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On functional properties of weak (p, q)-quasiconformal homeomorphisms

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(Presented by V. Gutlyanskiĭ)

The paper is dedicated to the 100th anniversary of G. D. Suvorov

Abstract. In this paper we study functional properties of weak (p, q)-quasiconformal homeomorphisms such as Liouville type theorems, the global integrability and the Hölder continuity. The proof of Liouville type theorems is based on the duality property of weak (p, q)-quasiconformal homeomorphisms.

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1. Introduction

Various generalizations of quasiconformal mappings such as mappings of the Dirichlet class [23] or mappings quasiconformal in mean, (see, for example, [15, 16, 21, 25]) play an important role in the geometric function theory. In the present article we study functional properties of generalized quasiconformal mappings which are connected with composition operators on Sobolev spaces [7, 26, 32]. Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is called a weak (p, q)-quasiconformal homeomorphism, $1 \leq q \leq p \leq \infty$, if $\varphi \in W^1_{q,\mathrm{loc}}(\Omega)$, has finite distortion and

$$K_{p,q}(\varphi; \Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,\ 1/q - 1/p = 1/\kappa\ (\kappa = \infty, if p = q),$$

where p-dilatation of a mapping φ at a point x is defined as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x, \varphi|^{\frac{1}{p}},\ x \in \Omega\}.$$

In the case $p = n$ we have the usual conformal dilatation and if $p = q = n$ this class coincides with quasiconformal mappings. In the

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case $p \neq n$ the p-dilatation arises in [4]. The weak (p,q)-quasiconformal homeomorphisms are natural generalization of (quasi)conformal mappings and have applications in the elliptic operators theory and in the elasticity theory. These applications are based in the composition operators on Sobolev spaces generated by weak (p,q)-quasiconformal homeomorphisms. Note that in the case $p = q = n$ a class of weak (p,q)-quasiconformal homeomorphisms coincides with the usual quasiconformal mappings and in the case $p = n$, $q = n-1$ these mappings are mappings of integrable distortions that were considered in [12, 14]. Weak quasiconformal homeomorphisms allow a capacitary description and on this way are closely connected with so-called Q-homeomorphisms [7]. Study of Q-homeomorphisms are based on the capacitary (moduli) distortion of these classes and is upon intensive development at last decades (see, for example, [15, 22]).

In the theory of weak (p,q)-quasiconformal homeomorphisms the significant role plays the composition duality property [26]. Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $n-1 < q \leq p < \infty$, that induces the bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ n-1 < q \leq p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ be a weak (q',p')-quasiconformal homeomorphism where $p' = p/(p-n+1)$, $q' = q/(q-n+1)$.

Using this composition duality we obtain self-improvement type theorems for weak quasiconformal homeomorphisms and Liouville type theorems.

The classical Liouville theorems states that does not exists a conformal mapping $\varphi : \mathbb{R}^2 \to \widetilde{\Omega}$ onto any bounded domain $\widetilde{\Omega} \subset \mathbb{R}^2$ and in the space $\mathbb{R}^n$, $n \geq 3$, the class of conformal mappings coincide with the Möbius group of transformation. In the present article we prove, in particular, that does not exists a weak (p,q)-quasiconformal homeomorphism, $n-1 < q < p \leq n$, $\varphi : \mathbb{R}^n \to \widetilde{\Omega}$ onto any domain of finite measure $\widetilde{\Omega} \subset \mathbb{R}^n$, $n \geq 2$.

Note, that in the case $p = n$ the Liouville type theorem was proved in [9]. In capacitory terms Liouville type theorems for mappings of bounded (p,q)-distortion were obtained in [28].

The global L_p-integrability of weak derivatives of quasiconformal mappings and its Hölder continuity represent an interesting part of the quasiconformal mapping theory [1, 5, 18]. In the second part of the paper we prove the property of the global integrability of weak derivatives of weak (p,q)-quasiconformal mappings and obtain a self-improvement type theorem.

2. Composition operators on Sobolev spaces

2.1. Sobolev spaces

Let E be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. The Lebesgue space $L_p(E)$, $1 \leq p \leq \infty$, is defined as a Banach space of p-summable functions $f : E \to \mathbb{R}$ equipped with the standard norm.

If Ω is an open subset of $\mathbb{R}^n$, the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|.$$

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

Sobolev spaces are Banach spaces of equivalence classes [20]. To clarify the notion of equivalence classes we use the non-linear p-capacity associated with Sobolev spaces. Recall the notion of the p-capacity of a set $E \subset \Omega$ [20]. Let Ω be a domain in $\mathbb{R}^n$ and a compact $F \subset \Omega$. The p-capacity of the compact F is defined by

$$\operatorname{cap}_p(F; \Omega) = \inf\{\|f|L^1_p(\Omega)\|^p,$$

where infimum is taken over all continuous functions with a compact support $f \in L^1_p(\Omega)$ such that $f \geq 1$ on F. By the similar way we can define p-capacity of open sets.

For arbitrary set $E \subset \Omega$ we define an inner p-capacity as

$$\underline{\operatorname{cap}}_p(E; \Omega) = \sup\{\operatorname{cap}_p(e; \Omega),\ e \subset E \subset \Omega,\ e \text{ is a compact}\},$$

and an outer p-capacity as

$$\overline{\operatorname{cap}}_p(E; \Omega) = \inf\{\operatorname{cap}_p(U; \Omega),\ E \subset U \subset \Omega,\ U \text{ is an open set}\}.$$

A set $E \subset \Omega$ is called p-capacity measurable, if $\underline{\operatorname{cap}}_p(E; \Omega) = \overline{\operatorname{cap}}_p(E; \Omega)$. The value

$$\operatorname{cap}_p(E; \Omega) = \underline{\operatorname{cap}}_p(E; \Omega) = \overline{\operatorname{cap}}_p(E; \Omega)$$

is called the p-capacity of the set $E \subset \Omega$.

The notion of p-capacity permits us to refine the notion of Sobolev functions. Let a function $f \in L^1_p(\Omega)$. Then refined function

$$\tilde{f}(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\, dy$$

is defined quasieverywhere i. e. up to a set of p-capacity zero and it is absolutely continuous on almost all lines [20]. This refined function $\tilde{f} \in L^1_p(\Omega)$ is called a unique quasicontinuous representation (*a canonical representation*) of function $f \in L^1_p(\Omega)$. Recall that a function $\tilde{f}$ is termed quasicontinuous if for any $\varepsilon > 0$ there is an open set U_ε such that the p-capacity of U_ε is less than ε and on the set $\Omega \setminus U_\varepsilon$ the function $\tilde{f}$ is continuous (see, for example [11, 20]). In what follows we will use the quasicontinuous (refined) functions only.

Note that the first weak derivatives of the function f coincide almost everywhere with the usual partial derivatives (see, e.g., [20]).

2.2. Composition operators and the composition duality property

Let $\varphi : \Omega \to \mathbb{R}^n$ be a weakly differentiable mapping. Then the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x,\varphi)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ is the operator norm of $D\varphi(x)$, i. e., $|D\varphi(x)| = \max\{D\varphi(x)\cdot h : h \in \mathbb{R}^n, |h| = 1\}$. Recall that a weakly differentiable mapping $\varphi : \Omega \to \mathbb{R}^n$ is a mapping of finite distortion if $D\varphi(x) = 0$ for almost all $x \in Z = \{x \in \Omega : J(x,\varphi)\} = 0\}$ [30].

Let us recall also the change of variable formula for the Lebesgue integral [3, 10]. Suppose that for a mapping $\varphi : \Omega \to \mathbb{R}^n$ there exists a collection of closed sets $\{A_k\}_1^\infty$, $A_k \subset A_{k+1} \subset \Omega$ for which restrictions $\varphi|_{A_k}$ are Lipschitz mappings on the sets A_k and

$$\left|\Omega \setminus \sum_{k=1}^{\infty} A_k\right| = 0.$$

Then there exists a measurable set $S \subset \Omega$, $|S| = 0$ such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^n$ has the Luzin N-property and the change of variable formula

$$\int\limits_E f \circ \varphi(x) |J(x,\varphi)|\, dx = \int\limits_{\mathbb{R}^n \setminus \varphi(S)} f(y) N_f(E,y)\, dy \qquad (2.1)$$

holds for every measurable set $E \subset \Omega$ and every nonnegative measurable function $f : \mathbb{R}^n \to \mathbb{R}$. Here $N_f(y, E)$ is the multiplicity function defined as the number of preimages of y under f in E.

Note, that Sobolev mappings of the class $W^1_{1,\mathrm{loc}}(\Omega)$ satisfy the conditions of the change of variable formula [10] and so for Sobolev mappings the change of variable formula (2.1) holds.

If the mapping φ possesses the Luzin N-property (the image of a set of measure zero has measure zero), then $|\varphi(S)| = 0$ and the second integral can be rewritten as the integral on $\mathbb{R}^n$. Note, that Sobolev homeomorphisms of the class $L^1_p(\Omega)$, $p \geq n$, possess the Luzin N-property.

Let Ω and $\widetilde{\Omega}$ be domains in $\mathbb{R}^n$, $n \geq 2$. We say that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\Omega) \|f \mid L^1_p(\widetilde{\Omega})\|.$$

The problem of composition operators on Sobolev spaces arises firstly in the work [19] where were introduced sub-areal mappings and in the Reshennyak's problem (1969) connected to quasiconformal mappings [29]. In connection with the geometric function theory we define p-dilatation of a mapping φ at a point x as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x,\varphi|^{\frac{1}{p}}, \; x \in \Omega\}.$$

If $p = n$ we have the usual conformal dilatation and in the case $p \neq n$ the p-dilatation arises in [4].

The geometric theory of composition operators on Sobolev spaces is based on the measure property of composition operators introduced in [26] (in the limit case $p = \infty$ in [27]).

Theorem 2.1. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leq q < p \leq \infty.$$

Then

$$\Phi(\widetilde{A}) = \sup_{f \in L^1_p(\widetilde{A}) \cap C_0(\widetilde{A})} \left(\frac{\|\varphi^*(f) \mid L^1_q(\Omega)\|}{\|f \mid L^1_p(\widetilde{A})\|} \right)^{\kappa},$$

(where $1/q - 1/p = 1/\kappa$) is a bounded monotone countably additive set function defined on open bounded subsets $\widetilde{A} \subset \widetilde{\Omega}$.

The following theorem allows refine this function Φ as a measure generated by the p-dilatation K_p.

Theorem 2.2. *A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leq q \leq p \leq \infty,$$

if and only if $\varphi \in W^1_{1,\mathrm{loc}}(\Omega)$, has finite distortion, and

$$K_{p,q}(\varphi; \Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,$$

where $1/q - 1/p = 1/\kappa$.

This theorem in the case $1 \leq q = p < \infty$ was proved in [7] and in the case $1 \leq q < p < \infty$ in [26] (see also [32]), case $p = \infty$ was considered in [9]. Homeomorphisms that satisfy conditions of Theorem 2.2 are called weak (p, q)-quasiconformal homeomorphisms [7, 31] and are a natural generalization of quasiconformal mappings ($p = q = n$).

In the geometric theory of composition operators on Sobolev spaces the significant role plays the following composition duality property [26]:

Theorem 2.3. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; n-1 < q \leq p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \;\; n-1 < p' \leq q' < \infty,$$

where $p' = p/(p-n+1)$ and $q' = q/(q-n+1)$.

Let us recall for readers convenience short highlights of the proof [26]. On the first step we need to check that the inverse mapping $\varphi^{-1} \in W^1_{1,\mathrm{loc}}(\widetilde{\Omega})$ [26, Theorem 3]. Because $\varphi^{-1} \in W^1_{1,\mathrm{loc}}(\widetilde{\Omega})$ then [24] (see, also, [2, 8, 13])

$$|D\varphi^{-1}(y)| = \begin{cases} \left(\frac{|\operatorname{adj} D\varphi|(x)}{|J(x,\varphi)|}\right)_{x=\varphi^{-1}(y)} & \text{if} \quad x \in \Omega \setminus (S \cup Z), \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$|D\varphi^{-1}(y)| \leq \frac{|D\varphi(x)|^{n-1}}{|J(x,\varphi)|},$$

for almost all $x \in \Omega \setminus (S \cup Z)$, $y = \varphi(x) \in \Omega' \setminus \varphi(S \cup Z)$, and

$$|D\varphi^{-1}(y)| = 0 \text{ for almost all } y \in \varphi(S).$$

Now, taking into account that

$$\frac{q'p'}{q'-p'} = \frac{pq}{(p-q)(n-1)}$$

we obtain

$$\int\limits_{\Omega'} \left(\frac{|D\varphi^{-1}(y)|^{q'}}{|J(y,\varphi^{-1})|} \right)^{p'/(q'-p')} dy \leq \int\limits_{\Omega} \left(\frac{|D\varphi(x)|^{p}}{|J(x,\varphi)|} \right)^{q/(p-q)} dx$$

(in the case $p = q$ we have $p' = q'$ and L_∞-norms instead integrals) and by Theorem 2.2 we have a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \ \ n-1 < p' \leq q' < \infty.$$

Remark 2.4. *In the case $n = 2$ we have $p' = p/(p-1)$, $q' = q/(q-1)$ and $p'' = p$, $q'' = q$. Hence the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ 1 < q \leq p < \infty,$$

if and only if the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \ \ 1 < p' \leq q' < \infty.$$

In the case $n \neq 2$ we have

$$p'' = \left(p'\right)' = \frac{p}{(n-1)^2 - p(n-2)} \neq p, \ \ \text{if}\, p' > n-1,$$

$$q'' = \left(q'\right)' = \frac{q}{(n-1)^2 - q(n-2)} \neq q, \ \ \text{if}\, q' > n-1,$$

and this case is more complicated.

Using this composition duality property we obtain the following self-improvement type theorem.

Theorem 2.5. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $n-1 < q \leq p < n$. Then φ induces a bounded composition operator*

$$\varphi^* : L^1_r(\widetilde{\Omega}) \to L^1_s(\Omega)$$

for all $s \leq r$ such that $q'' \leq s \leq q$ and $p'' \leq r \leq p$.

Proof. Because $\varphi : \Omega \to \widetilde{\Omega}$ is a weak (p,q)-quasiconformal homeomorphism, then the composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n-1 < q \leq p < \infty,$$

is bounded.

In the case $p < n$ and $q < n$ we have that

$$p'' = \big(p'\big)' = \frac{p}{(n-1)^2 - p(n-2)} < p$$

and

$$q'' = \big(q'\big)' = \frac{q}{(n-1)^2 - q(n-2)} < q.$$

So by the composition duality property we have that this weak (p,q)-quasiconformal homeomorphism generates also a bounded composition operator

$$\varphi^* : L^1_{p''}(\widetilde{\Omega}) \to L^1_{q''}(\Omega), \ \ 1 < q'' < q \leq p'' < p < \infty.$$

Using the Marcinkiewicz interpolation theorem [6] we obtain that

$$\varphi^* : L^1_r(\widetilde{\Omega}) \to L^1_s(\Omega)$$

is bounded for all $s \leq r$ such that $q'' \leq s \leq q$ and $p'' \leq r \leq p$. □

3. Liouville type theorems for weak (p,q)-quasiconformal homeomorphisms

The composition duality property allows us to obtain Liouville type theorems for weak (p,q)-quasiconformal homeomorphisms.

Theorem 3.1. *Let $n < p < \infty$ and suppose there exists a weak (p,n)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\widetilde{\Omega}| < \infty$.*

Proof. Because $\varphi : \Omega \to \widetilde{\Omega}$ is a weak (p,n)-quasiconformal homeomorphism, then the composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_n(\Omega), \ \ n < p < \infty,$$

is bounded. By the duality property the inverse composition operator

$$\big(\varphi^{-1}\big)^* : L^1_n(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \ \ p' < n,$$

will be bounded. Hence, for any function $f \in L^1_p(\widetilde{\Omega})$ the inequality

$$\|f \mid L^1_{p'}(\widetilde{\Omega})\| \leq \| \big(\varphi^{-1}\big)^* \| \|\varphi^*(f) \mid L^1_n(\Omega)\| \leq \| \big(\varphi^{-1}\big)^* \| \|\varphi^*\| \|f \mid L^1_p(\widetilde{\Omega})\|$$

holds. It means that the embedding

$$L^1_p(\widetilde{\Omega}) \hookrightarrow L^1_{p'}(\widetilde{\Omega}),\ n-1<p'<p<\infty,$$

holds. Hence $|\widetilde{\Omega}|<\infty$. □

From this theorem immediately follows

Corollary 3.2. *For any $n<p<\infty$ and any domain $\Omega\subset\mathbb{R}^n$ does not exist a weak (p,n)-quasiconformal homeomorphism $\varphi:\Omega\to\mathbb{R}^n$.*

Remark 3.3. *This corollary can be formulated in the strong form: for any domain $\Omega\subset\mathbb{R}^n$ and any $n<p<\infty$ does not exists a weak (p,n)-quasiconformal homeomorphism φ from Ω onto any domain of unbounded volume.*

In the case $n<q<p$ we have an additional assumption of finiteness of a measure of Ω.

Theorem 3.4. *Let $n<q<p<\infty$ and $|\Omega|<\infty$. Suppose there exists a weak (p,q)-quasiconformal homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$. Then $|\widetilde{\Omega}|<\infty$.*

Proof. Because $\varphi:\Omega\to\widetilde{\Omega}$ is a weak (p,q)-quasiconformal homeomorphism, then the composition operator

$$\varphi^*: L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),\ n<q<p<\infty,$$

is bounded. By the duality property the inverse composition operator

$$\left(\varphi^{-1}\right)^*: L^1_{q'}(\Omega)\to L^1_{p'}(\widetilde{\Omega}),\ n-1<p'<q'<n,$$

will be bounded also. Because $|\Omega|<\infty$, then this embedding

$$L^1_q(\Omega)\hookrightarrow L^1_{q'}(\Omega)$$

holds. Hence the embedding

$$L^1_p(\widetilde{\Omega}) \hookrightarrow L^1_{p'}(\widetilde{\Omega}),\ n-1<p'<p<\infty,$$

holds. Therefore $|\widetilde{\Omega}|<\infty$. □

From this theorem immediately follows

Corollary 3.5. *For any domain $\Omega\subset\mathbb{R}^n$, $|\Omega|<\infty$, and any $n<p<\infty$ does not exist a weak (p,q)-quasiconformal homeomorphism $\varphi:\Omega\to\mathbb{R}^n$.*

Remark 3.6. *This corollary can be formulated in the strong form: for any domain $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$, and any $n < p < \infty$ does not exists a weak (p, n)-quasiconformal homeomorphism φ from Ω onto any domain of unbounded volume.*

In the case $n - 1 < q < p \leq n$ using the dual composition property we obtain dual Liouville type theorems for weak (p, q)-quasiconformal homeomorphisms.

Theorem 3.7. *Let $n - 1 < q < n$ and suppose there exists a weak (n, q)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\Omega| < \infty$.*

From this theorem immediately follows

Corollary 3.8. *For any $n - 1 < q < n$ and any domain $\widetilde{\Omega}$ does not exist a weak (n, q)-quasiconformal homeomorphism $\varphi : \mathbb{R}^n \to \widetilde{\Omega}$.*

In the case $n - 1 < q \leq p < n$ we have an additional assumption of finiteness of a measure of $\widetilde{\Omega}$.

Theorem 3.9. *Let $n - 1 < q < p < n$ and $|\widetilde{\Omega}| < \infty$. Suppose there exists a weak (p, q)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\Omega| < \infty$.*

From this theorem immediately follows

Corollary 3.10. *For any domain $\widetilde{\Omega}$ such that $|\widetilde{\Omega}| < \infty$ and for any $n - 1 < q < p < n$ does not exist a weak (p, q)-quasiconformal homeomorphism $\varphi : \mathbb{R}^n \to \widetilde{\Omega}$.*

4. Composition operators and integrability of derivatives

The global L_p-integrability of weak derivatives of quasiconformal mappings and its Hölder continuity represent an interesting part of the quasiconformal mapping theory [1, 5, 18]. In the next theorem we consider the property of the global integrability of weak derivatives of weak (p, q)-quasiconformal mappings.

Theorem 4.1. *Let homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ 1 \leq q \leq p \leq \infty.$$

If $p \neq n$, then $|D\varphi|^{\frac{p-n}{p}} \in L_\kappa(\Omega)$ where $1/q - 1/p = 1/\kappa$.

Proof. The case $p=q$ was proved in [7] and the case $p=\infty$ was considered in [9].

Let $n<p<\infty$. We denote $Z=\{x\in\Omega : J(x,\varphi)=0\}$. Because φ generates a bounded composition operator

$$\varphi^{*} : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),$$

then by Theorem 2.2 φ is a mapping of finite distortion and so $D\varphi(x)=0$ for almost all $x\in Z$. Using Theorem 2.2 and the Hadamard's inequality:

$$|J(x,\varphi)| \leq |D\varphi(x)|^n, \text{ for almost all } x\in \Omega\setminus Z,$$

we have

$$\||D\varphi|^{\frac{p-n}{p}} \mid L_{\kappa}(\Omega)\| = \left(\int\limits_{\Omega} |D\varphi(x)|^{\frac{p-n}{p}\frac{pq}{p-q}}\ dx\right)^{\frac{p-q}{pq}}$$
$$= \left(\int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{p-n}{p}\frac{pq}{p-q}}\ dx\right)^{\frac{p-q}{pq}} = \left(\int\limits_{\Omega\setminus Z} \left(|D\varphi(x)|^{p-n}\right)^{\frac{q}{p-q}}\ dx\right)^{\frac{p-q}{pq}}$$
$$\leq \left(\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\ dx\right)^{\frac{p-q}{pq}} = \|K_p \mid L_{\kappa}(\Omega)\| < \infty.$$

Let $1\leq q<p<n$. Because $Z=\{x\in\Omega : J(x,\varphi)=0\}$, then by the change of variable formula for weakly differentiable mappings [10] we have $|\varphi(Z)|=0$. Since in the case $1\leq q<p<n$ the mapping φ possesses the Luzin N^{-1} property (preimage of a set of a measure zero has measure zero) [31,32] we have $|Z|=0$ and so $|J(x,\varphi)|\neq 0$ a.e. in Ω. Hence by the Hadamard's inequality:

$$\||D\varphi|^{\frac{p-n}{p}} \mid L_{\kappa}(\Omega)\| = \left(\int\limits_{\Omega} |D\varphi(x)|^{\frac{p-n}{p}\frac{pq}{p-q}}\ dx\right)^{\frac{p-q}{pq}}$$
$$= \left(\int\limits_{\Omega\setminus Z} \left(|D\varphi(x)|^{p-n}\right)^{\frac{q}{p-q}}\ dx\right)^{\frac{p-q}{pq}} \leq \left(\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\ dx\right)^{\frac{p-q}{pq}}$$
$$= \|K_p \mid L_{\kappa}(\Omega)\| < \infty.$$

□

Remark 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $p \neq n$. Then $|D\varphi|^{\frac{p-n}{p}} \in L_\alpha(\Omega)$ for any $\alpha \leq \kappa$.*

In [31] it was proved that weak (p,q)-quasiconformal homeomorphism, $n < q < p < \infty$, are locally Hölder continuous with the Hölder exponent $\alpha = p(q-n)/q(p-n)$. As a consequence of Theorem 4.1 we obtain the property of global Hölder continuity for weak (p,q)-quasiconformal homeomorphism in the case of continuous embedding domains. Note, that we call a domain $\Omega \subset \mathbb{R}^n$ as a Hölder continuous embedding domain if the embedding operator of the Sobolev space to the space of continuous functions

$$W^1_p(\Omega) \hookrightarrow C(\Omega), \;\; p > n,$$

is bounded. Examples of such domains are domains with Lipschitz boundaries or domains with the uniform interior cone condition (see, for example, [6]).

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a Hölder continuous embedding domain and homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; n < q \leq p \leq \infty.$$

Then φ belongs to the Hölder space $H^\alpha(\Omega)$, $\alpha = p(q-n)/q(p-n)$.

Proof. By Theorem 4.1 a mapping $\varphi \in L^1_s(\Omega)$ for

$$s = \frac{p-n}{p}\frac{pq}{p-q} = \frac{(p-n)q}{p-q}$$

In the case $n < q < p \leq \infty$ we have

$$\frac{(p-n)q}{p-q} > n$$

and using Sobolev embedding theorems into the spaces of Hölder continuous functions [20] we obtain that φ belongs to $H^\alpha(\Omega)$, $\alpha = p(q-n)/q(p-n)$. □

Corollary 4.4. *Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a continuous embedding domain and homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; n-1 < q < p < n.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to the Hölder space $H^\alpha(\widetilde{\Omega})$, $\alpha = q'(p'-n)/p'(q'-n)$.

Proof. By the duality theorem [26, 31] the inverse mapping generates a bounded composition operator

$$(\varphi^{-1})^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}),$$

where $q' = q/(q-n+1)$, $p' = p/(p-n+1)$, $n < p' \leq q' \leq \infty$.

By Corollary 4.3 we obtain that the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to the Hölder space $H^{\alpha}(\widetilde{\Omega})$, $\alpha = q'(p'-n)/p'(q'-n)$. □

The global integrability of derivatives allows us to obtain the second type of self improvement theorem for composition operators on Sobolev spaces. Namely if φ is a weak (p,q)-quasiconformal homeomorphism, then φ be also a weak (r,s)-quasiconformal mapping under some restrictions on r and s that depend on p and q.

Theorem 4.5. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $1 < q < p < \infty$. Then φ is a weak (r,s)-quasiconformal homeomorphism for all $1 < s < r < \infty$ such that $p/q \leq r/s$ and*

$$\frac{rs - ps}{rq - ps} = \frac{(p-n)}{p-q}.$$

Proof. By Theorem 2.2 it is sufficient to check that

$$K_{r,s}(\varphi;\Omega) = \|K_r \mid L_{\kappa'}(\Omega)\| < \infty,$$

where $1/s - 1/r = 1/\kappa'$.

Because φ is the weak (p,q)-quasiconformal mapping then φ is a mapping of finite distortion. Denote $Z = \{x \in \Omega : J(x,\varphi) = 0\}$. Then

$$\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^r}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} dx = \int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p |D\varphi(x)|^{r-p}}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} dx$$
$$= \int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} \left(|D\varphi(x)|^{r-p}\right)^{\frac{s}{r-s}} dx.$$

By conditions of the theorem we have that $s/(r-s) \leq q/(p-q)$. Hence, using the Hölder inequality we obtain

$$\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^r}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} dx$$
$$\leq \left(\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{p}{p-q}} dx\right)^{\frac{s(p-q)}{q(r-s)}} \left(\int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{qs(r-p)}{qr-ps}} dx\right)^{\frac{qr-ps}{q(r-s)}}.$$

Using the equality

$$\frac{qs(r-p)}{qr-ps} = \frac{q(p-n)}{p-q}$$

and Theorem 4.1 we have

$$\int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{qs(r-p)}{qr-ps}}\ dx = \int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{q(p-n)}{p-q}}\ dx < \infty.$$

Hence

$$K_{r,s}(\varphi;\Omega) = \|K_r \mid L_{\kappa'}(\Omega)\| < \infty,$$

where $1/s - 1/r = 1/\kappa'$. □

Remark 4.6. *Recall that for bounded domains* $|D\varphi|^{\frac{p-n}{p}} \in L_\alpha(\Omega)$ *for any* $\alpha \leq \frac{q(p-n)}{p-q}$. *Therefore for bounded domains the second condition of the previous theorem is*

$$\frac{qs(r-p)}{qr-ps} \leq \frac{q(p-n)}{p-q}.$$

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