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Filtering of stationary Gaussian statistical experiments

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Abstract. This article proposes a new filtering model for stationary Gaussian Markov statistical experiments, given by diffusion-type difference stochastic equations.

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1. Stationary statistical experiments

Much of the applied research is concerned with the dynamics of the frequency of some factor A in a complex system. Or, which is almost the same, with the dynamics of its concentration. Such important characteristics are described by the concept of a statistical experiment.

The *statistical experiment (SE)* is defined as the averaged sums:

$$S_N(k) = \frac{1}{N}\sum_{r=1}^{N} \delta_r(k),\ k \geq 0, \tag{1.1}$$

in which the random variables $\delta_r(k)$, $1 \leq r \leq N$, $k \geq 0$, are equally distributed and independent for each fixed $k \geq 0$, which take binary values 0 or 1.

In particular, let us consider

$$\delta_r(k) = I(A) = \begin{cases} 1, & \text{if event } A \text{ occurs;} \\ 0, & \text{if event } A \text{ does not occur.} \end{cases}, \quad 1 \leq r \leq N\ , \quad k \geq 0.$$

In this case, the random amount $S_N(k)$, $k \geq 0$, describes the relative frequencies of presence of the attribute A in a sample of fixed volume N at each time instant $k \geq 0$.

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Introducing the normalized fluctuations $\zeta_N(k) := \sqrt{N}(S_N(k) - \rho)$, where ρ be the equilibrium of SE [2, Sect. 3], one gets an important representation of SEs.

Namely, some natural conditions [2, Proposition 5.2], the SE (1.1) has the following diffusion approximation:

$$\Delta\zeta(k+1) = -V\zeta(k) + \sigma\Delta W(k+1)\ , \quad 0 \le V \le 2\ , \quad \sigma \ge 0, \tag{1.2}$$

where $\Delta\zeta(k+1) := \zeta(k+1) - \zeta_N(k)$ and $\Delta W(k+1)$ are the standard normally distributed martingale – differences. The solution of the difference stochastic equation (1.2) is called *discrete Markov diffusion* (DMD).

The next theorem [2, Thm.1] gives the necessary and sufficient conditions of the stationarity, in wide sense, of the DMD (1.2).

Theorem 1. *(Theorem on stationarity). The DMD* (1.2) *is a stationary random sequence in wide sense if and only if the following relations take place:*

$$E\zeta(0) = 0\ , \quad E\zeta^2(0) = R_\zeta = \sigma^2/(2V - V^2). \tag{1.3}$$

Now consider a stationary, in wide sense, two-component random sequence $\Big(\zeta(k),\ \Delta\zeta(k+1)\Big)$, $k \ge 0$, with the following joint covariances:

$$R_\zeta = E\big[\zeta(k)\big]^2\ , \quad R_\zeta^0 = E\big[\zeta(k)\Delta\zeta(k+1)\big]\ , \quad R_\zeta^\Delta = E\big[\Delta\zeta(k+1)\big]^2. \tag{1.4}$$

In filtration problems of Gaussian stationary DMD, the equivalence formulated in the following theorem is essentially used.

Theorem 2. *(Theorem on equivalence). Let the two-component Gaussian Markov random sequence* $\big(\zeta(k),\ \Delta\zeta(k+1)\big)$, $k \ge 0$, *with the mean value* $E[\zeta(k)] = 0$, $k \ge 0$, *and the joint covariances* (1.4) *that satisfy the stationarity condition*

$$\sigma^2 = \big(2V - V^2\big)R_\zeta. \tag{1.5}$$

Then the random sequence $\Big(\zeta(k),\ \Delta\zeta(k+1)\Big)$, $k \ge 0$, *is a solution of the stochastic difference equation* (1.2), *that is a DMD.*

Proof. By Theorem on normal correlation [1, Th. 13.1], one has:

$$E\big[\Delta\zeta(k+1)\,|\,\zeta(k)\big] = R_\zeta^0 R_\zeta^{-1}\zeta(k). \tag{1.6}$$

Hence

$$R_\zeta^0 = -VR_\zeta \quad \text{and} \quad R_\zeta^\Delta = 2VR_\zeta. \tag{1.7}$$

Considering the martingale-differences

$$\Delta W(k+1) = \frac{1}{\sigma}\Big(\Delta\zeta_N(k+1) + V\zeta(k)\Big). \tag{1.8}$$

Let's calculate its first two moments.

$$E\Big[\Delta W(k+1)\Big] = \frac{1}{\sigma}E\Big[\Delta\zeta_N(k+1) + V\zeta(k)\Big] = 0, \tag{1.9}$$

$$E\Big[\Delta W(k+1)\Big]^2 = 1. \tag{1.10}$$

Now it remains to prove that, the stochastic part covariations are:

$$E\Big[\Delta W(k+1)\Delta W(r+1)\Big] = \begin{cases} 1, & \text{if } k = r, \\ 0, & \text{otherwise.} \end{cases} \tag{1.11}$$

Suppose for determination, that $r < k$. Using the Markov property of the sequence $\zeta(k)$, $k \geq 0$, and the relation (1.9), one obtains:

$$E\Big[\Delta W(k+1)\,|\,\zeta(r),\zeta(k)\Big] = E\Big[\Big(\Delta\zeta(k+1)+V\zeta(k)\Big)\,|\,\zeta(k)\Big] = 0. \tag{1.12}$$

Theorem 2 is proved. □

2. Filtering of discrete Markov diffusion

In our constructions of the new filter, we proceed from the following basic principle: the presence of two normally distributed random sequences implies the presence of their covariances, which contain information about the filtering.

The task is to estimate the unknown parameters of a stationary Gaussian Markov signal process $\alpha(k)$ by using the trajectories of the signal $(\alpha(k),\, \Delta\alpha(k+1))$, and a stationary Gaussian Markov filtering process $(\beta(k),\, \Delta\beta(k+1))$, $k \geq 0$.

The signal with unknown parameters is determined by the next equation:

$$\Delta\alpha(k+1) = -V_0\alpha(k) + \sigma_0\Delta W^0(k+1)\ , \quad k \geq 0. \tag{2.1}$$

The filtering process – by the equation:

$$\Delta\beta(k+1) = -V\beta(k) + \sigma\Delta W(k+1)\ , \quad k \geq 0. \tag{2.2}$$

It is known that the best estimate (in the mean square sense) of the signal $(\alpha(k),\, \Delta\alpha(k+1))$, by observing the filtering proccess $(\beta(k),\, \Delta\beta(k+1))$, coincides with the conditional expectation [2]

$$(\widehat{\alpha}(k),\, \Delta\widehat{\alpha}(k+1)) = E\Big[(\alpha(k),\, \Delta\alpha(k+1))\,\Big|\,(\beta(k),\, \Delta\beta(k+1))\Big]. \quad (2.3)$$

The next calculation of filtering matrix $\mathbf{\Phi}_\beta$, determined by the conditional expectation (2.3), essentially uses Theorem 2 on equivalence and is based on Theorem on normal correlation by Liptser and Shiryaev [1].

Theorem 3. *The estimate* (2.3) *is determined by the filtering equation*

$$\big(\widehat{\alpha}(k), \quad \Delta\widehat{\alpha}(k+1)\big) = \mathbf{\Phi}_\beta \cdot \begin{pmatrix} \beta(k) \\ \Delta\beta(k+1) \end{pmatrix}, \quad (2.4)$$

with the filtering matrix

$$\mathbf{\Phi}_\beta = \begin{bmatrix} 1 & 0 \\ -V_0 & 0 \end{bmatrix} R_{\alpha\beta} R_\beta^{-1}. \quad (2.5)$$

where

$$R_{\alpha\beta} := E\big[\alpha(k)\beta(k)\big]\ , \quad R_\beta := E\big[\beta^2(k)\big]. \quad (2.6)$$

Corollary 1. *The interpolation of the signal $\alpha(k)$ by observing the filtering process $\beta(k)$ is*

$$\widehat{\alpha}(k) = \mathbf{\Phi}_{11}\beta(k)\ , \quad \mathbf{\Phi}_{11} := R_{\alpha\beta} R_\beta^{-1};$$
$$\Delta\widehat{\alpha}(k+1) = \mathbf{\Phi}_{21}\beta(k) = -V_0 \mathbf{\Phi}_{11}\Delta\beta(k+1).$$

Corollary 2. *One has the following statistical parameter estimation:*

$$V_0 \approx -\frac{\mathbf{\Phi}_{21}}{\mathbf{\Phi}_{11}}. \quad (2.7)$$

Corollary 3. *One has the following statistical parameter estimation:*

$$\sigma_\alpha \approx \sigma_\alpha^T = \mathcal{E}_0^T \cdot R_\alpha^T\ , \quad \mathcal{E}_0^T := 2V_0^T - \big(V_0^T\big)^2. \quad (2.8)$$

Proposition 1. *Under the assumption of mutual uncorrelatedness* (2.12)*, the statistical estimates* (2.7)[1] – (2.8) *are unbiased and strongly consistent, as $T \to \infty$.*

[1] For the filtering parameter estimation see the Section 4.

Proof of Theorem 3. By the Theorem on normal correlation [1, Th. 13.1], the filtering matrix introduced in (2.4) has the following form:

$$\mathbf{\Phi}_\beta = \mathbb{R}_{\alpha\beta}\mathbb{R}_\beta^{-1}, \tag{2.9}$$

where

$$\mathbb{R}_{\alpha\beta} := \begin{bmatrix} R_{\alpha\beta} & R^0_{\alpha\beta} \\ R^0_{\beta\alpha} & R^\Delta_{\alpha\beta} \end{bmatrix} , \quad \mathbb{R}_\beta := \begin{bmatrix} R_\beta & R^0_\beta, \\ R^0_\beta & R^\Delta_\beta \end{bmatrix}, \tag{2.10}$$

and the covariances are defined as follows:

$$\begin{aligned} &R^0_{\alpha\beta} := E(\alpha(k)\Delta\beta(k+1)), \\ &R^0_{\beta\alpha} := E(\beta(k)\Delta\alpha(k+1)) , \quad R^\Delta_{\alpha\beta} := E(\Delta\alpha(k+1)\Delta\beta(k+1)), \\ &R^0_\beta := E(\beta(k)\Delta\beta(k+1)) , \quad R^\Delta_\beta := E[(\Delta\beta(k+1))^2]. \end{aligned} \tag{2.11}$$

Under the assumption of mutual uncorrelatedness

$$E\big[\alpha(k)\cdot W(k)\big] = 0, \quad E\big[\beta(k)\cdot W^0(k)\big] = 0$$

$$\text{and } E\big[W(k)\cdot W^0(k)\big] = 0, \quad k \geq 0, \tag{2.12}$$

the equations (2.1)–(2.2) imply the following representations:

$$\begin{aligned} &R^0_{\alpha\beta} = -VR_{\alpha\beta} , \quad R^0_{\beta\alpha} = -V_0 R_{\alpha\beta} , \quad R^\Delta_{\alpha\beta} = VV_0R_{\alpha\beta}; \\ &R^0_\beta = -VR_\beta , \quad R^\Delta_\beta := 2VR_\beta. \end{aligned} \tag{2.13}$$

So the matrix

$$\mathbb{R}_\beta = \begin{bmatrix} R_\beta & -VR_\beta \\ -VR_\beta & 2VR_\beta \end{bmatrix} \tag{2.14}$$

has the following inversion:

$$\mathbb{R}_\beta^{-1} = \begin{bmatrix} 2VR_\beta & VR_\beta \\ VR_\beta & R_\beta \end{bmatrix} \cdot d_\beta^{-1} , \quad d_\beta := (2V - V^2)R^2_\beta. \tag{2.15}$$

Now let's calculate the elements of the filtering matrix

$$\mathbf{\Phi}_\beta = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{2.16}$$

By (2.9), taking into account (2.15), one has:

$$\mathbf{\Phi}_\beta = \begin{bmatrix} R_{\alpha\beta} & -VR_{\alpha\beta} \\ -V_0R_{\alpha\beta} & VV_0R_{\alpha\beta} \end{bmatrix} \cdot \begin{bmatrix} 2VR_\beta & VR_\beta \\ VR_\beta & R_\beta \end{bmatrix} \cdot d_\beta^{-1}. \tag{2.17}$$

Hence, taking into account the relation $d_\beta := (2V - V^2)R_{\alpha\beta}R_\beta^2$, one obtains

$$\begin{aligned}\Phi_{11} &= R_{\alpha\beta}R_\beta^{-1},\\ \Phi_{21} &= -V_0 R_{\alpha\beta}R_\beta^{-1},\\ \Phi_{12} &= \Phi_{22} = 0.\end{aligned} \tag{2.18}$$

So the matrix

$$\mathbf{\Phi}_\beta = \begin{bmatrix} R_{\alpha\beta}R_\beta^{-1} & 0\\ -V_0 R_{\alpha\beta}R_\beta^{-1} & 0\end{bmatrix}, \tag{2.19}$$

which is equivalent to (2.5). Theorem 3 is proved. □

3. The filtering error

Let's denote the filtering mean square estimation error

$$\Gamma(k) = E\Big(\alpha(k) - \widehat{\alpha}(k)\Big)^2 + E\Big(\Delta\alpha(k+1) - \Delta\widehat{\alpha}(k+1)\Big)^2. \tag{3.1}$$

By stationarity of the processes $\alpha(k)$ and $\beta(k)$, $k \geq 0$, we shall skip the parameter k where it is considered possible and convenient.

By the normal correlation theorem [1, Theorem 13.3], the mean square error of the filtering is expressed as the trace of the following error matrix:

$$\begin{aligned}\mathbf{\Gamma} = \begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{21} & \Gamma_{22}\end{bmatrix} &= cov[\alpha(k), \Delta\alpha(k+1) \,|\, \beta(k), \Delta\beta(k+1)]\\ &= \mathbb{R}_\alpha - \mathbb{R}_{\alpha\beta}\mathbb{R}_\beta^{-1}\mathbb{R}_{\alpha\beta}^* = \mathbb{R}_\alpha[\mathbb{I} - \underbrace{\mathbb{R}_\alpha^{-1}\mathbb{R}_{\alpha\beta}}_{\mathbf{\Phi}_\alpha}\underbrace{\mathbb{R}_\beta^{-1}\mathbb{R}_{\alpha\beta}^*}_{\mathbf{\Phi}_\beta^*}], \quad \forall k \geq 0.\end{aligned} \tag{3.2}$$

Let's denote $\mathcal{F}_k^\beta$ the natural increasing sequence of σ-algebras of events, generated by the trajectories of the filtering DMD $\beta(k)$, $k \geq 0$. Then the elements of error matrix $\mathbf{\Gamma}$ are defined as:

$$\left.\begin{aligned}\Gamma_{11} &= E\big[\big(\alpha(k) - \widehat{\alpha}(k)\big)^2 \,|\, \mathcal{F}_k^\beta\big],\\ \Gamma_{12} &= E\big[(\alpha(k) - \widehat{\alpha}(k))(\Delta\alpha(k+1) - \Delta\widehat{\alpha}(k+1)) \,|\, \mathcal{F}_k^\beta\big],\\ \Gamma_{21} &= E\big[(\Delta\alpha(k+1) - \Delta\widehat{\alpha}(k+1))(\alpha(k) - \widehat{\alpha}(k)) \,|\, \mathcal{F}_k^\beta\big],\\ \Gamma_{22} &= E\big[\big(\Delta\alpha(k+1) - \Delta\widehat{\alpha}(k+1)\big)^2 \,|\, \mathcal{F}_k^\beta\big],\end{aligned}\right\}, \quad \forall k \geq 0,$$

and

$$\mathbb{R}_\alpha = \begin{bmatrix} 1 & -V_0\\ -V_0 & 2V_0\end{bmatrix} \cdot R_\alpha^2,$$

the covariation matrix $\mathbb{R}_{\alpha\beta}$ is defined in (2.10) and the term $\mathbf{\Phi}_\beta$ is defined in formula (2.19).

Let's calculate the term $\mathbf{\Phi}_\alpha$.

$$\mathbf{\Phi}_\alpha = \mathbb{R}_\alpha^{-1}\mathbb{R}_{\alpha\beta} = \begin{bmatrix} R_\alpha^\Delta & R_\alpha^0 \\ R_\alpha^0 & R_\alpha \end{bmatrix} \cdot \begin{bmatrix} R_{\alpha\beta} & R_{\alpha\beta}^0 \\ R_{\alpha\beta}^0 & R_{\alpha\beta}^\Delta \end{bmatrix} \cdot d_\alpha^{-1}, \qquad d_\alpha := V_0(2-V_0)R_\alpha^2. \tag{3.3}$$

So that

$$\mathbf{\Phi}_\alpha = \begin{bmatrix} 1 & -V \\ 0 & 0 \end{bmatrix} \cdot R_{\alpha\beta}R_\alpha^{-1}. \tag{3.4}$$

Next, using (2.5), one obtains

$$\mathbf{\Phi}_\beta^* = \mathbb{R}_\beta^{-1}\mathbb{R}_{\alpha\beta}^* = \begin{bmatrix} 1 & -V_0 \\ 0 & 0 \end{bmatrix} \cdot R_{\alpha\beta}R_\beta^{-1}. \tag{3.5}$$

So

$$\begin{aligned} \mathbf{\Phi}_\alpha \cdot \mathbf{\Phi}_\beta^* &= \begin{bmatrix} 1 & -V \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -V_0 \\ 0 & 0 \end{bmatrix} \cdot R_{\alpha\beta}^2 R_\alpha^{-1} R_\beta^{-1} \\ &= \begin{bmatrix} 1 & -V_0 \\ 0 & 0 \end{bmatrix} \cdot R_{\alpha\beta}^2 R_\alpha^{-1} R_\beta^{-1}. \end{aligned} \tag{3.6}$$

Hence

$$\begin{aligned} \mathbb{I} - \mathbf{\Phi}_\alpha\mathbf{\Phi}_\beta^* &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -V_0 \\ 0 & 0 \end{bmatrix} \underbrace{R_{\alpha\beta}^2 R_\alpha^{-1} R_\beta^{-1}}_{=:\Gamma_{\alpha\beta}} \\ &= \begin{bmatrix} 1-\Gamma_{\alpha\beta} & V_0\Gamma_{\alpha\beta} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.7}$$

So the filtering error matrix (3.2) has the following form:

$$\begin{aligned} \mathbf{\Gamma} = \mathbb{R}_\alpha(\mathbb{I} - \mathbf{\Phi}_\alpha\mathbf{\Phi}_\beta^*) &= \begin{bmatrix} 1 & -V_0 \\ -V_0 & 2V_0 \end{bmatrix} R_\alpha^2 \cdot \begin{bmatrix} 1-\Gamma_{\alpha\beta} & V_0\Gamma_{\alpha\beta} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-\Gamma_{\alpha\beta} & -V_0(1-\Gamma_{\alpha\beta}) \\ -V_0(1-\Gamma_{\alpha\beta}) & V_0(2-V_0\Gamma_{\alpha\beta}) \end{bmatrix} \cdot R_\alpha^2. \end{aligned} \tag{3.8}$$

Using the trivial identity $2 - V_0\Gamma = 2 - V_0 + V_0(1-\Gamma)$, one has:

$$\mathbf{\Gamma} = R_\alpha^2 \cdot (1-\Gamma_{\alpha\beta}) \cdot \begin{bmatrix} 1 & -V_0 \\ -V_0 & V_0^2 \end{bmatrix} + R_\alpha^2 \cdot \begin{bmatrix} 0 & 0 \\ 0 & V_0(2-V_0) \end{bmatrix}. \tag{3.9}$$

and considering the stationarity condition $\sigma_0^2 = R_\alpha V_0(2-V_0)$, one obtains the following equivalence of (3.9):

$$\mathbb{\Gamma} = R_\alpha^2 \cdot (1-\Gamma_{\alpha\beta}) \cdot \begin{bmatrix} 1 & -V_0 \\ -V_0 & V_0^2 \end{bmatrix} + R_\alpha \cdot \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0^2 \end{bmatrix}. \tag{3.10}$$

Hence

$$\Gamma_{11} = R_\alpha^2 \cdot (1-\Gamma_{\alpha\beta}), \qquad \Gamma_{12} = -V_0 R_\alpha^2 \cdot (1-\Gamma_{\alpha\beta}),$$
$$\Gamma_{21} = -V_0 R_\alpha^2 \cdot (1-\Gamma_{\alpha\beta}), \quad \Gamma_{22} = V_0^2 R_\alpha^2 \cdot (1-\Gamma_{\alpha\beta}) + R_\alpha^2 \cdot (2V_0 - V_0^2).$$

4. The filtering empirical estimation

In real physical observations, the condition of mutual uncorrelatedness (2.12) is practically not satisfied. Therefore, the covariance characteristics (2.12) should be taken into account in the covariance analysis of filtering, if they are nonzero. The corresponding correction terms are subject to estimates, based on the filtering equation (2.4).

We will explore the best estimate (in the mean square sense), determined by the following empirical filtering equation:

$$\big(\widehat{\alpha}(k), \quad \Delta\widehat{\alpha}(k+1)\big) = \mathbf{\Phi}_\beta^T \cdot \begin{pmatrix} \beta(k) \\ \Delta\beta(k+1) \end{pmatrix}, \quad k \geq 0, \tag{4.1}$$

with the empirical filtering matrix

$$\mathbf{\Phi}_\beta^T = \mathbb{R}_{\alpha\beta}^T (\mathbb{R}_\beta^T)^{-1}, \tag{4.2}$$

where

$$\mathbb{R}_{\alpha\beta}^T := \begin{bmatrix} R_{\alpha\beta}^T & R_{\alpha\beta}^{0T} \\ R_{\beta\alpha}^{0T} & R_{\alpha\beta}^{\Delta T} \end{bmatrix}, \quad \mathbb{R}_\beta^T := \begin{bmatrix} R_\beta^T & R_\beta^{0T}, \\ R_\beta^{0T} & R_\beta^{\Delta T} \end{bmatrix}. \tag{4.3}$$

The following empirical covariances, corresponding to (2.6) and (2.11), are used here:

$$R_{\alpha\beta}^T := \frac{1}{T}\sum_{k=0}^{T-1}(\alpha(k)\beta(k)), \quad R_{\alpha\beta}^{0T} := \frac{1}{T}\sum_{k=0}^{T-1}(\alpha(k)\Delta\beta(k+1)),$$
$$R_{\beta\alpha}^{0T} := \frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)\Delta\alpha(k+1)), \quad R_{\alpha\beta}^{\Delta T} := \frac{1}{T}\sum_{k=0}^{T-1}(\Delta\alpha(k+1)\Delta\beta(k+1)),$$
$$R_\beta^T := \frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)^2), \quad R_\beta^{0T} := \frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)\Delta\beta(k+1)),$$
$$R_\beta^{\Delta T} := \frac{1}{T}\sum_{k=0}^{T-1}[(\Delta\beta(k+1))^2]. \tag{4.4}$$

Note that for one-component correlations one has the representation

$$R_{\beta}^{0T} = -VR_{\beta}^{T} \ , \quad R_{\beta}^{\Delta} := 2VR_{\beta}^{T}. \tag{4.5}$$

So the factor matrix

$$\mathbb{R}_{\beta}^{T} = \begin{bmatrix} R_{\beta}^{T} & -VR_{\beta}^{T} \\ -VR_{\beta}^{T} & 2VR_{\beta}^{T} \end{bmatrix} \tag{4.6}$$

has the following inversion:

$$(\mathbb{R}_{\beta}^{T})^{-1} = \begin{bmatrix} 2VR_{\beta}^{T} & VR_{\beta}^{T} \\ VR_{\beta}^{T} & R_{\beta}^{T} \end{bmatrix} \cdot (d_{\beta}^{T})^{-1} \ , \quad d_{\beta}^{T} := (2V - V^{2})(R_{\beta}^{T})^{2}. \tag{4.7}$$

Supposing that the mutual correlations (2.12) in reality are not null, the empirical covariances are connected by more complex relations, namely

$$\begin{aligned} R_{\alpha\beta}^{0T} &= -VR_{\alpha\beta}^{T} + \sigma\frac{1}{T}\sum_{k=0}^{T-1}(\alpha(k)\Delta W(k+1)), \\ R_{\beta\alpha}^{0T} &= -V_{0}R_{\alpha\beta}^{T} + \sigma_{0}\frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)\Delta W^{0}(k+1)), \\ R_{\alpha\beta}^{\Delta T} &= VV_{0}R_{\alpha\beta}^{T} - \sigma\frac{1}{T}\sum_{k=0}^{T-1}(\alpha(k)\Delta W(k+1)) \\ &\quad - \sigma_{0}\frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)\Delta W^{0}(k+1)) \\ &\quad + \sigma\sigma_{0}\frac{1}{T}\sum_{k=0}^{T-1}(\Delta W(k+1)\Delta W^{0}(k+1)). \end{aligned} \tag{4.8}$$

Taking into account (4.3) and (4.6), one has

$$\mathbb{R}_{\alpha\beta}^{T} = \begin{bmatrix} R_{\alpha\beta}^{T} & -VR_{\alpha\beta}^{T} + A^{T} \\ -V_{0}R_{\alpha\beta}^{T} + B^{T} & VV_{0}R_{\alpha\beta}^{T} + C^{T} \end{bmatrix} \tag{4.9}$$

where

$$
\begin{aligned}
A^T &:= \sigma \frac{1}{T}\sum_{k=0}^{T-1}(\alpha(k)\Delta W(k+1)),\\
B^T &:= \sigma_0 \frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)\Delta W^0(k+1)),\\
C^T &:= -\sigma \frac{1}{T}\sum_{k=0}^{T-1}(\alpha(k)\Delta W(k+1)) - \sigma_0 \frac{1}{T}\sum_{k=0}^{T-1}(\beta(k)\Delta W^0(k+1)) + \\
&\qquad + \sigma\sigma_0 \frac{1}{T}\sum_{k=0}^{T-1}(\Delta W(k+1)\Delta W^0(k+1)).
\end{aligned}
\tag{4.10}
$$

Now our task is to express the filtering matrix in the terms of empirical covariances.
By the definition (4.2) one has

$$
\Phi_\beta^T = \begin{bmatrix} R_{\alpha\beta}^T & -VR_{\alpha\beta}^T + A^T \\ -V_0 R_{\beta\alpha}^T + B^T & VV_0 R_{\alpha\beta}^T + C^T \end{bmatrix} \cdot \begin{bmatrix} 2VR_\beta^T & VR_\beta^T, \\ VR_\beta^T & R_\beta^T \end{bmatrix} \cdot d_\beta^{-1}. \tag{4.11}
$$

Taking into account the relation $d_\beta^T = \mathcal{E}(R_\beta^T)^2 = \sigma \cdot R_\beta^T$, one obtains:

$$
\begin{aligned}
\Phi_{11}^T &= R_{\alpha\beta}^T (R_\beta^T)^{-1} + VA^T \cdot \left(\sigma R_\beta^T\right)^{-1};\\
\Phi_{12}^T &= \qquad A^T \cdot (\mathcal{E} R_\beta^T)^{-1};\\
\Phi_{21}^T &= -V_0 R_{\alpha\beta}^T (R_\beta^T)^{-1} + V(2B^T + C^T)\cdot(\sigma R_\beta^T)^{-1};\\
\Phi_{22}^T &= \qquad (VB^T + C^T)\cdot(\sigma R_\beta^T)^{-1}.
\end{aligned}
\tag{4.12}
$$

which one can rewrite in the matrix form as

$$
\Phi_\beta^T = \begin{bmatrix} 1 & 0 \\ -V_0 & 0 \end{bmatrix} R_{\alpha\beta}^T (R_\beta^T)^{-1} + \begin{bmatrix} VA^T & A^T, \\ V(2B^T + C^T) & VB^T + C^T \end{bmatrix} (\sigma R_\beta^T)^{-1}. \tag{4.13}
$$

The empirical matrix representation (4.13) contains two terms. The first addendum defines the filtering matrix under conditions of uncorrelatedness (2.12) of the stochastic components of signal and filter. The second addendum defines additional statistical estimates, generated by the correlation of the stochastic components of signal and filter.

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