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$\mathfrak{B}_1$ classes of De Giorgi, Ladyzhenskaya and Ural'tseva and their application to elliptic and parabolic equations with nonstandard growth

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Abstract. The article provides an application of generalized De Giorgi functional classes to the proof of the Hölder continuity of weak solutions to quasilinear elliptic and parabolic equations with nonstandard growth conditions.

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1. Introduction

We will consider the question of regularity of weak solutions to quasilinear elliptic and parabolic equations with nonstandard (p, q) growth. Such properties as the local boundedness of weak solutions, their continuity, and the Harnack inequality for positive solutions are indispensable in the qualitative theory of second-order elliptic and parabolic equations. The local boundedness and Hölder continuity of weak solutions to linear divergence-type second-order elliptic and parabolic equations with measurable coefficients are known since the famous results by De Giorgi [32] and Nash [88], and the Harnack inequality is in use since Moser's celebrated papers [86, 87]. It were Ladyzhenskaya, Ural'tseva [67], and Serrin [90] who generalized De Giorgi's and Moser's results to the case of quasilinear elliptic equations. Particularly, if Ω is a bounded open set of $\mathbb{R}^n$, $n \geqslant 2$, then the so-called $\mathfrak{B}_p(\Omega)$ class, $p > 1$, was defined in [67]: the

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function $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ belongs to the class $\mathfrak{B}_p(\Omega)$, if, for any ball $B_{8\rho}(x_0) \subset \Omega$, any $k \in \mathbb{R}$, and $\sigma \in (0,1)$, the following relation holds:

$$\int_{A^{\pm}_{k,\rho(1-\sigma)}} |\nabla u|^p \, dx \leqslant c\bigg(\frac{M^p(k)}{(\sigma\rho)^p} + 1\bigg)|A^{\pm}_{k,\rho}|, \tag{1.1}$$

where c is a positive constant, $|A^{\pm}_{k,\rho}|$ is the n-dimensional Lebesgue measure of the set $A^{\pm}_{k,\rho} := B_\rho(x_0) \cap \{(u-k)_\pm > 0\}$, and $M(k) = M(k,\rho) := \sup\limits_{B_\rho(x_0)} (u-k)_\pm$. It was proved in [67] that $\mathfrak{B}_p(\Omega) \subset C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with some $\alpha \in (0,1)$ depending only on n, p, and c. Later, DiBenedetto and Trudinger [37] proved the validity of Harnack's inequality for functions in the $\mathfrak{B}_p(\Omega)$ classes.

For second-order linear parabolic equations with measurable coefficients, the Hölder continuity of solutions was first proved by Nash [88]. This result was extended to the case of quasilinear parabolic equations with linear growth in [66].

The parabolic theory for degenerate and singular quasilinear equations differs substantially from the "linear" case which can be already realized looking at the Barenblatt solution to the parabolic p-Laplace equation. DiBenedetto developed an innovative intrinsic scaling method (see [34] and references therein). He introduced parabolic classes $\mathcal{B}_p$ and proved that the functions from these classes are locally Hölder-continuous [33]. The classes $\mathcal{B}_p$, $p \geqslant 2$, can be considered as an extension of the classes $\mathcal{B}_2$ introduced in [66]. The further expansion of the $\mathcal{B}_p$ classes to parabolic equations was carried out in the works by Ivanov [56, 57], DiBenedetto and Gianazza [35, 36], Gianazza, Surnachev, and Vespri [51], Gianazza and Vespri [52], Skrypnik [95].

The study of the regularity of minima of the functionals with nonstandard growth of the (p,q)-type has been initiated by Marcellini [76–80]. In the last thirty years, the qualitative theory of second-order equations with nonstandard growth has been actively developed (see, e.g., [1–10, 20, 24, 25, 28–31, 39–46, 48–50, 58–61, 64, 65, 69, 70, 72–75, 81–83, 85, 93, 99, 105–108]). Moreover, the parabolic equations and systems with a variable growth exponent $p(x,t)$ were studied intensively in the last years (see, e.g., [12, 14–16, 21, 22, 38, 91, 92, 101–104, 111]). Equations of this type and systems of such equations arise in various problems of mathematical physics. Their description can be found in the books by Antontsev–Díaz–Shmarev [13], Růžička [89], and Weickert [100]. At the same time, to prove the regularity of solutions to the corresponding equations, the classes $\mathfrak{B}_p(\Omega)$ and their generalizations such as the $\mathfrak{B}_{p(x)}$ classes [47] and the $\mathfrak{B}_G$ classes [71] were used.

We will make attempt to unify the De Giorgi approach to establish the local regularity of solutions to elliptic and parabolic equations with nonstandard growth. We give extension of the well-known elliptic and parabolic $\mathfrak{B}_p$ classes defined by Ladyzhenskaya and Ural'tseva [66,67] and DiBenedetto [33]. Particularly, in Section 3, we define elliptic $\mathfrak{B}_{1,\varphi}$ classes and prove the Hölder continuity and Harnack's inequality for functions in the $\mathfrak{B}_{1,\varphi}$ classes. In addition, we will prove that the solutions to the equations

$$\Delta_{p(x)} u := \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = 0,$$

$$\Delta_g u := \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = 0,$$

$$\frac{g(u)}{g(v)} \geqslant \left(\frac{u}{v}\right)^{p-1}, \quad u > v > 0, \quad p > 1,$$

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = 0, \quad a(x) \geqslant 0, \quad 1 < p < q,$$

$$\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\big(1 + \ln(1 + |\nabla u|)\big)\Big) = 0, \quad p > 1,$$

$$(-1)^m \sum_{|\alpha|=m} D^\alpha\left[\Big(\sum_{|\beta|=m} |D^\beta u|^2\Big)^{\frac{p-2}{2}} D^\alpha u\right] - \Delta_q u = 0, \quad q > mp,$$

belong to $\mathfrak{B}_{1,\varphi}$ with the correspondent choice of φ. Moreover, in some sense, we will improve the result by Lieberman (see Remark 3.1).

In Section 4, we define parabolic $\mathcal{B}_{1,\varphi}$ classes and prove the Hölder continuity for functions in $\mathcal{B}_{1,\varphi}$. In addition, we prove that the solutions to the equations

$$u_t - \operatorname{div}\big(|\nabla u|^{p(x,t)-2}\nabla u\big) = 0,$$

$$u_t - \Delta_g u = 0,$$

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x,t)|\nabla u|^{q-2}\nabla u) = 0,$$

$$u_t - \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\big(1 + \ln(1 + |\nabla u|)\big)\Big) = 0$$

belong to $\mathcal{B}_{1,\varphi}$ with the corresponding choice of φ. Moreover, we give answer to the still open problem on the regularity of solutions to parabolic equations with (p,q)-growth in the case $p \leqslant 2 \leqslant q$.

2. Notation and auxiliary propositions

Everywhere below, Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. For arbitrary $\rho > 0$ and $y \in \mathbb{R}^n$, $B_\rho(y) := \{x \in \mathbb{R}^n : |x - y| < \rho\}$ is the open n-dimensional ball centered at y with radius ρ. For every Lebesgue measurable set $E \subset \mathbb{R}^n$, we denote, by $|E|$, the n-dimensional Lebesgue measure of E (or the $n+1$-dimensional measure, if $E \subset \mathbb{R}^{n+1}$). We will also use the well-known notation for function spaces and for their elements (see [34, 66, 67] and references therein).

The following two lemmas will be used in the sequel. The first one is the well-known DeGiorgi–Poincaré lemma (see [67, Chap. II, Lemma 3.9]).

Lemma 2.1. *Let $u \in W^{1,1}(B_\rho(x_0))$. Then, for any $s \geqslant 1$ and for any $k, l \in \mathbb{R}$, $k < l$, the following inequalities hold:*

$$(l-k)^s \, |A^+_{l,\rho}|^{1-\frac{1}{n}} \, |A^-_{k,\rho}| \leqslant cs\rho^n \int\limits_{A^+_{k,\rho} \setminus A^+_{l,\rho}} |\nabla u|(u-k)_+^{s-1} dx,$$

$$(l-k)^s \, |A^-_{k,\rho}|^{1-\frac{1}{n}} \, |A^+_{l,\rho}| \leqslant cs\rho^n \int\limits_{A^-_{l,\rho} \setminus A^-_{k,\rho}} |\nabla u|(l-u)_+^{s-1} dx, \tag{2.1}$$

where $A^+_{k,\rho} := B_\rho(x_0) \cap \{u > k\}$, $A^-_{k,\rho} := B_\rho(x_0) \cap \{u < k\}$, and c is a positive constant depending only on n.

The following lemma can also be found in [67, Chap. II, Lemma 4.7].

Lemma 2.2. *Let y_j, $j = 0, 1, 2, \ldots$, be a sequence of nonnegative numbers satisfying*

$$y_{j+1} \leqslant cb^j y_j^{1+\delta}, \quad j = 0, 1, 2, \ldots,$$

with some constants $\delta > 0$ and $c, b > 1$. Then

$$y_j \leqslant c^{\frac{(1+\delta)^j - 1}{\delta}} b^{\frac{(1+\delta)^j - 1}{\delta^2} - \frac{j}{\delta}} y_0^{(1+\delta)^j}, \quad j = 0, 1, 2, \ldots.$$

Particularly, if $y_0 \leqslant \nu := c^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$, then

$$y_j \leqslant \nu b^{-\frac{j}{\delta}} \quad \text{and} \quad \lim_{j \to \infty} y_j = 0.$$

3. Elliptic $\mathfrak{B}_1$ classes

We assume that, for every $v \in \mathbb{R}_+$, the function $x \to \varphi(x, v)$ is measurable and, for every $x \in \Omega$, the function $v \to \varphi(x, v)$ is increasing and continuous, $\varphi(x,0)=\lim\limits_{v\to+0}\varphi(x,v)=0$. We say that a measurable function $u:\Omega\to\mathbb{R}$ belongs to the $\mathfrak{B}_{1,s,\varphi}(\Omega)$ class for some $s\geqslant 1$, if $|u|^s\in W^{1,1}_{\rm loc}(\Omega)$, $u\in L^\infty(\Omega)$, $M:=\sup_\Omega|u|$, and there exist positive numbers K_1, c_1, c_2, and c_3 such that, for all $k,l\in\mathbb{R}$, $k<l$, $|k|<M$, $|l|<M$, for any $\varepsilon\in(0,1]$, $\sigma\in(0,1)$, and for any ball $B_\rho(x_0)$ such that $B_{8\rho}(x_0)\subset\Omega$, the following inequalities hold:

$$\begin{aligned}\int\limits_{A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}} &|\nabla u|(u-k)_+^{s-1}dx\\ \leqslant K_1\frac{M_+^s(k)}{\rho}&\bigg\{\frac{1}{\varepsilon}|A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}|+\varepsilon^{c_1}\sigma^{-c_3}|A^+_{k,\rho}|\\ +\frac{\varepsilon^{c_1}\sigma^{-c_3}}{\varphi\big(x_0,\frac{M_+(k)}{\rho}\big)}&\int_{A^+_{k,\rho}}\varphi\Big(x_0,K_1\frac{(u-k)_+}{\sigma\rho\zeta}\Big)\zeta^{c_2-1}dx\bigg\}\\ +\frac{K_1\varepsilon^{c_1}}{\varphi\big(x_0,\frac{M_+(k)}{\rho}\big)}&|A^+_{k,\rho}|,\end{aligned}\tag{3.1}$$

$$\begin{aligned}\int\limits_{A^-_{l,\rho(1-\sigma)}\setminus A^-_{k,\rho(1-\sigma)}} &|\nabla u|(u-l)_-^{s-1}dx\\ \leqslant K_1\frac{M_-^s(l)}{\rho}&\bigg\{\frac{1}{\varepsilon}|A^-_{l,\rho(1-\sigma)}\setminus A^-_{k,\rho(1-\sigma)}|+\varepsilon^{c_1}\sigma^{-c_3}|A^-_{l,\rho}|\\ +\frac{\varepsilon^{c_1}\sigma^{-c_3}}{\varphi\big(x_0,\frac{M_-(l)}{\rho}\big)}&\int_{A^-_{l,\rho}}\varphi\Big(x_0,K_1\frac{(u-k)_-}{\sigma\rho\zeta}\Big)\zeta^{c_2-1}dx\bigg\}\\ +\frac{K_1\varepsilon^{c_1}}{\varphi\big(x_0,\frac{M_-(l)}{\rho}\big)}&|A^-_{l,\rho}|,\end{aligned}\tag{3.2}$$

where $\zeta\in C_0^\infty(B_\rho(x_0))$, $0\leqslant\zeta\leqslant 1$, $\zeta=1$ in $B_{\rho(1-\sigma)}(x_0)$, $|\nabla\zeta|\leqslant(\sigma\rho)^{-1}$,

$$(u-k)_\pm:=\max\{\pm(u-k),0\},$$

$$A^\pm_{k,\rho}:=B_\rho(x_0)\cap\{(u-k)_\pm>0\},$$

$$M_\pm(k)=M_\pm(k,\rho):=\sup_{B_\rho(x_0)}(u-k)_\pm.$$

We now consider some examples, where the solutions of quasilinear elliptic equations with non-standard growth belong to the classes $\mathfrak{B}_{1,s,\varphi}(\Omega)$.

Example 3.1. Let $p(x)$ be a measurable function on Ω satisfying the condition

$$1<p\leqslant p(x)\leqslant q<+\infty,\tag{3.3}$$

let $W^{1,p(x)}(\Omega) = \{u : u \in W^{1,1}(\Omega), |\nabla u|^{p(x)} \in L^1(\Omega)\}$, and let, for every $u \in W^{1,p(x)}(\Omega)$,

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^1(\Omega)} + \inf\left\{\lambda : \lambda > 0, \int_\Omega |\nabla u|^{p(x)}\lambda^{-p(x)}dx \leqslant 1\right\}.$$

We note that $W_0^{1,p(x)}(\Omega)$ is the closure in $W^{1,p(x)}(\Omega)$ of the set of functions from $W^{1,p(x)}(\Omega)$ with compact support.

By definition, a function $u \in W^{1,p(x)}(\Omega)$ is a solution to the equation

$$\Delta_{p(x)}u := \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = 0, \tag{3.4}$$

if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\eta\, dx = 0, \quad \forall\, \eta \in W_0^{1,p(x)}(\Omega). \tag{3.5}$$

Interest in Eq. (3.4) with a variable growth exponent $p(x)$ arose in the works [105,106] in connection with the study of variational functionals with integrands of the form $|\nabla u|^{p(x)}/p(x)$, and with the Lavrent'ev effect [68] for such functionals [107,108]. In particular, in [108], it was established that if

$$|p(x) - p(y)| \leqslant L/\ln|x-y|^{-1} \ \text{ for } |x-y| < 1/2, \tag{3.6}$$

then the sets of smooth functions $C^\infty(\Omega)$ ($C_0^\infty(\Omega)$) are dense in $W(\Omega)$ ($W_0(\Omega)$), and condition (3.6) is exact for the validity of this assertion. If condition (3.6) is violated, then there is a counterexample of a non-Hölder solution to Eq. (3.4). Subsequently, the inner Hölder continuity of solutions of Eq. (3.4) with condition (3.6) was established in [6,47,64]. The Wiener-type criterion for the regularity of boundary points for Eq. (3.4) under assumptions (3.3), (3.6) was obtained in [7].

Testing the integral identity (3.5) by $\eta = (u-k)_\pm\zeta^{c_2}$, $c_2 \geqslant q$, where ζ is the same as in (3.1), (3.2), using the Young inequality and (3.6), we obtain

$$\int_{A^\pm_{k,\rho(1-\sigma)}} |\nabla u|^{p(x)}dx$$
$$\leqslant C(n,p,q,L,M)\frac{M_\pm(k)}{\sigma\rho}\int_{A^\pm_{k,\rho}}\left(\frac{(u-k)_\pm}{\sigma\rho\zeta}\right)^{p^- -1}\zeta^{c_2-p^-+1}dx,$$

where $p^- := \min_{B_\rho(x_0)} p(x)$. From this by the Young inequality, we arrive at

$$\int\limits_{A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}} |\nabla u| dx \leqslant \frac{M_+(k)}{\varepsilon\rho}|A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}|$$
$$+\left(\frac{\varepsilon\rho}{M_+(k)}\right)^{p^- -1}\int_{A^+_{k,\rho(1-\sigma)}}|\nabla u|^{p(x)}dx + \left(\frac{\varepsilon\rho}{M_+(k)}\right)^{p^- -1}|A^+_{k,\rho}|$$
$$\leqslant \frac{M_+(k)}{\varepsilon\rho}|A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}| + C(n,p,q,L,M)\frac{\varepsilon^{p^- -1}M_+(k)}{\sigma^q\rho}|A^+_{k,\rho}|$$
$$+\left(\frac{\varepsilon\rho}{M_+(k)}\right)^{p^- -1}|A^+_{k,\rho}| \leqslant \frac{M_+(k)}{\varepsilon\rho}|A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}|$$
$$+ C(n,p,q,L,M)\varepsilon^{p-1}|A^+_{k,\rho}|\left(\frac{M_+(k)}{\sigma^q\rho}+\left(\frac{\rho}{M_+(k)}\right)^{p(x_0)-1}\right),$$

which yields (3.1) with $s=1$, $\varphi(x,u)=u^{p(x)-1}$, $c_1=p-1$, $c_3=q$, and K_1 depending on n, p, q, L, and M. The proof of inequality (3.2) is completely similar.

Example 3.2. Let us consider the equation $\Delta_g u = 0$, where g satisfies

$$\frac{g(w)}{g(v)} \geqslant \left(\frac{w}{v}\right)^{p-1}, \quad w>v>0, \quad p>1. \tag{3.7}$$

We set $G(w) := g(w)w$ for $w>0$ and write $W^{1,G}(\Omega)$ for the class of functions which are weakly differentiable in Ω with $\int_\Omega G(|\nabla u|)\,dx < +\infty$. $W^{1,G}_0(\Omega) := \{u \in W^{1,G}(\Omega) : u \text{ has a compact support in } \Omega\}$.

By a solution to the equation $\Delta_g u = 0$, we mean a function $u \in W^{1,G}(\Omega)$ that satisfies the integral identity:

$$\int_\Omega g(|\nabla u|)\nabla u \nabla \eta\, dx = 0, \quad \forall\, \eta \in W^{1,G}_0(\Omega).$$

The study of the equation $\Delta_g u = 0$ goes back to the work by Lieberman [72], which generalizes the natural structural conditions proposed by Ladyzhenskaya and Ural'tseva [67] for the coefficients of second-order quasilinear elliptic equations to ensure the regularity of their weak solutions.

Testing the previous integral identity by $\eta=(u-k)_\pm\zeta^{c_2}$, where ζ is the same as in (3.1), (3.2) and using the evident inequality

$$g(a)b \leqslant \varepsilon g(a)a + g(b/\varepsilon)b, \quad \forall\, a,b,\varepsilon>0, \tag{3.8}$$

we obtain

$$\int_{A^{\pm}_{k,\rho(1-\sigma)}} G(|\nabla u|)\,dx \leqslant \gamma(n,p)\frac{M_{\pm}(k)}{\sigma\rho}\int_{A^{\pm}_{k,\rho}} g\left(\gamma\frac{(u-k)_{\pm}}{\sigma\rho\zeta}\right)\zeta^{c_2-1}\,dx.$$

From whence, by (3.8), we arrive at (3.1), (3.2) with $s=1$, $\varphi(x,u)=\varphi(u)=g(u)$, $c_1=p-1$, $c_3=1$ and K_1 depending only on n and p.

Example 3.3. Let us consider the two-phase elliptic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)=0, \quad x\in\Omega, \tag{3.9}$$

which is the Euler–Lagrange equation of the functional

$$\mathcal{P}_{p,q}(u,\Omega) := \int_{\Omega}\big(|\nabla u|^p + a(x)|\nabla u|^q\big)dx,$$

where $0\leqslant a(x)\in C^{0,\alpha}(\Omega)$, $1<p<q\leqslant p+\alpha$. The functional $\mathcal{P}_{p,q}$ is a part of the family of variational integrals introduced by Zhikov [105, 106] in order to develop models for strongly anisotropic materials. They intervene in the homogenization theory and elasticity theory, where the modulating coefficient $a(x)$ dictates the geometry of a composite made by two different materials, with hardening exponents p and q, respectively. They can also be used in order to provide new examples of the Lavrent'ev phenomenon [107, 108]. From the global viewpoint, the integrand of the functional $\mathcal{P}_{p,q}$ satisfies the so-called (p,q)-growth conditions:

$$|\nabla u|^p \leqslant |\nabla u|^p + a(x)|\nabla u|^q \leqslant c(1+|\nabla u|^q).$$

Now, the regularity theory of minima of such functionals is well developed, starting with the pioneering contributions of Marcellini [76–80] (see Sec. 1 and [84] for a survey). New non-trivial phenomena appear due to the fact that the integrand of the functional $\mathcal{P}_{p,q}$ switches between two different types (phases) of elliptic behavior, according to the coefficient $a(x)$. Specifically, on the set $\{a(x)>0\}$, the growth of the integrand with respect to the gradient is polynomial with order q. Whereas, on the zero set $\{a(x)=0\}$, the growth occurs at a rate of p. As a result, Eq. (3.9) demonstrates a new type of non-uniform and doubly degenerate ellipticity that mixes up two different kinds of p-Laplace operators. The study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point $x\in\Omega$ has been continued in a series of remarkable papers by Mingione et al. [17–19, 26, 27]. In this regard, we also cite Skrypnik and Buryachenko [93].

Let $u \in W^{1,q}(\Omega)$ be a weak solution to Eq. (3.9), i.e. u satisfies the integral identity

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u\nabla\eta + a(x)|\nabla u|^{q-2}\nabla u\nabla\eta\big)\,dx = 0, \quad \forall\,\eta \in W_0^{1,q}(\Omega).$$

Set $g_a(v) := v^{p-1} + av^{q-1}$ and

$$[a]_\alpha := \sup_{x,y\in\Omega,\,x\neq y} \frac{|a(x)-a(y)|}{|x-y|^\alpha}.$$

Testing the previous integral identity by $\eta = (u-k)_\pm \zeta^{c_2+q-p}$, where ζ is the same as in (3.1) and (3.2), we obtain

$$\begin{aligned}\int_{A^\pm_{k,\rho(1-\sigma)}} & g_{a(x)}(|\nabla u|)|\nabla u|\,dx \\ &\leqslant \gamma(n,p,q)\frac{M_\pm(k)}{\sigma\rho}\int_{A^\pm_{k,\rho}} g_{a(x)}\bigg(\frac{(u-k)_\pm}{\sigma\rho\zeta}\bigg)\zeta^{c_2+q-p-1}dx. \quad (3.10)\end{aligned}$$

The following two cases are possible: $a(x_0) = 0$ or $a(x_0) > 0$.

In the first one, we have

$$\begin{aligned} a(x)\bigg(\frac{(u-k)_\pm}{\sigma\rho\zeta}\bigg)^{q-1} &\leqslant 3[a]_\alpha\rho^\alpha\bigg(\frac{(u-k)_\pm}{\sigma\rho\zeta}\bigg)^{q-1} \\ &\leqslant 3[a]_\alpha\bigg(\frac{2M}{\sigma\zeta}\bigg)^{q-p}\bigg(\frac{(u-k)_\pm}{\sigma\rho\zeta}\bigg)^{p-1}, \quad x\in B_\rho(x_0).\end{aligned}$$

Therefore, inequality (3.10) implies that

$$\begin{aligned}\int_{A^\pm_{k,\rho(1-\sigma)}} & |\nabla u|^p\,dx \\ &\leqslant \gamma(n,p,q,[a]_\alpha,M)\sigma^{p-q-1}\frac{M_\pm(k)}{\rho}\int_{A^\pm_{k,\rho}}\bigg(\frac{(u-k)_\pm}{\sigma\rho\zeta}\bigg)^{p-1}\zeta^{c_2-1}dx.\end{aligned}$$

If $a(x_0) > 0$, we set $R = \frac{1}{8}(a(x_0)/2[a]_\alpha)^{1/\alpha}$. Then $\frac{1}{2}a(x_0) \leqslant a(x) \leqslant \frac{3}{2}a(x_0)$, $x \in B_\rho(x_0) \subset B_R(x_0)$, and inequality (3.10) implies that

$$\begin{aligned}\int_{A^\pm_{k,\rho(1-\sigma)}} & g_{a(x_0)}(|\nabla u|)|\nabla u|\,dx \\ &\leqslant \gamma(n,p,q)\frac{M_\pm(k)}{\sigma\rho}\int_{A^\pm_{k,\rho}} g_{a(x_0)}\bigg(\frac{(u-k)_\pm}{\sigma\rho\zeta}\bigg)\zeta^{c_2-1}dx.\end{aligned}$$

From this, using (3.8), we arrive at (3.1), (3.2) with $s=1$, $\varphi(x,u)=g_{a(x)}(u)$, $c_1=p-1$, $c_3=1$, and K_1 depending only on n, p, and q.

The case of the equation

$$\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\big(1+a(x)\ln^{\beta}(1+|\nabla u|)\big)\Big)=0,\quad 0<\beta\leqslant 1,$$
$$0\leqslant a(x)\in C^{0,\alpha}(\Omega),\quad \beta\leqslant\alpha\leqslant 1$$

can be considered almost similarly.

Example 3.4. We consider the equation

$$(-1)^m\sum_{|\alpha|=m}D^{\alpha}\bigg[\Big(\sum_{|\beta|=m}|D^{\beta}u|^2\Big)^{\frac{p-2}{2}}D^{\alpha}u\bigg]-\Delta_q u=0,\quad q>mp,\quad m\geqslant 2,$$

which comes from work [94] (see also recent works [23, 62, 63, 97–99] and references therein).

Let $u\in W^{1,q}(\Omega)\cap W^{m,p}(\Omega)$ be a weak solution to this equation, i.e. u satisfies the integral identity

$$\sum_{|\alpha|=m}\int_{\Omega}\Big(\sum_{|\beta|=m}|D^{\beta}u|^2\Big)^{\frac{p-2}{2}}D^{\alpha}uD^{\alpha}v\,dx+\int_{\Omega}|\nabla u|^{q-2}\nabla u\nabla v dx=0$$

for all $v\in W_0^{1,q}(\Omega)\cap W_0^{m,p}(\Omega)$, and let $\zeta\in C_0^{\infty}(\Omega)$ be a function with the properties $0\leqslant\zeta\leqslant 1$, $\zeta\equiv 1$ in $B_{\rho(1-\sigma)}(x_0)$ and $|D^{\alpha}\zeta|\leqslant\gamma_{n,m}(\sigma\rho)^{-|\alpha|}$, $|\alpha|\leqslant m$, where $B_\rho(x_0)$ and $B_{\rho(1-\sigma)}(x_0)$ are the same balls as in (3.1) and (3.2).

Testing the integral identity by $(u-k)_{\pm}^{s}\zeta^{c_2}$, where s is a sufficiently large positive number, and using the relations (see, e.g., [96, Lemma 4])

$$D^{\alpha}\big((u-k)_{\pm}^{s}\zeta^{c_2}\big)=s(u-k)_{\pm}^{s-1}D^{\alpha}u\zeta^{c_2}+R(\alpha,s,c_2)$$

and

$$|R(\alpha,s,c_2)|\leqslant\gamma(\alpha,s,c_2)\sum_{|\beta|=1}^{|\alpha|-1}(u-k)_{\pm}^{s-\frac{|\alpha|}{|\beta|}}|D^{\beta}u|^{\frac{|\alpha|}{|\beta|}}\zeta^{c_2}$$
$$+\gamma(\alpha,s,c_2)(\sigma\rho)^{-|\alpha|}(u-k)_{\pm}^{s}\zeta^{c_2-|\alpha|},\quad 2\leqslant|\alpha|\leqslant m,$$

we obtain

$$\int_{A^{\pm}_{k,\rho}}\Big\{\sum_{|\alpha|=m}|D^{\alpha}u|^{p}+\sum_{|\alpha|=1}|D^{\alpha}u|^{q}\Big\}(u-k)_{\pm}^{s-1}\zeta^{c_2}dx$$
$$\leqslant\frac{C_1}{(\sigma\rho)^q}\int_{A^{\pm}_{k,\rho}}(u-k)_{\pm}^{s+q-1}\zeta^{c_2-q}dx+C_1\int_{A^{\pm}_{k,\rho}}(u-k)_{\pm}^{s-\varkappa_1}\zeta^{c_2}dx\qquad(3.11)$$
$$+C_1\int_{A^{\pm}_{k,\rho}}\Big\{\sum_{2\leqslant|\alpha|\leqslant m-1}|D^{\alpha}u|^{p_\alpha}\Big\}(u-k)_{\pm}^{s-1}\zeta^{c_2}dx,$$

where

$$\frac{1}{p_\alpha} = \frac{|\alpha|-1}{m-1}\,\frac{1}{p} + \frac{m-|\alpha|}{m-1}\,\frac{1}{q_1} \quad \text{for} \quad 2 \leqslant |\alpha| \leqslant m-1,$$

$$mp < q_1 < q \leqslant n,$$

and $C_1 = C_1(n,m,p,q,q_1,s,c_2)$ and $\varkappa_1 = \varkappa_1(m,p,q,q_1)$ are positive constants. We use the integration by parts as in [96, Lemma 4] to estimate the last integral on the right-hand side of (3.11) as follows:

$$\begin{aligned}
&\int_{A^{\pm}_{k,\rho}} \Big\{ \sum_{2\leqslant|\alpha|\leqslant m-1} |D^\alpha u|^{p_\alpha} \Big\} (u-k)_\pm^{s-1} \zeta^{c_2}\, dx \\
&\leqslant \delta \int_{A^{\pm}_{k,\rho}} \Big\{ \sum_{|\alpha|=m} |D^\alpha u|^{p} + \sum_{|\alpha|=1} |D^\alpha u|^{q} \Big\} (u-k)_\pm^{s-1} \zeta^{c_2}\, dx \\
&+ \frac{C_2}{\delta^{\varkappa_2}} \bigg[\frac{1}{(\sigma\rho)^q} \int_{A^{\pm}_{k,\rho}} (u-k)_\pm^{s+q-1} \zeta^{c_2-q}\, dx + \int_{A^{\pm}_{k,\rho}} (u-k)_\pm^{s-\varkappa_3} \zeta^{c_2}\, dx \bigg]
\end{aligned} \tag{3.12}$$

with arbitrary $\delta \in (0,1)$ and some positive constants $\varkappa_2 = \varkappa_2(m,p,q,q_1)$, $\varkappa_3 = \varkappa_3(m,p,q,q_1)$ and $C_2 = C_2(n,m,p,q,q_1,s,c_2)$. Inequalities (3.11) and (3.12) for an appropriate choice of δ give the following:

$$\begin{aligned}
&\int_{A^{\pm}_{k,\rho}} \Big\{ \sum_{|\alpha|=m} |D^\alpha u|^{p} + \sum_{|\alpha|=1} |D^\alpha u|^{q} \Big\} (u-k)_\pm^{s-1} \zeta^{c_2}\, dx \\
&\qquad \leqslant C_3 \bigg[\frac{1}{(\sigma\rho)^q} \int_{A^{\pm}_{k,\rho}} (u-k)_\pm^{s+q-1} \zeta^{c_2-q}\, dx + \int_{A^{\pm}_{k,\rho}} (u-k)_\pm^{s-\varkappa_4} \zeta^{c_2}\, dx \bigg]
\end{aligned}$$

with some positive constants $C_3(n,m,p,q,q_1,s,c_2)$ and $\varkappa_4(m,p,q,q_1)$. From this, we arrive at (3.1), (3.2) with some sufficiently large s, $\varphi(x,u) = \varphi(u) = u^{q-1}$, $c_1 = q-1$, $c_3 = q$, and K_1 depending only on n, m, p, and q.

The main result of this section reads as follows:

Theorem 3.1. *Let $u \in \mathfrak{B}_{1,s,\varphi}(\Omega)$ with some $s \geqslant 1$, and let φ satisfies*

$$\frac{\varphi(x,w)}{\varphi(x,v)} \leqslant 2\Big(\frac{w}{v}\Big)^{\mu}, \quad w \geqslant v \geqslant s_0 > 0, \tag{3.13}$$

for all $x \in \Omega$, with some $\mu > 0$ and some $s_0 > 0$. Then u is locally Hölder-continuous in Ω.

Theorem 3.2. *Let* $0 \leqslant u \in \mathfrak{B}_{1,s,\varphi}(\Omega)$ *with some* $s \geqslant 1$, *and let* φ *additionally satisfies* (3.13). *Then* u *satisfies the Harnack inequality:*

$$\max_{B_\rho(x_0)} u \leqslant c\,(\min_{B_\rho(x_0)} +\rho^{1/s}).$$

Theorems 3.1 and 3.2 are immediate consequences of the following two lemmas (see [67] for details).

Lemma 3.1. *Let* $0 \leqslant u \in \mathfrak{B}_{1,s,\varphi}(\Omega)$ *with some* $s \geqslant 1$, *and let* φ *satisfies* (3.13). *Fix* $a \in (0,1)$, $N \in (0,M)$. *Then there exists* $\nu \in (0,1)$ *which depends only on* s, n, c_1, c_2, c_3, K_1 *and* a, *and is such that if*

$$|\{x \in B_r(\overline{x}) : u(x) \leqslant N\}| \leqslant \nu |B_r(\overline{x})|,$$

then either

$$N \leqslant r^{1/s} \tag{3.14}$$

or

$$u(x) \geqslant aN \quad \text{for a.a. } x \in B_{r/2}(\overline{x}),$$

provided that $B_{8r}(\overline{x}) \subset \Omega$.

Lemma 3.2. *Let* $0 \leqslant u \in \mathfrak{B}_{1,s,\varphi}(\Omega)$ *with some* $s \geqslant 1$, *and let* φ *satisfies* (3.13). *Assume that, with some* $N \in (0,M)$ *and some* $\beta \in (0,1)$, *the following relation holds:*

$$|\{x \in B_r(\overline{x}) : u(x) \leqslant N\}| \leqslant (1-\beta)|B_r(\overline{x})|.$$

Then there exists a number $\xi \in (0,1)$ *which depends only on* s, n, c_1, c_2, c_3, K_1, *and* β *and is such that either*

$$\xi N \leqslant r^{1/s} \tag{3.15}$$

or

$$u(x) \geqslant \xi N \quad \text{for a.a. } x \in B_{2r}(\overline{x}),$$

provided that $B_{8r}(\overline{x}) \subset \Omega$.

The proofs of Lemmas 3.1 and 3.2 are completely similar to that of [67, Lemmas 6.1, 6.2, Chapt. II]. Particularly, in the proof of Lemma 3.1, we use (2.1), Lemma 2.2, and inequality (3.2) with

$$k = \frac{N}{2^{j+1}}, \quad l = \frac{N}{2^j}, \quad \varepsilon = \left(\frac{|A^-_{N/2^j,r} \setminus A^-_{N/2^{j+1},r}|}{|B_r(\overline{x})|}\right)^{\frac{1}{1+c_1}}, \; j = 0,1,2,\dots j_*,$$

where j_* is a sufficiently large positive number. In addition, assuming that (3.14) and (3.15) are violated, and ξ is so small that $\xi^{-1} \geqslant s_0$, we can use (3.13).

Remark 3.1. We note that Theorems 3.1 and 3.2 improve Lieberman's results [72], since condition (3.13) in [72] is assumed with $s_0 = 0$. We give an example of the function φ which satisfies (3.7) and (3.13) with $s_0 = 1$, but, at the same time, condition (3.13) with $s_0 = 0$ will not be fulfilled.

Let $\varphi_1 : (0, 1) \to (0, +\infty)$ be a function with the properties: φ_1 is nondecreasing on $(0, 1)$, $\lim\limits_{s\to 1-0} \varphi_1(s) = +\infty$, and $\int_0^1 \varphi_1(s)ds < +\infty$. Consider the function

$$\varphi(w) = \begin{cases} \int\limits_0^w \varphi_1(s)ds & \text{if } \ 0 < w < 1, \\ w\int\limits_0^1 \varphi_1(s)ds & \text{if } \ w > 1. \end{cases}$$

From the definition of φ_1, it follows that, for $0 < v < w \leqslant 1$,

$$\int\limits_0^v \varphi_1(s)ds = \frac{v}{w}\int\limits_0^w \varphi_1\Big(\frac{v}{w}s\Big)ds \leqslant \frac{v}{w}\int\limits_0^w \varphi_1(s)ds.$$

Moreover,

$$\frac{\varphi(w)}{\varphi(v)} \geqslant \frac{w}{v} \ \text{ for any } \ 0 < v < w,$$

so (3.7) is fulfilled. We also note that (3.13) is evidently fulfilled for $s_0 = 1$, but

$$\lim_{u\to 1-0} \frac{u\varphi'(u)}{\varphi(u)} = \lim_{u\to 1-0} \frac{u\varphi_1(u)}{\int_0^u \varphi_1(s)ds} = +\infty.$$

So, condition (3.13) will be violated for $s_0 = 0$.

Remark 3.2. We will consider a possible generalization of $\mathfrak{B}_1$ classes. Particularly, we assume that the following inequalities hold instead of (3.1) and (3.2):

$$\begin{aligned} &\int\limits_{A^+_{k,\rho(1-\sigma)}\setminus A^+_{l,\rho(1-\sigma)}} |\nabla u| dx \\ &\leqslant K_1 \frac{M_+(k)}{\rho}\Big\{\frac{1}{\varepsilon}|A^+_{k,\rho(1-\sigma)} \setminus A^+_{l,\rho(1-\sigma)}| + \varepsilon^{c_1}\rho^{-\theta(\rho)}|A^+_{k,\rho}|\Big\} \\ &+ K_1 \frac{\varepsilon^{c_1}\sigma^{-c_3}\rho^{-\theta(\rho)}}{\varphi\big(x_0, \frac{M_+(k)}{\rho}\big)}\Big\{|A^+_{k,\rho}| \\ &+ \frac{M_+(k)}{\rho}\int_{A^+_{k,\rho}} \varphi\Big(x_0, K_1\frac{(u-k)_+}{\sigma\rho\zeta}\Big)\zeta^{c_2-1}dx\Big\}, \end{aligned} \tag{3.16}$$

$$\int\limits_{A^-_{l,\rho(1-\sigma)}\setminus A^-_{k,\rho(1-\sigma)}} |\nabla u| dx$$
$$\leqslant K_1 \frac{M_-(l)}{\rho}\Big\{\frac{1}{\varepsilon}|A^-_{l,\rho(1-\sigma)}\setminus A^-_{k,\rho(1-\sigma)}| + \varepsilon^{c_1}\rho^{-\theta(\rho)}|A^-_{l,\rho}|\Big\}$$
$$+ K_1 \frac{\varepsilon^{c_1}\sigma^{-c_3}\rho^{-\theta(\rho)}}{\varphi\big(x_0, \frac{M_-(l)}{\rho}\big)}\Big\{|A^-_{l,\rho}| + \frac{M_-(l)}{\rho}\int_{A^-_{l,\rho}} \varphi\Big(x_0, K_1\frac{(u-l)_-}{\sigma\rho\zeta}\Big)\zeta^{c_2-1}dx\Big\}, \tag{3.17}$$

where φ satisfies condition (3.13) and

$$0 \leqslant \theta(\rho) \leqslant L\frac{\ln\ln\frac{\lambda(\rho)}{\rho}}{\ln\frac{1}{\rho}}, \quad 0 < L < \frac{c_1}{1+n(c_1+1)},$$

$\lambda(\rho)$ is nondecreasing, and $\dfrac{\lambda(\rho)}{\rho}$ is nonincreasing for sufficiently small ρ,

$$\lim_{\rho\to 0}\lambda(\rho) = 0, \quad \lim_{\rho\to 0}\frac{\lambda(\rho)}{\rho} = +\infty, \quad \int_0 \frac{d\rho}{\lambda(\rho)} = +\infty.$$

As an example, the function $\lambda(\rho) = \rho\log\dfrac{1}{\rho}$ satisfies the above conditions.

These classes can be used in the study of the equations $\Delta_{p(x)}u = 0$, $1 < p \leqslant p(x) \leqslant q < +\infty$, when condition (3.6) fails (see, e.g., [8, 11, 109, 110] and references therein). Assuming $\mathrm{osc}\{p(x); B_\rho(x_0)\} \leqslant \theta(\rho)$ instead of (3.6) and testing (3.5) by $\eta = (u-k)_\pm\zeta^{c_2}$, $c_2 \geqslant q$, similarly to Example 3.1, we arrive at (3.16), (3.17) with $\varphi(x,u) = u^{p(x)-1}$, $c_1 = p-1$, $c_3 = q$, and K_1 depending on n, p, q, L, and M.

Lemmas 3.1, 3.2 can be reformulated as follows.

Lemma 3.3. *Let $u \geqslant 0$ satisfies* (3.16) *and* (3.17), *and let φ satisfies* (3.13). *Fix $a, \xi \in (0,1)$, $N \in (0, M)$. Then there exists $\nu_1 \in (0,1)$ which depends only on n, c_1, c_2, c_3, K_1, and a and is such that if*

$$|\{x \in B_r(\overline{x}) : u(x) \leqslant \xi N\}| \leqslant \nu_1 r^{n\theta(r)}|B_r(\overline{x})|,$$

then either $\xi N \leqslant s_0 r$ or

$$u(x) \geqslant aN \quad \textit{for a.a. } x \in B_{r/2}(\overline{x}),$$

provided that $B_{8r}(\overline{x}) \subset \Omega$.

Fix ρ_0 by the condition

$$C\nu_1^{-\frac{1+c_1}{c_1}}\Big(\log\frac{\lambda(\rho_0)}{\rho_0}\Big)^{L\frac{1+n(c_1+1)}{c_1}} \leqslant \ln\frac{\lambda(\rho_0)}{\rho_0},$$

where $C > 0$ is a fixed number depending only on n, c_1, c_2, c_3, and K_1, and ν_1 is defined in Lemma 3.3.

Lemma 3.4. *Let $u \geqslant 0$ satisfies* (3.16) *and* (3.17), *and let φ satisfies* (3.13). *Assume that, with some $N \in (0, M)$ and some $\beta \in (0, 1)$,*

$$|\{x \in B_r(\overline{x}) : u(x) \leqslant N\}| \leqslant (1-\beta)|B_r(\overline{x})|.$$

Fix j_* by the condition

$$j_* = C\nu_1^{-\frac{1+c_1}{c_1}} r^{-\theta(r)\frac{1+n(c_1+1)}{c_1}}.$$

Then either

$$N \leqslant s_0\lambda(r) \tag{3.18}$$

or

$$u(x) \geqslant N/2^{j_*+1} \quad \text{for a.a. } x \in B_{2r}(\overline{x}),$$

provided that $B_{8r}(\overline{x}) \subset B_{\rho_0}(\overline{x}) \subset \Omega$.

The proof of Lemmas 3.3 and 3.4 is completely similar to that of Lemmas 3.1 and 3.2. Particularly, in the proof of Lemma 3.4, we use (2.1), Lemma 2.2, inequality (3.17) with $k = N/2^{j+1}$, $l = N/2^j$, and

$$\varepsilon = \left(\frac{|A^-_{k,r} \setminus A^-_{l,r}|}{|B_r(\overline{x})|}\right)^{\frac{1}{1+c_1}} r^{\frac{\theta(r)}{1+c_1}}, \quad j = 0, 1, 2, \dots j_*.$$

In addition, assume that (3.18) is violated. Since

$$2^{j_*} r \leqslant 2^{C\nu_1^{-\frac{1+c_1}{c_1}}\left(\log\frac{\lambda(r)}{r}\right)^{L\frac{1+n(c_1+1)}{c_1}}} \leqslant r2^{\log\frac{\lambda(r)}{r}} = \lambda(r) \tag{3.19}$$

for $r \leqslant \rho_0$, we can use (3.13).

We give a sketch of the proof of the continuity of the function u satisfying (3.16) and (3.17).

Fix $r \leqslant \rho_0$. The following two alternative cases are possible:

$$|\{x \in B_r(x_0) : u(x) \geqslant \mu_r^+ - \omega_r/2\}| \leqslant \frac{1}{2}|B_r(x_0)|$$

or

$$|\{x \in B_r(x_0) : u(x) \leqslant \mu_r^- + \omega_r/2\}| \leqslant \frac{1}{2}|B_r(x_0)|,$$

where

$$\mu_r^+ = \sup_{B_r(x_0)} u, \quad \mu_r^- = \inf_{B_r(x_0)} u, \quad \omega_r = \mu_r^+ - \mu_r^-.$$

Assume, for example, the first one. Then, by Lemma 3.4, we obtain

$$\omega_{r/2} \leqslant (1 - 2^{-1-j_*})\omega_r + s_0\lambda(r), \quad r \leqslant \rho_0.$$

From this, using (3.19), we obtain

$$\omega_{r/2} \leqslant \left(1 - \frac{r}{2\lambda(r)}\right)\omega_r + s_0\lambda(r), \quad r \leqslant \rho_0.$$

Iterating the previous inequality, we arrive at

$$\omega_r \leqslant \omega_{\rho_0} \exp\left(-\gamma \int_{2r}^{\rho} \frac{ds}{\lambda(s)}\right) + \gamma\lambda(\rho)$$

for any $2r < \rho \leqslant \rho_0$. This implies the continuity of u.

4. Parabolic $\mathcal{B}_1$ classes

4.1. Definition of the class and the main result

For simplicity, we consider the parabolic $\mathcal{B}_{1,\varphi}$ classes that correspond to second-order parabolic equations with nonstandard growth.

We assume that, for every $v \in \mathbb{R}_+$, the function $(x,t) \to \varphi(x,t,v)$ is measurable, and, for every $(x,t) \in \Omega_T := \Omega \times (0,T)$, $0 < T < +\infty$, the function $v \to \varphi(x,t,v)$ is increasing and continuous, $\varphi(x,t,0) := \lim\limits_{v \to +0} \varphi(x,t,v) = 0$.

We say that the measurable function $u(x,t)$ belongs to the parabolic class $\mathcal{B}_{1,\varphi}(\Omega_T)$,

$$u \in C_{\mathrm{loc}}(0,T; L^2_{\mathrm{loc}}(\Omega)) \cap L^1_{\mathrm{loc}}(0,T; W^{1,1}_{\mathrm{loc}}(\Omega)) \cap L^\infty(\Omega_T), \quad M := \sup_{\Omega_T} |u|,$$

and there exist constants $\delta \geqslant 0$, K_1, K_2, c_1, c_2, and $c_3 > 0$ such that, for any $(x_0,t_0) \in \Omega_T$, any cylinder $Q_{8\rho,8\theta}(x_0,t_0) := B_{8\rho}(x_0) \times (t_0 - 8\theta, t_0) \subset \Omega_T$, $\theta \leqslant K_2\rho^{1+\delta}$, for all $k,l \in \mathbb{R}$, $|k|$, $|l| < M$, $k < l$, for any $\varepsilon \in (0,1]$, $\sigma, \varepsilon_1 \in (0,1)$, for any $\zeta \in C_0^\infty(B_\rho(x_0))$, $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $B_{\rho(1-\sigma)}(x_0)$ and $|\nabla\zeta| \leqslant (\sigma\rho)^{-1}$, for any $\lambda(t) \in C^1(\mathbb{R}_+)$, $0 \leqslant \lambda(t) \leqslant 1$, the following inequalities hold:

$$\begin{aligned}
&\iint\limits_{A^{+}_{k,\rho(1-\sigma),\theta}\setminus A^{+}_{l,\rho(1-\sigma),\theta}} |\nabla u|\lambda(t)\,dx\,dt \leqslant \frac{K_1\varepsilon^{c_1}\sigma^{-c_3}}{\varphi\Big(x_0,t_0,\dfrac{M_+(k)}{\rho}\Big)}\times\\
&\times\Big\{\iint\limits_{A^{+}_{k,\rho,\theta}}(u-k)_+^2\,|\lambda_t|\,dx\,dt+\int\limits_{B_\rho(x_0)\times\{t_0-\theta\}}(u-k)_+^2\,\lambda(t_0-\theta)\,dx\\
&+\rho^{-1}\iint\limits_{A^{+}_{k,\rho,\theta}}\varphi\Big(x_0,t_0,K_1\frac{(u-k)_+}{\sigma\rho\zeta}\Big)(u-k)_+\zeta^{c_2-1}\,dx\,dt+|A^{+}_{k,\rho,\theta}|\Big\}\\
&+K_1\frac{M_+(k)}{\varepsilon\rho}|A^{+}_{k,\rho(1-\sigma),\theta}\setminus A^{+}_{l,\rho(1-\sigma),\theta}|+K_1\frac{\varepsilon^{c_1}M_+(k)}{\sigma^{c_3}\rho}|A^{+}_{k,\rho,\theta}|,
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
&\iint\limits_{A^{-}_{l,\rho(1-\sigma),\theta}\setminus A^{-}_{k,\rho(1-\sigma),\theta}} |\nabla u|\lambda(t)\,dx\,dt \leqslant \frac{K_1\varepsilon^{c_1}\sigma^{-c_3}}{\varphi\Big(x_0,t_0,\dfrac{M_-(l)}{\rho}\Big)}\times\\
&\times\Big\{\iint\limits_{A^{-}_{l,\rho,\theta}}(u-l)_-^2\,|\lambda_t|\,dx\,dt+\int\limits_{B_\rho(x_0)\times\{t_0-\theta\}}(u-l)_-^2\,\lambda(t_0-\theta)\,dx\\
&+\rho^{-1}\iint\limits_{A^{-}_{l,\rho,\theta}}\varphi\Big(x_0,t_0,K_1\frac{(u-l)_-}{\sigma\rho\zeta}\Big)(u-l)_-\zeta^{c_2-1}\,dx\,dt+|A^{-}_{l,\rho,\theta}|\Big\}\\
&+K_1\frac{M_-(l)}{\varepsilon\rho}|A^{-}_{l,\rho(1-\sigma),\theta}\setminus A^{-}_{k,\rho(1-\sigma),\theta}|+K_1\frac{\varepsilon^{c_1}M_-(l)}{\sigma^{c_3}\rho}|A^{-}_{l,\rho,\theta}|,
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
&\int\limits_{B_{\rho(1-\sigma)}(x_0)\times\{t\}}(u-k)_\pm^2\lambda(t)\,dx\\
&\leqslant\int\limits_{B_\rho(x_0)\times\{t_0-\theta\}}(u-k)_\pm^2\lambda(t_0-\theta)\,dx+K_1\sigma^{-c_3}\\
&\times\Big\{\iint\limits_{A^{\pm}_{k,\rho,\theta}}(u-k)_\pm^2|\lambda_t|\,dx\,dt+\varphi\Big(x_0,t_0,\frac{M_\pm(k)}{\rho}\Big)\frac{M_\pm(k)}{\rho}|A^{\pm}_{k,\rho,\theta}|\\
&+\rho^{-1}\iint\limits_{A^{\pm}_{k,\rho,\theta}}\varphi\Big(x_0,t_0,K_1\frac{(u-k)_\pm}{\sigma\rho\zeta}\Big)(u-k)_\pm\zeta^{c_2-1}dx\,dt+|A^{\pm}_{k,\rho,\theta}|\Big\},\\
&\qquad\qquad\forall\, t\in(t_0-\theta,t_0),
\end{aligned}\tag{4.3}$$

$$\int\limits_{B_{\rho(1-\sigma)}(x_0)\times\{t\}} \ln_+^2 \frac{M_\pm(k)}{w_k}\,dx \leqslant \int\limits_{B_\rho(x_0)\times\{t_0-\theta\}} \ln_+^2 \frac{M_\pm(k)}{w_k}\,dx$$

$$+K_1 \ln\frac{1}{\varepsilon_1}\bigg\{(\sigma\rho)^{-2}\iint\limits_{A^\pm_{k,\rho,\theta}} \psi\Big(x_0,t_0,\frac{K_1 w_k}{\sigma\rho\zeta}\Big)\zeta^{c_2-2}dx\,dt + |A^\pm_{k,\rho,\theta}|\bigg\}, \tag{4.4}$$

$$\forall\, t\in(t_0-\theta,t_0),$$

where $\psi(x,t,s)=\varphi(x,t,s)/s$,

$$(u-k)_\pm := \max\{\pm(u-k),0\},$$
$$w_k := (1+\varepsilon_1)M_\pm(k) - (u-k)_\pm,$$
$$A^\pm_{k,\rho,\theta} := Q_{\rho,\theta}(x_0,t_0)\cap\{(u-k)_\pm>0\},$$
$$M_\pm(k) = M_\pm(k,\rho,\theta) := \sup_{Q_{\rho,\theta}(x_0,t_0)} (u-k)_\pm.$$

We will assume that there exists $s_0>0$ such that the function $\psi(x,t,s)=\varphi(x,t,s)/s$ satisfies one of the following conditions:

$$\psi(x,t,s) \text{ is nondecreasing for all } (x,t)\in\Omega_T \text{ and for } s\geqslant s_0, \tag{4.5}$$

$$\psi(x,t,s) \text{ is nonincreasing for all } (x,t)\in\Omega_T \text{ and for } s\geqslant s_0, \tag{4.6}$$

and $\varphi(x,t,1)\asymp 1$ for any $(x,t)\in\Omega_T$, where $\asymp$ means that there exists constant $K>0$ such that $K^{-1}\leqslant\varphi(x,t,1)\leqslant K$. We also suppose the existence of a constant $c>0$ such that $\varphi(x_1,t_1,v/\rho)\leqslant c\,\varphi(x_2,t_2,v/\rho)$ for any (x_1,t_1), $(x_2,t_2)\in Q_{\rho,\theta}(x_0,t_0)$ and every $v\in(0,M)$.

In the case (4.6), we additionally assume that, for all $t\in(t_0-\theta,t_0)$ and $\varepsilon\in(0,1)$, the following inequality holds:

$$D^-\int\limits_{B_\rho(x_0)\times\{t\}} G_k(u)\frac{t-t_0+\theta}{\theta}\,\zeta^{c_2}dx$$
$$+\frac{\rho}{2\varepsilon}\int\limits_{B_\rho(x_0)\times\{t\}} \left|\nabla\ln\frac{(1+\varepsilon_1)M_\pm(k)}{w_k}\right|\frac{t-t_0+\theta}{\theta}\,\zeta^{c_2}dx$$
$$\leqslant\frac{K_1}{\theta}\int\limits_{B_\rho(x_0)\times\{t\}} G_k(u)\,\zeta^{c_2}dx + K_1(\varepsilon\sigma)^{-c_3}\bigg(1+\Big(\frac{\rho}{\varepsilon_1 M_\pm(k)}\Big)^{c_3}\bigg)|B_\rho(x_0)|$$
$$+K_1(\varepsilon\sigma)^{-c_3}\int\limits_{B_\rho(x_0)\times\{t\}} \frac{\varphi\Big(x_0,t_0,\dfrac{K_1 w_k}{\sigma\rho\zeta}\Big)w_k}{\Phi\Big(\dfrac{w_k}{\rho}\Big)}\,\zeta^{c_2-1}dx, \tag{4.7}$$

where $\Phi(w) := \Phi(x_0, t_0, w) := \int\limits_0^w \varphi(x_0, t_0, s)ds$,

$$G_k(u) := \int\limits_0^{(u-k)_\pm} \frac{(1+\varepsilon_1)M_\pm(k) - s}{\Phi\left(\dfrac{(1+\varepsilon_1)M_\pm(k) - s}{\rho}\right)}\, ds.$$

Here, we used the notation D^- for the derivative

$$D^- f(t) := \limsup_{h\to 0} \frac{f(t) - f(t-h)}{h}\,.$$

Example 4.1. Let u be a solution to the equation $u_t - \Delta_{p(x,t)} u = 0$,

$$\frac{2n}{n+1} < p \leqslant p(x,t) \leqslant q,$$

$$|p(x,t) - p(x,\tau)| \leqslant \frac{L}{\big|\ln(|x-y| + |t-\tau|)\big|}, \quad (x,t) \neq (y,\tau).$$

Solution $u \in C_{\rm loc}(0,T; L^2_{\rm loc}(\Omega)) \cap L^{p(x,t)}_{\rm loc}(0,T; W^{1,p(x,t)}_{\rm loc}(\Omega))$ satisfies the integral identity

$$\int_\Omega u(x,t_2)\eta(x,t_2)\,dx - \int_\Omega u(x,t_1)\eta(x,t_1)\,dx \\ + \int_{t_1}^{t_2}\int_\Omega \{-u\eta_t + |\nabla u|^{p(x,t)-2}\nabla u \nabla \eta\}\,dx\,dt = 0, \quad (4.8)$$

for any $\eta \in C(0,T; L^2(\Omega)) \cap L^{p(x,t)}(0,T; W^{1,p(x,t)}(\Omega))$, η, $|\eta_t| \in L^\infty(\Omega)$, and for any $t_2 > t_1 > 0$.

We test the integral identity by $\eta = (u-k)_\pm \zeta^q(x)\lambda(t)$, where ζ, λ as in (4.1), (4.2). By the Young inequality we arrive at

$$\int\limits_{B_{\rho(1-\sigma)}(x_0)\times\{t\}} (u-k)^2_\pm \lambda(t)dx + \iint\limits_{A^\pm_{k,\rho(1-\sigma),\theta}} |\nabla u|^{p(x,t)}\lambda(t)\,dx\,dt \\ \leqslant \int_{B_\rho(x_0)} (u-k)^2_\pm \lambda(t_0-\theta)\,dx \\ + \gamma(n,p,q,M,L) \iint\limits_{A^\pm_{k,\rho,\theta}} \left((u-k)_\pm |\lambda_t| + \left(\frac{(u-k)_\pm}{\sigma\rho}\right)^{p^-}\right) dx\,dt,$$

where $p^- = \min\limits_{Q_{\rho,\theta}(x_0,t_0)} p(x,t)$. From this we arrive at (4.3) and, using the Young inequality similarly to Example 3.1, we arrive at (4.1), (4.2) with $\varphi(x,t,s) = s^{p(x,t)-1}$.

To prove (4.4), we test the integral identity by

$$\eta = \left[\ln_+^2 \frac{M_\pm(k)}{(1+\varepsilon_1)M_\pm(k) - (u-k)_\pm} \right]'_u \zeta^q(x).$$

Then, by the Young inequality, the standard calculations give (4.4) with $\psi(x,t,s) = s^{p(x,t)-2}$.

To prove (4.7), we note that the integral identity (4.8) can be rewritten in the form (for details, we refer the reader to [34, Chap. II]):

$$\int_\Omega (u-k)_\pm \eta \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \Big\{ -(u-k)_\pm \eta_t + |\nabla (u-k)_\pm|^{p(x,t)-2} \nabla (u-k)_\pm \nabla \eta \Big\} \, dxdt \leqslant 0, \quad t_2 > t_1 > 0, \tag{4.9}$$

for any $k > 0$ and any $\eta \geqslant 0$, η is the same as in (4.8).

Testing (4.9) by $\eta = \rho^{-p^-} w_k^{1-p^-} \dfrac{t - t_0 + \theta}{\theta} \zeta^q(x)$, and using the Young inequality, we obtain

$$D^- \int\limits_{B_\rho(x_0)\times\{t\}} G_k(u) \frac{t-t_0+\theta}{\theta} \zeta^q dx + \int\limits_{B_\rho(x_0)\times\{t\}} \Big(\frac{\rho}{w_k}\Big)^{p^-} |\nabla w_k|^{p^-} \frac{t-t_0+\theta}{\theta} \zeta^q \, dx$$

$$\leqslant D^- \int\limits_{B_\rho(x_0)\times\{t\}} G_k(u) \frac{t-t_0+\theta}{\theta} \zeta^q dx + \int\limits_{B_\rho(x_0)\times\{t\}} \Big(\frac{\rho}{w_k}\Big)^{p^-} |\nabla w_k|^{p(x,t)} \frac{t-t_0+\theta}{\theta} \zeta^q \, dx$$

$$+ \int\limits_{B_\rho(x_0)\times\{t\}} \Big(\frac{\rho}{w_k}\Big)^{p^-} \frac{t-t_0+\theta}{\theta} \zeta^q \, dx$$

$$\leqslant \frac{1}{\theta} \int\limits_{B_\rho(x_0)\times\{t\}} G_k(u) \, \zeta^q \, dx + \gamma(n,p,q,M,L) \bigg(1 + \Big(\frac{\rho}{\varepsilon_1 M_\pm(k)}\Big)^{p^-} \bigg) |B_\rho(x_0)|.$$

From this, using the Young inequality with arbitrary $\varepsilon > 0$, we arrive at (4.7).

Example 4.2. Let u be a solution to the equation

$$u_t - \Delta_g u = 0, \tag{4.10}$$

where $g : (0, +\infty) \to (0, +\infty)$ satisfies the following condition:

$$\frac{g(w)}{g(v)} \geqslant \left(\frac{w}{v}\right)^{p-1}, \quad w > v > 0, \quad p > 1.$$

For every $s > 0$, we set $G(s) := g(s)s$.

We say that the function $u \in C_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega)) \cap L^G_{\text{loc}}(0, T; W^{1,G}_{\text{loc}}(\Omega))$ is a solution to Eq. (4.10), if the following integral identity holds:

$$\begin{gathered}\int_\Omega u(x, t_2)\eta(x, t_2)\, dx - \int_\Omega u(x, t_1)\eta(x, t_1)\, dx \\ + \int_{t_1}^{t_2} \int_\Omega \left\{ - u\eta_t + g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\eta \right\} dx\, dt = 0, \quad t_2 > t_1 > 0,\end{gathered} \tag{4.11}$$

for any $\eta \in C(0, T; L^2(\Omega)) \cap L^G(0, T; W^{1,G}_0(\Omega))$, $\eta, \eta_t \in L^\infty(\Omega_T)$.

We test the integral identity (4.11) by $\eta = (u-k)_\pm \zeta^{c_2}(x)\lambda(t)$ and use (3.8) and the Young inequality to arrive at (4.1), (4.2), and (4.3) with $\varphi(x, t, u) = \varphi(u) = g(u)$.

To prove (4.4), we test the integral identity (4.11) by

$$\eta = \left[\ln^2_+ \frac{M_\pm(k)}{(1+\varepsilon_1)M_\pm(k) - (u-k)_\pm}\right]'_u \zeta^{c_2}(x)\lambda(t).$$

Using (3.8), we arrive at (4.4) with $\psi(s) = g(s)/s$.

To prove (4.7), we note that the integral identity (4.11) can be rewritten in the form

$$\begin{aligned}\int_\Omega (u-k)_\pm \eta\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \Big\{ & - (u-k)_\pm \eta_t \\ & + g(|\nabla(u-k)_\pm|)\frac{\nabla(u-k)_\pm}{|\nabla(u-k)_\pm|}\nabla\eta \Big\} dx\, dt \leqslant 0, \quad t_2 > t_1 > 0,\end{aligned} \tag{4.12}$$

for any $k > 0$ and any $\eta \geqslant 0$, η is the same as in (4.8).

Testing (4.12) by

$$\eta = \frac{w_k}{\Phi\left(\frac{w_k}{\rho}\right)} \frac{t - t_0 + \theta}{\theta} \zeta^q,$$

using (3.7) and the fact that $\Phi(\frac{w_k}{\rho}) \leqslant \frac{1}{\rho} G(\frac{w_k}{\rho})$, we arrive at

$$D^{-} \int\limits_{B_\rho(x_0)\times\{t\}} G_k(u)\,\frac{t-t_0+\theta}{\theta}\,\zeta^q\,dx + \int\limits_{B_\rho(x_0)\times\{t\}} \frac{G(|\nabla w_k|)}{G(\frac{w_k}{\rho})}\,\frac{t-t_0+\theta}{\theta}\,\zeta^q\,dx$$
$$\leqslant \frac{1}{\theta} \int\limits_{B_\rho(x_0)\times\{t\}} G_k(u)\,\zeta^q\,dx + \gamma \int\limits_{B_\rho(x_0)\times\{t\}} \frac{G\Big(\frac{\gamma w_k}{\sigma\rho\zeta}\Big)}{\Phi\Big(\frac{w_k}{\rho}\Big)}\,\zeta^q\,dx. \quad (4.13)$$

By (3.7), we have

$$\frac{\rho}{\varepsilon}\,\frac{|\nabla w_k|}{w_k} = \frac{1}{\varepsilon}\,\frac{|\nabla w_k|}{G\Big(\frac{w_k}{\rho}\Big)}\,g\Big(\frac{w_k}{\rho}\Big) \leqslant \frac{G(|\nabla w_k|)}{G\Big(\frac{w_k}{\rho}\Big)} + \varepsilon^{-\frac{p}{p-1}}. \quad (4.14)$$

Combining (4.13) and (4.14), we arrive at (4.7).

Example 4.3. Let u be a solution to the equation

$$u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u + a(x,t)|\nabla u|^{q-2}\nabla u\big) = 0,$$

$0 \leqslant a(x,t) \in C^{0,\alpha,\alpha/p^-}(\Omega_T)$, $p^- = \min(2,p)$, $\frac{2n}{n+1} < p < q \leqslant 2$, or $p \geqslant 2$. We note (see [24]) that the constant δ from the definition of the $\mathcal{B}_{1,\varphi}(\Omega_T)$ classes is equal to $p^- - 1$.

We set

$$G_a(s) := g_a(s)s, \quad g_a(s) := s^{p-1} + a s^{q-1}, \quad s > 0.$$

The solution $u \in C_{\rm loc}(0,T;L^2_{\rm loc}(\Omega)) \cap L^{G_a}_{\rm loc}(0,T;W^{1,G_a}_{\rm loc}(\Omega))$ satisfies the integral identity

$$\int_\Omega u(x,t_2)\eta(x,t_2)dx - \int_\Omega u(x,t_1)\eta(x,t_1)dx$$
$$+ \int_{t_1}^{t_2}\int_\Omega \Big\{ - u\eta_t + g_{a(x,t)}(|\nabla u|)\,\frac{\nabla u}{|\nabla u|}\,\nabla\eta\Big\}\,dxdt = 0, \quad t_2 > t_1 > 0,$$

for any $\eta \in C(0,T;L^2(\Omega)) \cap L^{G_a}(0,T;W^{1,G_a}_0(\Omega))$, η, $\eta_t \in L^\infty(\Omega_T)$.

The proof of inequalities (4.1)–(4.4), (4.7) is completely similar to that of Example 4.2 in two alternative cases: $a(x_0,t_0) = 0$ or $a(x_0,t_0) > 0$ (see Example 3.3).

The case of the equation

$$u_t - \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\big(1 + a(x,t)\ln^{\beta}(1+|\nabla u|)\big)\Big) = 0, \quad 0 < \beta \leqslant 1,$$

$$0 \leqslant a(x,t) \in C^{0,\alpha,\alpha/p-}(\Omega_T), \quad \beta \leqslant \alpha \leqslant 1.$$

can be considered almost similarly.

The main result of this section reads

Theorem 4.1. *Let $u \in \mathcal{B}_{1,\varphi}(\Omega_T)$ and φ satisfies* (4.5) *or* (4.6)*. In case* (4.5)*, we assume that there exists $s_0 > 0$ such that*

$$\frac{\varphi(x,t,w)}{\varphi(x,t,v)} \leqslant 2\left(\frac{w}{v}\right)^{\mu}, \quad w \geqslant v \geqslant s_0 > 0, \tag{4.15}$$

for all $(x,t) \in \Omega_T$, with some $\mu > 0$ and some $s_0 > 0$. Then u is locally Hölder-continuous in Ω_T.

Remark 4.1. We note that Theorem 4.1 improves the results in [53–55], since only the case $s_0 = 0$ in conditions (4.6) and (4.15) was considered there. The example from Remark 3.1 can be used also in the parabolic case. We also give an answer to the question on the Hölder continuity of solutions to the parabolic equations with (p,q)-growth, $p \leqslant 2 \leqslant q$. So, the function $\varphi(s) = s^{p-1} + s^{q-1}$, $s > 0$ and $p \leqslant 2 < q$, satisfies conditions (4.5) and (4.15) with $s_0 = \left(\frac{2-p}{q-2}\right)^{1/(q-p)}$ and $\mu = q-1$. On the other hand, the function $\varphi(s) = s^{p-1}\big(1+\ln(1+s)\big)$, $s > 0$, and $p < 2$, $q = p+1$, satisfies condition (4.6) with $s_0 = \exp\left(\frac{p-1}{2-p}\right) - 1$.

4.2. DeGiorgi-type lemmas

Below, we use the simplified notations $\varphi(u)$ and $\psi(u)$ instead of $\varphi(x_0,t_0,u)$ and $\psi(x_0,t_0,u)$.

The main result of this section is

Theorem 4.2. *Let $0 \leqslant u \in \mathcal{B}_{1,\varphi}(\Omega_T)$, and let inequality* (4.15) *and condition* (4.5) *or* (4.6) *be fulfilled. Fix $N \in (0,M)$, $a \in (0,1)$. Then there exist numbers $B \geqslant \max\{1, s_0\}$ and $\nu \in (0,1)$ which depend only on K_1, c, c_1, c_2, c_3, n, τ, ρ, a, and N and are such that if*

$$|Q_{\rho,\tau}(x_0,t_0) \cap \{u \leqslant N\}| \leqslant \nu |Q_{\rho,\tau}(x_0,t_0)|,$$

then either

$$N \leqslant B\rho \tag{4.16}$$

or

$$u(x,t) \geqslant aN \quad \textit{for all} \quad (x,t) \in Q_{\rho/2,\tau/2}(x_0,t_0).$$

Proof. For $j = 0, 1, 2, \ldots$, we define the sequences

$$k_j := aN + \frac{(1-a)N}{2^j}, \quad \rho_j := \frac{\rho}{2}(1+2^{-j}), \quad \overline{\rho}_j := \frac{\rho_j + \rho_{j+1}}{2},$$

$$\tau_j := \frac{\tau}{2}(1+2^{-j}), \quad \overline{\tau}_j := \frac{\tau_j + \tau_{j+1}}{2}, \quad B_j := B_{\rho_j}(x_0),$$

$$\overline{B}_j := B_{\overline{\rho}_j}(x_0), \quad Q_j := Q_{\rho_j,\tau_j}(x_0,t_0), \quad \overline{Q}_j := Q_{\overline{\rho}_j,\overline{\tau}_j}(x_0,t_0),$$

$$A_{j,k_j} := A^-_{k_j,\rho_j,\tau_j}, \quad \overline{A}_{j,k_j} := A^-_{k_j,\overline{\rho}_j,\overline{\tau}_j}.$$

Let $\lambda_j(t)$ be such that $\lambda_j(t) = 1$ for $t > \overline{t} - \tau_{j+1}$, $\lambda_j(t) = 0$ for $t < \overline{t} - \overline{\tau}_j$, $0 \leqslant \lambda_j \leqslant 1$ and $|d\lambda_j/dt| \leqslant 2^{j+2}\tau^{-1}$. Let also $\zeta_j \in C_0^\infty(\overline{B}_j)$, be such that $0 \leqslant \zeta_j \leqslant 1$, $\zeta_j = 1$ in B_{j+1} and $|\nabla \zeta_j| \leqslant 2^{j+2}\rho^{-1}$.

We also assume that condition (4.16) is violated, i.e.

$$N \geqslant B\rho. \tag{4.17}$$

Then inequality (4.2) with $\varepsilon = 1$ implies that

$$\begin{aligned}
&\iint\limits_{\overline{A}_{j,k_j}} |\nabla\big((k_j - \max\{u, k_{j+1}\})_+ \zeta_j\big)|dxdt \\
&\quad \leqslant \gamma \iint\limits_{\overline{A}_{j,k_j}\setminus \overline{A}_{j,k_{j+1}}} |\nabla u| dxdt + \iint\limits_{\overline{A}_{k_j,k_j}} \big(k_j - \max\{u,k_{j+1}\}\big)_+ |\nabla \zeta_j| dxdt \\
&\qquad \leqslant \gamma 2^{j\gamma} \frac{N}{\rho}\bigg(1 + \frac{N\rho}{\tau\varphi\big(\frac{N}{\rho}\big)}\bigg)|A_{j,k_j}|.
\end{aligned}$$

Here, we also used our assumption that φ is nondecreasing, and the evident inequality $\big(k_j - \max\{u, k_{j+1}\}\big)_+ \leqslant (k_j - u)_+$ holds.

By (4.17), inequality (4.3) can be rewritten in the form

$$\sup_{\overline{t}-\overline{\tau}_j<t<\overline{t}} \int_{\overline{B}_j} (k_j - u)_+^2 \lambda_j(t)dx \leqslant \gamma\, 2^{j\gamma} \frac{N}{\rho}\varphi\Big(\frac{N}{\rho}\Big)\bigg(1 + \frac{N\rho}{\tau\varphi\big(\frac{N}{\rho}\big)}\bigg)|A_{j,k_j}|.$$

From this by the Sobolev embedding theorem and Hölder's inequality, we obtain

$$(k_j - k_{j+1})|A_{j+1,k_{j+1}}| \leqslant \iint\limits_{\overline{A}_{j,k_j}} (k_j - \max\{u, k_{j+1}\})_+ \zeta_j \lambda_j \, dxdt$$

$$\leqslant \gamma \bigg(\sup_{\bar{t}-\bar{\tau}_j < t < \bar{t}} \int_{\overline{B}_j} (k_j - u)_+^2 \lambda_j(t) dx \bigg)^{\frac{1}{n+2}}$$

$$\times \bigg(\iint\limits_{\overline{A}_{j,k_j}} \big| \nabla \big((k_j - \max\{u, k_{j+1}\})_+ \zeta_j \big) \big| \, dxdt \bigg)^{\frac{n}{n+2}} |A_{j,k_j}|^{\frac{2}{n+2}}$$

$$\leqslant \gamma 2^{j\gamma} \Big(\varphi\Big(\frac{N}{\rho}\Big) \Big)^{\frac{1}{n+1}} \Big(\frac{N}{\rho}\Big)^{\frac{n+1}{n+2}} \bigg(1 + \frac{N\rho}{\tau \varphi\big(\frac{N}{\rho}\big)} \bigg)^{\frac{n+1}{n+2}} |A_{j,k_j}|^{1+\frac{1}{n+2}}.$$

From this, we get

$$y_{j+1} := \frac{|A_{j+1,k_{j+1}}|}{|Q_{\rho,\tau}(x_0,t_0)|} \leqslant \frac{\gamma 2^{j\gamma}}{1-a} \bigg(\frac{\tau\varphi\big(\frac{N}{\rho}\big)}{N\rho} \bigg)^{\frac{1}{n+2}} \bigg(1 + \frac{N\rho}{\tau\varphi\big(\frac{N}{\rho}\big)} \bigg)^{\frac{n+1}{n+2}} y_j^{1+\frac{1}{n+2}}.$$

This inequality together with Lemma 2.2 implies that $\lim\limits_{j\to\infty} y_j = 0$, provided ν is chosen to satisfy

$$\nu := \gamma^{-2-n} 2^{-\gamma(n+2)^2} (1-a)^{n+2} \frac{N\rho}{\tau\varphi\big(\frac{N}{\rho}\big)} \bigg(1 + \frac{N\rho}{\tau\varphi\big(\frac{N}{\rho}\big)} \bigg)^{-n-1}.$$

This proves Theorem 4.2. □

Theorem 4.3 (DeGiorgi type lemma involving initial data)**.** *Let $0 \leqslant u \in \mathcal{B}_{1,\varphi}(\Omega_T)$. Fix $N \in (0, M)$, $a \in (0,1)$. Then there exist numbers $B \geqslant \max\{1, s_0\}$ and $\nu_1 \in (0,1)$ which depend only on K_1, c, c_1, c_2, c_3, n, τ, ρ, a, and N and are such that if*

$$\tau \leqslant \frac{N\rho}{\tau\varphi\big(\frac{N}{\rho}\big)}, \quad u(x, t_0 - \tau) \geqslant N,$$

and

$$|Q_{\rho,\tau}(x_0,t_0) \cap \{u \leqslant N\}| \leqslant \nu_1 |Q_{\rho,\tau}(x_0,t_0)|,$$

then either (4.16) *holds true or*

$$u(x,t) \geqslant aN \quad \textit{for all} \quad (x,t) \in Q_{\rho/2,\tau}(x_0,t_0).$$

Proof. The proof is similar to that of Theorem 4.2. Taking $\eta(t) \equiv 1$ into account, using (4.2) and (4.3), and repeating the same arguments as in the previous proof, we prove Theorem 4.3. The precise choice of ν_1 is

$$\nu_1 := \gamma^{-2-n} 2^{-\gamma(n+2)^2} (1-a)^{n+2}.$$

□

4.3. Hölder continuity. Proof of Theorem 4.1

In the proof, we follow [34] (see also [53–55]). Further, we need the following lemmas.

Lemma 4.1. *Let* $0 \leqslant u \in \mathcal{B}_{1,\varphi}(\Omega_T)$, *and let inequality* (4.15) *and condition* (4.5) *or* (4.6) *be fulfilled. Assume also that, with some* $N \in (0, M)$ *and some* $\beta \in (0,1)$, *the following relation holds:*

$$|\{x \in B_r(\overline{x}) : u(x,\overline{t}) \leqslant N\}| \leqslant (1-\beta)|B_r(\overline{x})|. \tag{4.18}$$

Then there exists a number $j_1 > 1$ *which depends only on* K_1, n, c, c_1, c_2, c_3, μ, *and* β *and is such that either*

$$\frac{N}{2^{j_1}} \leqslant r$$

or

$$\left|\left\{x \in B_r(\overline{x}) : u(x,\overline{t}) \leqslant \frac{N}{2^{j_1+2}}\right\}\right| \leqslant \left(1 - \frac{\beta^2}{4}\right)|B_r(\overline{x})| \tag{4.19}$$

for all $t \in (\overline{t}, \overline{t}+\theta)$, *where* $\theta = r^2[\psi(N/2r)]^{-1}$ *in case* (4.5), *and* $\theta = r^2[\psi(N/2^{j_1+2}r)]^{-1}$ *in case* (4.6).

Proof. We use inequality (4.4) with $\rho = r$, $k = N/4$, $\varepsilon_1 = 2^{-j_1}$. In case (4.5), since $\frac{1}{4}N(1+\varepsilon_1) - (N-u)_+ \leqslant \frac{1}{4}N(1+\varepsilon_1)$, we estimate the second term on the right-hand side of (4.4) as follows:

$$\begin{aligned} &r^{-2}\ln\frac{1}{\varepsilon_1} \iint\limits_{A^-_{N/4,r,\theta}} \psi\left(\frac{\frac{1}{4}N(1+\varepsilon_1) - (N-u)_+}{r}\right) dxdt \\ &\qquad \leqslant \gamma \ln\frac{1}{\varepsilon_1}\,\frac{\psi(\frac{1}{4}N(1+\varepsilon_1)/r)}{\psi(\frac{1}{2}N/r)}\,|B_r(\overline{x})| \leqslant \gamma j_1 |B_r(\overline{x})|. \end{aligned} \tag{4.20}$$

In case (4.6), since $\frac{1}{4}N(1+\varepsilon_1) - (N-u)_+ \geqslant \frac{1}{4}N\varepsilon_1$, we estimate the second term on the right-hand side of (4.3) as follows:

$$\begin{aligned} &r^{-2}\ln\frac{1}{\varepsilon_1} \iint\limits_{A^-_{N/4,r,\theta}} \psi\left(\frac{\frac{1}{4}N(1+\varepsilon_1) - (N-u)_+}{r}\right) dxdt \\ &\qquad \leqslant \gamma r^{-2}\theta \ln\frac{1}{\varepsilon_1}\,\psi\left(\frac{N\varepsilon_1}{4r}\right)|B_r(\overline{x})| = \gamma j_1 |B_r(\overline{x})|. \end{aligned} \tag{4.21}$$

The term on the left-hand side of (4.4) can be estimated by

$$\begin{aligned} &\int\limits_{B_{r(1-\sigma)}(\overline{x})\times\{t\}} \ln^2_+ \frac{N/4}{\frac{1}{4}N(1+\varepsilon_1) - (\frac{1}{4}N - u)_+}\, dx \\ &\geqslant (j_1-1)^2|\{x \in B_r(\overline{x}) : u(x,t) < 2^{-j_1-2}N\}| - n\sigma(j-1)^2|B_r(\overline{x})|. \end{aligned}$$

Then, from (4.4), (4.18), (4.20), and (4.21), we arrive at

$$|\{x \in B_r(\overline{x}) : u(x,t) \leqslant 2^{-j_1-2}N\}| \leqslant |B_r(\overline{x})| \bigg\{ \Big(\frac{j_1}{j_1-1}\Big)^2 (1-\beta) + n\sigma + \gamma\sigma^{-c_3}\frac{j_1}{(j_1-1)^2} \bigg\}.$$

Choosing σ so small that $n\sigma < \beta^2/4$ and then choosing j so large that

$$\Big(\frac{j_1}{j_1-1}\Big)^2 < 1+\beta, \quad \gamma\sigma^{-c_3}\frac{j_1}{(j_1-1)^2} < \frac{\beta^2}{4},$$

we get the required relation (4.19). □

Lemma 4.2. *Let* $0 \leqslant u \in \mathcal{B}_{1,\varphi}(\Omega_T)$, *and let condition* (4.5) *and inequality* (4.15) *be satisfied. Assume also that, with some* $N \in (0, M)$ *and some* $B > 1$, *the following inequalities hold:*

$$u(x,\overline{t}) \geqslant N \quad \text{for} \quad x \in B_r(\overline{x}), \tag{4.22}$$

$$N \geqslant Br.$$

Then, for any ξ, $\nu_1 \in (0,1)$, *there exists a number* j_2 *which depends only on* K_1, n, c, c_1, c_2, c_3, μ, *and* ν_1 *and is such that*

$$|\{x \in B_{r/2}(\overline{x}) : u(x,t) < 2^{-j_2}N\}| \leqslant \nu_1 |B_{r/2}(\overline{x})| \tag{4.23}$$

for any

$$t \in (\overline{t}, \overline{t}+\theta), \quad \theta = \frac{r^2}{\psi(\xi N/r)}.$$

Proof. We use inequality (4.4) with

$$\rho = r, \quad k = \xi N/4, \quad \sigma = 1/2, \quad \varepsilon_1 = 2^{-j}.$$

By (4.22), the first term on the right-hand side of (4.4) is equal to zero. Choosing $B \geqslant s_0$ large enough, since

$$\frac{\xi N}{4}(1+\varepsilon_1) - \Big(\frac{\xi N}{4} - u\Big)_+ \leqslant \frac{\xi N}{4}(1+\varepsilon_1),$$

we estimate the second term on the right-hand side of (4.4) as follows:

$$r^{-2}\ln\frac{1}{\varepsilon_1}\iint\limits_{A^-_{\xi N/4,r,\theta}} \psi\bigg(\frac{\frac14\xi N(1+\varepsilon_1) - (\frac14\xi N - u)_+}{r}\bigg)dxdt \leqslant \gamma r^{-2}\theta\ln\frac{1}{\varepsilon_1}\,\psi\bigg(\frac{\xi N(1+\varepsilon_1)}{4r}\bigg)\,|B_r(\overline{x})| = \gamma j|B_r(\overline{x})|.$$

The term on the left-hand side of (4.4) can be estimated by

$$\int\limits_{B_{r/2}(\overline{x})\times\{t\}} \ln_+^2 \frac{\xi N/4}{\frac{1}{4}\xi N(1+\varepsilon_1)-(\frac{1}{4}\xi N-u)_+}\,dx \geqslant (j-2)^2\ln 2\,|\{x\in B_{r/2}(\overline{x}) : u(x,t)\leqslant 2^{-j}\xi N\}|.$$

Choosing j so large that $\gamma j/(j-1)^2=\nu_1$, we obtain the required relation (4.23) with $j_2=j+[\ln\frac{1}{\xi}]+1$. □

4.4. Proof of Theorem 4.1 in cases (4.5) and (4.15)

Let $(x_0,t_0)\in\Omega_T$ be arbitrary. We fix a number $R>0$ so that

$$Q_R(x_0,t_0):=B_R(x_0)\times(t_0-R^2/\psi(s_0),t_0)\subset\Omega_T$$

and set

$$\mu_+:=\sup_{Q_R(x_0,t_0)}u,\quad \mu_-:=\inf_{Q_R(x_0,t_0)}u,\quad \omega:=\mu_+-\mu_-.$$

Fix positive numbers $B>s_0$ and $j_*<\ln B$ which will be specified later and depend only on K_1, n, c, c_1, c_2, c_3, and μ. If

$$\omega\geqslant BR, \tag{4.24}$$

then

$$Q_{R,\theta}(x_0,t_0)\subset Q_R(x_0,t_0),\quad \theta=\frac{R^2}{\psi\Big(\dfrac{\omega}{2^{s_*}R}\Big)}.$$

We consider the cylinders $Q_{R,\eta}(x_0,\overline{t})\subset Q_{R,\theta}(x_0,t_0)$, where

$$\eta=\frac{1}{4}\frac{R^2}{\psi\Big(\dfrac{\omega}{2R}\Big)}\quad\text{and}\quad t_0-\frac{R^2}{\psi\Big(\dfrac{\omega}{2^{s_*}R}\Big)}\leqslant\overline{t}-\frac{R^2}{4\psi\Big(\dfrac{\omega}{2R}\Big)}<\overline{t}\leqslant t_0.$$

The following two alternative cases are possible.

First alternative. There exists a cylinder $Q_{R,\eta}(x_0,\overline{t})\subset Q_{R,\theta}(x_0,t_0)$ such that

$$|\{(x,t)\in Q_{R,\eta}(x_0,\overline{t}) : u(x,t)\leqslant\mu_-+\omega/2\}|\leqslant\nu|Q_{R,\eta}(x_0,\overline{t})|. \tag{4.25}$$

Second alternative. For all cylinders $Q_{R,\eta}(x_0,\overline{t})\subset Q_{R,\theta}(x_0,t_0)$, the opposite inequality holds:

$$|\{(x,t)\in Q_{R,\eta}(x_0,\overline{t}) : u(x,t)\leqslant\mu_-+\omega/2\}|>\nu|Q_{R,\eta}(x_0,\overline{t})|. \tag{4.26}$$

Further, we assume that inequality (4.24) holds.

4.5. Analysis of the first alternative

By Theorem 4.2 with $\xi = 1/2$, $a = 1/2$, $\rho = R$, $\tau = \eta$, we obtain from (4.25) that

$$u(x, \bar{t}) \geqslant \mu_- + \omega/4 \quad \text{for all} \quad x \in B_{R/2}(x_0). \tag{4.27}$$

Using Lemma 4.2 with $N = \omega/4$, we conclude in view of (4.27) that

$$|\{(x,t) \in Q_{R/4,\theta}(x_0,t_0) : u(x,t) \leqslant \mu_- + \omega/2^{j_2}\}| \leqslant \nu_1 |Q_{R/4}(x_0,t_0)| \tag{4.28}$$

with $j_2 > j_*$. With regard for Theorem 4.3 and (4.28), we get

$$u(x,t) \geqslant \mu_- + \omega/2^{j_2+1} \quad \text{for} \quad (x,t) \in Q_{R/8,\eta}(x_0,t_0). \tag{4.29}$$

4.6. Analysis of the second alternative

Lemma 4.3. *Fix a cylinder $Q_{R,\eta}(x_0, \bar{t})$ and suppose that* (4.26) *holds. Then there exists $t_* \in (\bar{t} - \eta, \bar{t} - \nu\eta/2)$ such that*

$$|\{x \in B_R(x_0) : u(x, t_*) \geqslant \mu_+ - \omega/2\}| \leqslant \frac{1-\nu}{1-\nu/2} |B_R(x_0)|. \tag{4.30}$$

Proof. Suppose that the statement of the lemma is false. Then, for all $t \in (\bar{t} - \eta, \bar{t} - \nu\eta/2)$, the following relation holds:

$$|\{x \in B_R(x_0) : u(x,t) \geqslant \mu_+ - \omega/2\}| > \frac{1-\nu}{1-\nu/2} |B_R(x_0)|.$$

Hence,

$$\begin{aligned} &|\{(x,t) \in Q_{R,\eta}(x_0,\bar{t}) : u(x,t) \geqslant \mu_+ - \omega/2\}| \\ &\geqslant \int\limits_{\bar{t}-\eta}^{\bar{t}-\nu\eta/2} |\{x \in B_R(x_0) : u(x,t) \geqslant \mu_+ - \omega/2\}|\, dt > (1-\nu)|Q_{R,\eta}(x_0,\bar{t})|, \end{aligned}$$

which contradicts (4.30). □

By Lemma 4.1, we obtain from (4.30) that there exists $j_1 \in (1, j_*)$ which depends only on K_1, n, c, c_1, c_2, c_3, μ, and ν and is such that, for all $t \in (t_*, \bar{t})$,

$$|\{x \in B_R(x_0) : u(x,t) \geqslant \mu_+ - \omega/2^{j_1+2}\}| \leqslant (1 - \nu^2/4)|B_R(x_0)|. \tag{4.31}$$

Lemma 4.4. *For any $\nu \in (0,1)$, there exists a number j_* which depends only on K_1, n, c, c_1, c_2, c_3, μ, and ν and is such that*

$$|\{(x,t) \in Q_{R,\theta}(x_0,t_0) : u(x,t) \leqslant \mu_+ - \omega/2^{j_*}\}| \leqslant \nu |Q_{R,\theta}(x_0,t_0)|. \quad (4.32)$$

Proof. For $j = j_1+2, \dots, j_*$, we set $k_j = \mu_+ - \omega/2^j$. We use inequality (4.1) with $k = k_j$, $l = k_{j+1}$, $\rho = 2R$, $\sigma = 1/2$, and with

$$\varepsilon = \left(\frac{|A^+_{k_j,R,\theta} \setminus A^+_{k_{j+1},R,\theta}|}{|Q_{R,\theta}(x_0,t_0)|}\right)^{\frac{1}{1+c_1}}.$$

We also choose $\eta(t)$ such that $0 \leqslant \eta(t) \leqslant 1$, $\eta(t) = 0$ for $t \leqslant t_0 - \theta$, $\eta(t) = 1$ for $t \geqslant t_0 - \theta/2$, and $|\eta_t| \leqslant 2\theta^{-1}$. By these choices and by the fact that

$$\psi\left(\frac{\omega}{2^{j_*}R}\right) \leqslant \psi\left(\frac{\omega}{2^j R}\right),$$

inequality (4.1) yields

$$\iint\limits_{A^+_{k_j,R,\theta} \setminus A^+_{k_{j+1},R,\theta}} |\nabla u| dxdt \leqslant \frac{\gamma\omega}{2^j R} |Q_{R,\theta}(x_0,t_0)| \left(\frac{|A^+_{k_j,R,\theta} \setminus A^+_{k_{j+1},R,\theta}|}{|Q_{R,\theta}(x_0,t_0)|}\right)^{\frac{c_1}{1+c_1}}.$$

From Lemma 2.1 with $s = 1$, (4.31), and (4.32), we get

$$\left(\frac{|A^+_{k_{j+1},R,\theta}|}{|Q_{R,\theta}(x_0,t_0)|}\right)^{\frac{1+c_1}{c_1}} \leqslant \gamma \frac{|A^+_{k_j,R,\theta} \setminus A^+_{k_{j+1},R,\theta}|}{|Q_{R,\theta}(x_0,t_0)|}.$$

Summing up the previous inequality for $j = j_1+2, \dots, j_*$ and choosing j_* from the condition

$$\gamma (j_* - j_1 - 2)^{-\frac{c_1}{1+c_1}} = \nu,$$

we arrive at the required relation (4.32). □

By Theorem 4.2, inequality (4.32) yields

$$u(x,t) \leqslant \mu_+ - \omega/2^{j_*+1} \quad \text{for} \quad (x,t) \in Q_{R/2,\theta/2}(x_0,t_0).$$

Combining this inequality and (4.29), we conclude that

$$\operatorname{osc}\{u; Q_{R/8,\eta/8}(x_0,t_0)\} \leqslant (1-\xi)\omega, \quad \xi = 2^{-j_2-1}. \quad (4.33)$$

Define the sequences $R_j := \left(\frac{1}{8}(1-\xi)\right)^j R$, $\omega_j := (1-\xi)^j \omega$,

$$Q_j := Q_{R_j,\theta_j}(x_0,t_0), \quad \theta_j := \frac{R_j^2}{\psi(\omega_j/R_j)}, \quad j = 0,1,2,\dots.$$

If $\omega \geqslant BR$, then $\omega_j \geqslant BR_j$, $j = 1, 2, \ldots$. Since

$$\frac{\eta}{8} = \frac{1}{32}\frac{R^2}{\psi\Big(\dfrac{\omega}{2R}\Big)} = \frac{2}{(1-\xi)^2}\frac{R_1^2}{\psi\Big(\dfrac{\omega_1}{16R_1}\Big)} \geqslant \frac{R_1^2}{\psi\Big(\dfrac{\omega_1}{R_1}\Big)},$$

we have from (4.33) that $\operatorname{osc}\{u; Q_1\} \leqslant \omega_1$. Repeating the previous procedure, we obtain $\operatorname{osc}\{u; Q_j\} \leqslant \omega_j$, $j = 1, 2, \ldots$. Note that, by (4.5), the constant μ defined in (4.15) satisfies $\mu \geqslant 1$. Hence, $Q_j \supset \widetilde{Q}_j$, where $\widetilde{Q}_j := Q_{R_j, R_j^{1+\mu}\omega^{1-\mu}}(x_0, t_0)$. This proves the Hölder continuity of u (for details, we refer the reader to [34, Chap. III, Proposition 3.1]).

4.7. Proof of Theorem 4.1 in case (4.6)

Let $(x_0, t_0) \in \Omega_T$ be arbitrary. We fix number $R > 0$ so that

$$Q_R^{(M)}(x_0, t_0) := B_R(x_0) \times \Big(t_0 - \frac{2MR}{\varphi(s_0)}, t_0\Big) \subset \Omega_T$$

and set

$$\mu_+ := \sup_{Q_R^{(M)}(x_0,t_0)} u, \quad \mu_- := \inf_{Q_R^{(M)}(x_0,t_0)} u, \quad \omega = \mu_+ - \mu_-.$$

We also fix positive numbers $\delta \in (0,1)$, $B > s_0/\delta$ which will be specified later and depend only on K_1, n, c, c_1, c_2, c_3 and μ. If

$$\omega \geqslant BR, \tag{4.34}$$

then

$$\frac{R^2}{\psi\Big(\dfrac{\delta\omega}{R}\Big)} = \frac{\delta\omega R}{\varphi\Big(\dfrac{\delta\omega}{R}\Big)} \leqslant \frac{\delta\omega R}{\varphi(s_0)} \leqslant \frac{2MR}{\varphi(s_0)}$$

and

$$Q_{R,\theta}(x_0, t_0) \subset Q_R^{(M)}(x_0, t_0) \ \text{ for } \ \theta = \frac{R^2}{\psi\Big(\dfrac{\delta\omega}{R}\Big)}.$$

Further, we will assume that inequality (4.34) holds.

The following two alternative cases are possible:

$$|\{x \in B_R(x_0) : \mu_+ - u(x, t_0 - \theta) \leqslant \omega/2\}| \leqslant \frac{1}{2}|B_R(x_0)|$$

or

$$|\{x \in B_R(x_0) : u(x, t_0 - \theta) - \mu_- \leqslant \omega/2\}| \leqslant \frac{1}{2}|B_R(x_0)|.$$

Both of these cases can be considered in a similar way. Assume, for example, the first one. Then, by Lemma 4.1, there exists j_1 which depends only on K_1, n, c, c_1, c_2, c_3, and μ and is such that if $B \geqslant 2^{j_1} s_0$, then

$$|\{x \in B_R(x_0) : \mu_+ - u(x,t) \leqslant \omega/2^{j_1+2}\}| \leqslant \frac{3}{4}|B_R(x_0)| \quad \text{for all } \ t \in \Big(t_0 - \theta, t_0 - \theta + R^2/\psi\Big(\frac{\omega}{2^{j_1+2}R}\Big)\Big).$$

Choosing $\delta = 2^{-j_1-2}$, we can rewrite the previous inequality as

$$|\{x \in B_R(x_0) : \mu_+ - u(x,t) \leqslant \delta\omega\}| \leqslant \frac{3}{4}|B_R(x_0)| \tag{4.35}$$

for all $t \in (t_0 - \theta, t_0)$, $\theta = R^2/\psi(\delta\omega/R)$.

Now, for the function $v = \mu_+ - u$, we will use inequality (4.7) with $\sigma = 1/2$, $\rho = R$, $N = \delta\omega$, $k = \varepsilon_1^j N$, $j = 1, 2, \ldots$, where ε_1 is small enough and will be determined later. We set

$$A_j(t) := \{x \in B_R(x_0) : v(x,t) < \varepsilon_1^j N\},$$

$$Y_j(t) := \frac{1}{|B_R(x_0)|}\int_{A_j(t)} \frac{t - t_0 + \theta}{\theta}\,\zeta^{c_2}(x)dx, \quad y_j := \sup_{t_0 - \frac{\theta}{2} < t < t_0} Y_j(t).$$

Fix $t_1 \in (t_0 - \frac{\theta}{2}, t_0)$ such that $y_{j+1} = Y_j(t_1)$. First, we assume that

$$D^- \int\limits_{B_R(x_0)\times\{t_1\}} G_{\varepsilon_1^j N}(v)\,\frac{t_1 - t_0 + \theta}{\theta}\,\zeta^{c_2} dx \geqslant 0. \tag{4.36}$$

Let us estimate the first term on the right-hand side of (4.7). By (4.6), we have

$$\Phi(u) = \int\limits_0^u \varphi(s)ds \geqslant \frac{1}{2}u\varphi(u), \quad u > 0,$$

and

$$
\begin{aligned}
G_{\varepsilon_1^j N}(v) &= \int\limits_0^{\varepsilon_1^j N - v} \frac{(1+\varepsilon_1)\varepsilon_1^j N - s}{\Phi\Big(\dfrac{(1+\varepsilon_1)\varepsilon_1^j N - s}{R}\Big)}\, ds \\
&\leqslant 2R \int\limits_0^{\varepsilon_1^j N - v} \frac{ds}{\varphi\Big(\dfrac{(1+\varepsilon_1)\varepsilon_1^j N - s}{R}\Big)} \\
&\leqslant \frac{2R^2}{\psi\Big(\dfrac{\varepsilon_1^{1+j} N}{R}\Big)} \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+} \\
&\leqslant 2\theta \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+} .
\end{aligned}
$$

So, we estimate the first term on the right-hand side of (4.7) as follows:

$$
\begin{aligned}
&\frac{1}{\theta} \int\limits_{B_R(x_0)\times\{t_1\}} G_{\varepsilon_1^j N}(v)\, \zeta^{c_2} dx \\
&\leqslant 2 \int\limits_{B_R(x_0)\times\{t_1\}} \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+}\, \zeta^{c_2} dx \\
&\leqslant 4 \int\limits_{B_R(x_0)\times\{t_1\}} \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+}\, \frac{t_1 - t_0 + \theta}{\theta}\, \zeta^{c_2} dx.
\end{aligned}
$$

Choosing B so large that $B \geqslant \delta^{-1}\varepsilon_1^{-1-j}$, we obtain from (4.6), (4.7), (4.34), and (4.36) that

$$
\begin{aligned}
&\frac{R}{\varepsilon} \int\limits_{B_R(x_0)\times\{t_1\}} \left| \nabla \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+} \right| \frac{t_1 - t_0 + \theta}{\theta}\, \zeta^{c_2} dx \\
&\leqslant 4 \int\limits_{B_R(x_0)\times\{t_1\}} \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+}\, \frac{t_1 - t_0 + \theta}{\theta}\, \zeta^{c_2} dx \\
&\qquad + \gamma \varepsilon^{-\gamma} |B_R(x_0)|.
\end{aligned}
$$

From this, by (4.35) and the Poincaré inequality, we obtain

$$\begin{aligned}
\frac{\gamma^{-1}}{\varepsilon} \int\limits_{B_R(x_0)\times\{t_1\}} & \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+} \frac{t_1 - t_0 + \theta}{\theta} \zeta^{c_2} dx \\
&\leqslant 4 \int\limits_{B_R(x_0)\times\{t_1\}} \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+} \frac{t_1 - t_0 + \theta}{\theta} \zeta^{c_2} dx \\
&\qquad + \gamma \varepsilon^{-\gamma} |B_R(x_0)|.
\end{aligned}$$

Choosing ε to be small enough and such that $\gamma^{-1}/\varepsilon = 8$, we conclude from the previous inequality that

$$\int\limits_{B_R(x_0)\times\{t_1\}} \ln \frac{\varepsilon_1^j N(1+\varepsilon_1)}{\varepsilon_1^j N(1+\varepsilon_1) - (\varepsilon_1^j N - v)_+} \frac{t_1 - t_0 + \theta}{\theta} \zeta^{c_2} dx \leqslant \gamma |B_R(x_0)|.$$

If ε_1 is sufficiently small, this implies that

$$Y_{j+1}(t) \leqslant \gamma \left(\ln \frac{1+\varepsilon_1}{2\varepsilon_1} \right)^{-1} \leqslant \nu. \tag{4.37}$$

Assume now that

$$D^- \int\limits_{B_R(x_0)\times\{t_1\}} G_{\varepsilon_1^j N}(v) \frac{t_1 - t_0 + \theta}{\theta} \zeta^{c_2} dx < 0.$$

In this case, we define

$$t_* := \sup \left\{ \tau \in (t_0 - \frac{\theta}{2}, t_1) : D^- \int\limits_{B_R(x_0)\times\{\tau\}} G_{\varepsilon_1^j N}(v) \frac{\tau - t_0 + \theta}{\theta} \zeta^{c_2} dx \geqslant 0 \right\}$$

and note that this set is nonempty. From the definition of t_*, we have

$$\begin{aligned}
I(t_1) = & \int\limits_{B_R(x_0)\times\{t_1\}} G_{\varepsilon_1^j N}(v) \frac{t_1 - t_0 + \theta}{\theta} \zeta^{c_2} dx \\
\leqslant & \int\limits_{B_R(x_0)\times\{t_*\}} G_{\varepsilon_1^j N}(v) \frac{t_* - t_0 + \theta}{\theta} \zeta^{c_2} dx = I(t_*).
\end{aligned} \tag{4.38}$$

By Fubini's theorem, we conclude that

$$I(t_*) = \int\limits_{B_R(x_0)\times\{t_*\}} \frac{t_* - t_0 + \theta}{\theta}\,\zeta^{c_2} dx$$

$$\times \int\limits_0^{\varepsilon_1^j N} \frac{\chi_{\{\varepsilon_1^j N - v > s\}}((1+\varepsilon_1)\varepsilon_1^j N - s)}{\Phi\Big(\dfrac{(1+\varepsilon_1)\varepsilon_1^j N - s}{R}\Big)} ds$$

$$= \int\limits_0^{\varepsilon_1^j N} \frac{(1+\varepsilon_1)\varepsilon_1^j N - s}{\Phi\Big(\dfrac{(1+\varepsilon_1)\varepsilon_1^j N - s}{R}\Big)} \int\limits_{B_R(x_0)\times\{t_*\}} \frac{t_* - t_0 + \theta}{\theta}\,\zeta^{c_2}\chi_{\{\varepsilon_1^j N - v > s\}} dx ds.$$

Similarly to (4.37), we have

$$\int\limits_{B_R(x_0)\times\{t_*\}} \frac{t_* - t_0 + \theta}{\theta}\,\zeta^{c_2}\chi_{\{\varepsilon_1^j N - v > s\}} dx \leqslant \gamma\Big(\ln\frac{1+\varepsilon_1}{1+\varepsilon_1 - s}\Big)^{-1} |B_R(x_0)|.$$

So, choosing s_* by the condition

$$s_* = (1+\delta_*)\Big(1 - \exp\Big(-\frac{\gamma}{\nu(1-\delta_*)}\Big)\Big), \quad 0 < \varepsilon_1 < \delta_*$$

with $0 < \delta_* < \exp(-2\gamma/\nu)$ to be chosen later and assuming that $y_j \geqslant \nu$, we obtain

$$I(t_*) \leqslant y_j |B_R(x_0)| (\varepsilon_1^j N)^2$$

$$\times \Bigg(\int_0^{s_*} \frac{(1+\varepsilon_1 - s)\,ds}{\Phi\Big(\dfrac{\varepsilon_1^j N(1+\varepsilon_1 - s)}{R}\Big)} + (1-\delta_*)\int_{s_*}^1 \frac{(1+\varepsilon_1 - s)\,ds}{\Phi\Big(\dfrac{\varepsilon_1^j N(1+\varepsilon_1 - s)}{R}\Big)}\Bigg).$$

Set $f(s) = s/\Phi(s)$. We note that $f(\varepsilon_1^j N(s+\varepsilon_1)/R)$ is a decreasing function for $s \geqslant s_0$. In view of the inequality

$$\int_0^\sigma f\Big(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\Big) ds$$

$$= \sigma\int_0^1 f\Big(\frac{\varepsilon_1^j N(\sigma s+\varepsilon_1)}{R}\Big) ds \geqslant \sigma\int_0^1 f\Big(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\Big) ds \quad (4.39)$$

with $\sigma \in (0,1)$, we obtain

$$\begin{aligned} I(t_*) &\leqslant y_j\,|B_R(x_0)|\,\varepsilon_1^j N \\ &\quad\times\left\{\int\limits_0^1 f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds - \frac{1}{2}\int\limits_0^{1-s_*} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds\right\} \\ &\leqslant y_j\,|B_R(x_0)|\,\varepsilon_1^j N\left(1-\frac{1-s_*}{2}\right)\int\limits_0^1 f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds. \end{aligned} \tag{4.40}$$

Now, we estimate the left-hand side of inequality (4.38). For $v < \varepsilon_1^{j+1}N$, we have

$$\begin{aligned} G_{\varepsilon_1^j N}(v) &= \int\limits_0^{(\varepsilon_1^j N - v)_+} \frac{(1+\varepsilon_1)\varepsilon_1^j N - s}{\Phi\left(\dfrac{(1+\varepsilon_1)\varepsilon_1^j N - s}{R}\right)}\,ds \\ &\geqslant \int\limits_0^{\varepsilon_1^j N(1-\varepsilon_1)} \frac{(1+\varepsilon_1)\varepsilon_1^j N - s}{\Phi\left(\dfrac{(1+\varepsilon_1)\varepsilon_1^j N - s}{R}\right)}\,ds = \varepsilon_1^j N R\int\limits_{\varepsilon_1}^1 f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds \\ &= \varepsilon_1^j N R\left\{\int\limits_0^1 f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds - \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds\right\}. \end{aligned}$$

So, we need to obtain the upper bound for the last integral on the right-hand side of the previous inequality.

Choose L from the condition $2^{-L-1} \leqslant \varepsilon_1 < 2^{-L}$. Then we have

$$\begin{aligned} \int\limits_0^1 f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds &\geqslant \int\limits_0^{2^L\varepsilon_1} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds \\ &= \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds + \sum_{l=0}^{L-1}\int\limits_{2^l\varepsilon_1}^{2^{l+1}\varepsilon_1} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right)ds, \end{aligned}$$

which yields

$$\int\limits_0^1 f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right) ds - \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right) ds$$

$$= \sum_{l=0}^{L-1} \int\limits_{2^l\varepsilon_1}^{2^{l+1}\varepsilon_1} f\left(\frac{\varepsilon_1^j N(s+\varepsilon_1)}{R}\right) ds = \sum_{l=0}^{L-1} 2^l \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^j N(\varepsilon_1 + 2^l(s+\varepsilon_1)}{R}\right) ds$$

$$\geqslant \sum_{l=0}^{L-1} \frac{2^l}{2^l+1} \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^{jN}(\varepsilon_1+s)}{R}\right) ds \geqslant \frac{L}{2} \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^{jN}(\varepsilon_1+s)}{R}\right) ds$$

$$\geqslant \frac{1}{2}\log\frac{1}{2\varepsilon_1} \int\limits_0^{\varepsilon_1} f\left(\frac{\varepsilon_1^j N(\varepsilon_1+s)}{R}\right) ds.$$

From this, we obtain

$$G_{\varepsilon_1^j N}(v) \geqslant \varepsilon_1^j NR\big(1 - K(\varepsilon_1)\big) \int\limits_0^1 f\left(\frac{\varepsilon_1^j N(\varepsilon_1+s)}{R}\right) ds, \tag{4.41}$$

where $K(\varepsilon_1) = \left(1 + \frac{1}{2}\log\frac{1}{2\varepsilon_1}\right)^{-1}$.

Combining (4.38)–(4.41), we arrive at

$$Y_{j+1}(t) \leqslant \frac{1-(1-s_*)/2}{1-K(\varepsilon_1)}\, y_j = \sigma_* y_j\,.$$

We choose ε_1 to be small enough and such that $K(\varepsilon_1) < (1-s_*)/2$ and get

$$y_{j+1} \leqslant \max\{\nu, \sigma_* y_j\}, \quad j = 1, 2, \dots .$$

The method of induction implies that

$$y_j \leqslant \max\{\nu, \sigma_*^{j+1} y_1\}, \quad j = 1, 2, \dots .$$

So, there is a positive number j_* which is determined by ν, n, K_1, c, c_1, c_2, c_3 and is such that $y_{j_*} \leqslant 2\nu$.

By definition of y_j, for all $t \in (t_0 - \theta/2, t_0)$, we have

$$|\{x \in B_{R/2}(x_0) : \mu_+ - u(x,t) \leqslant \varepsilon_1^{j_*} 2^{-j_1-2}\omega\}| \leqslant 2^{n+1}\nu.$$

By Theorem 4.2, we arrive at

$$\operatorname{osc}\{u; Q_{R/4,\theta_*/4}(x_0,t_0)\} \leqslant (1-\xi)\omega, \tag{4.42}$$

where

$$\theta_* = R^2/\psi\Big(\frac{\xi\omega}{R}\Big), \quad \xi = \varepsilon_1^{j_*} 2^{-j_1-3}.$$

Define the sequences $R_j := \sigma^j(1-\xi)^j R$, $\omega_j := (1-\xi)^j\omega$,

$$Q_j := Q_{R_j,\theta_j}(x_0,t_0), \quad \theta_j = R_j^2/\psi\Big(\frac{\omega_j}{R_j}\Big), \quad j = 0,1,2\dots.$$

If $\omega \geqslant BR$, then $\omega_j \geqslant BR_j$, $j = 1,2,\dots$. We note that

$$\frac{\theta_*}{4} = \frac{\xi\omega R}{4g\Big(\dfrac{\xi\omega}{R}\Big)} = \frac{\xi}{4(1-\xi)^2\sigma}\,\frac{\omega_1 R_1}{g\Big(\dfrac{\xi\sigma\omega_1}{R_1}\Big)} \geqslant \frac{\xi}{4\sigma}\,\frac{\omega_1 R_1}{g\Big(\dfrac{\omega_1}{R_1}\Big)} = \frac{R_1^2}{\psi\Big(\dfrac{\omega_1}{R_1}\Big)}.$$

If σ is chosen so that $\sigma = \xi/4$, then inequality (4.42) yields

$$\operatorname{osc}\{u; Q_1\} \leqslant \omega_1.$$

Repeating the previous procedure, we obtain

$$\operatorname{osc}\{u; Q_j\} \leqslant \omega_j, \quad j = 1,2,\dots.$$

Note that, by our assumptions, $\psi(\omega_j/R_j) \leqslant \psi(s_0)$. Hence, $Q_j \supset \widetilde{Q}_j$, where $\widetilde{Q}_j = Q_{R_j,R_j^2/\psi(s_0)}(x_0,t_0)$. This proves the Hölder continuity of u (for details, we refer the reader to [34, Chap. III, Proposition 3.1]).

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