^{n} M_k(D, A_n) + o\left(\left(\frac{1}{\log \frac{1}{t}} \right)^2 \right), \quad t \to 0. \quad (2.12)$$

In turn, allows (2.12) to receive the following asymptotic representation

$$\left(\sum_{k=1}^{n} |C_k(t, D, A_n)|^{-1} \right)^{-1}$$

$$= \frac{\log \frac{1}{t}}{4\pi n} \cdot \left(1 - \frac{4\pi}{n \log \frac{1}{t}} \cdot \sum_{k=1}^{n} M_k(D, A_n) + o\left(\frac{1}{\log \frac{1}{t}} \right) \right)^{-1}$$

$$= \frac{\log \frac{1}{t}}{4\pi n} + \frac{1}{n^2} \cdot \sum_{k=1}^{n} M_k(D, A_n) + o(1), \quad t \to 0. \quad (2.13)$$

Inequalities, (2.8) and (2.9) taking into (2.3) and (2.13) allow to notice that

$$\frac{1}{2\pi} \cdot \frac{1}{n} \cdot \log \frac{1}{t} + M(D, A_n) + o(1) \leqslant \frac{\log \frac{1}{t}}{2\pi n} + \frac{2}{n^2} \cdot \sum_{k=1}^{n} M_k(D, A_n) + o(1). \quad (2.14)$$

From (2.14) at $t \to 0$ we receive that

$$M(D, A_n) \leqslant \frac{2}{n^2} \cdot \sum_{k=1}^{n} M_k(D, A_n). \quad (2.15)$$

Formulas (2.4), (2.10) and (2.15) lead to the following expression

$$\frac{1}{2\pi} \cdot \frac{1}{n^2} \cdot \left[\sum_{k=1}^{n} \log r(D, a_k) + 2 \sum_{k \neq q} g_D(a_k, a_q) \right]$$

$$\leq \frac{1}{4\pi n^2} \cdot \left[\sum_{k=1}^{n} \log \frac{r\left(\mathbf{\Omega}_k^{(1)}, \zeta_k(a_k)\right)}{\frac{1}{\sigma_k} |a_k|^{\frac{1}{\sigma_k} - 1}} + \sum_{k=1}^{n} \log \frac{r\left(\mathbf{\Omega}_{k-1}^{(2)}, \zeta_{k-1}(a_k)\right)}{\frac{1}{\sigma_{k-1}} |a_k|^{\frac{1}{\sigma_{k-1}} - 1}} \right].$$

Thus, taking into (2.1), we receive

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k|^\alpha \cdot r(D, a_k) \right) \leq 2^{n\alpha} \cdot \prod_{k=1}^{n} \frac{\sigma_k |a_k|}{|a_k|^{\frac{1}{2\sigma_k}} \cdot |a_k|^{\frac{1}{2\sigma_{k-1}}}}$$

$$\times \prod_{k=1}^{n} \sin^\alpha \frac{\pi \sigma_k}{2} \cdot \prod_{k=1}^{n} \left(r\left(\mathbf{\Omega}_k^{(1)}, \zeta_k(a_k)\right) \cdot r\left(\mathbf{\Omega}_k^{(2)}, \zeta_k(a_{k+1})\right) \right)^{\frac{1}{2}}. \quad (2.16)$$

The equality obtain in this inequality, when points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential

$$Q(\zeta)d\zeta^2 = \frac{d\zeta^2}{(\zeta^2 + 1)^2}. \quad (2.17)$$

Using the Lemma 2.1 that the function $\alpha \ln \sin \frac{\pi \sigma_k}{2}$, is convex for $\sigma_k \in (0; 2)$, $\alpha \geq 0$. Hence, when $\sigma_k \in (0; 2)$, then

$$\frac{\alpha}{n} \cdot \sum_{k=1}^{n} \ln \sin \frac{\pi \sigma_k}{2} \leq \alpha \ln \sin \left(\frac{\pi}{2} \cdot \frac{1}{n} \sum_{k=1}^{n} \sigma_k \right).$$

Given that

$$\sum_{k=1}^{n} \sigma_k = 2,$$

we obtain

$$\prod_{k=1}^{n} \sin^\alpha \frac{\pi \sigma_k}{2} \leq \sin^{n\alpha} \frac{\pi}{n}. \quad (2.18)$$

The equality obtain in this inequality, if and only if

$$\sigma_1 = \sigma_2 = ... = \sigma_n = \frac{2}{n}.$$

Then from (2.16) using formulas (2.18) it is received the following ratio

$$\prod_{k=1}^{n} \left(|a_{k+1} - a_k|^\alpha \cdot r(B_k, a_k) \right) \leq \left(\frac{2^{\alpha+2}}{n} \right)^n \cdot \mathrm{T}(A_n) \cdot \sin^{n\alpha} \frac{\pi}{n}.$$

The equality obtain in this inequality, when points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential (2.2). It is derived from the square of the quadratic differential (2.17) conversion using

$$\zeta = -iw^{\frac{n}{2}}.$$

$\square$

3. The case of partially non-overlapping domains

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \geq 0$. Then for all system points $A_n = \{a_k\}_{k=1}^n$, the satisfied condition (1.1) and condition (2.1), arbitrary system partially non-overlapping domains B_k, the satisfied condition (1.3), $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, be satisfied inequality*

$$\prod_{k=1}^n \left(|a_{k+1} - a_k|^\alpha \cdot r\left(B_k, a_k\right) \right) \leq \left(\frac{2^{\alpha+2}}{n} \cdot \sin^\alpha \frac{\pi}{n} \right)^n \cdot \mathrm{T}\left(A_n\right).$$

The equality obtain in this inequality, where points a_k and domains B_k are, conformity, the poles and the circular domains of the quadratic differential (2.2).

Proof. In case of partially non-overlapping domains, the open set is entered by a representation (1.3), which satisfies (1.2). From here, we have

$$B_k \subset D, \quad k = \overline{1,n}. \tag{3.1}$$

From (3.1), we receive, using results works [5, 6, 21]

$$r\left(B_k, a_k\right) \leq r\left(D, a_k\right), \quad k = \overline{1,n}. \tag{3.2}$$

Multiplying inequalities (3.2), we draw a conclusion

$$\prod_{k=1}^n \left(|a_{k+1} - a_k|^\alpha \cdot r\left(B_k, a_k\right) \right) \leq \prod_{k=1}^n \left(|a_{k+1} - a_k|^\alpha \cdot r\left(D, a_k\right) \right).$$

We receive an end result using the theorem 2.1. $\square$

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