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On compressions of self-adjoint extensions of a symmetric linear relation with unequal deficiency indices

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(Presented by V. A. Derkach)

Abstract. Let A be a symmetric linear relation in the Hilbert space $\mathfrak{H}$ with unequal deficiency indices $n_-A < n_+(A)$. A self-adjoint linear relation $\widetilde{A} \supset A$ in some Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ is called an (exit space) extension of A. We study the compressions $C(\widetilde{A}) = P_{\mathfrak{H}}\widetilde{A} \upharpoonright \mathfrak{H}$ of extensions $\widetilde{A} = \widetilde{A}^*$. Our main result is a description of compressions $C(\widetilde{A})$ by means of abstract boundary conditions, which are given in terms of limit value of the Nevanlinna parameter $\tau(\lambda)$ from the Krein formula for generalized resolvents. We describe also all extensions $\widetilde{A} = \widetilde{A}^*$ of A with the maximal symmetric compression $C(\widetilde{A})$ and all extensions $\widetilde{A} = \widetilde{A}^*$ of the second kind in the sense of M.A. Naimark. These results generalize the recent results by A. Dijksma, H. Langer and the author obtained for symmetric operators A with equal deficiency indices $n_+(A) = n_-(A)$.

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1. Introduction

Assume that A is a closed not necessarily densely defined symmetric operator in a Hilbert space $\mathfrak{H}$. Recall that a self-adjoint linear relation (in particular operator) $\widetilde{A} \supset A$ in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ is called an (exit space) extension of A and a linear relation $C(\widetilde{A}) := P_{\mathfrak{H}}\widetilde{A} \upharpoonright \mathfrak{H}$ is called a compression of $\widetilde{A}$. A description of all extensions $\widetilde{A} = \widetilde{A}^*$ and their compressions $C(\widetilde{A})$ is an important problem in the extension theory of symmetric operators (note that $C(\widetilde{A})$ is a symmetric extension of A). In [9,20,21] all extensions $\widetilde{A} = \widetilde{A}^*$ of an operator A with arbitrary (equal or

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unequal) deficiency indices $n_\pm(A) \le \infty$ and their compressions $C(\widetilde{A})$ were described by means of holomorphic operator-functions $F(\lambda)(\lambda \in \mathbb{C}_+)$, whose values are contractions between defect subspaces of A. In the case $n_+(A) = n_-(A)$ another description of extensions $\widetilde{A} = \widetilde{A}^*$ of A is given by the Krein formula for generalized resolvents [11,12]. This formula gives a parametrization $\widetilde{A} = \widetilde{A}_\tau$ of all extensions $\widetilde{A} = \widetilde{A}^*$ by means of Nevanlinna functions $\tau = \tau(\lambda)$, whose values are linear relations in the auxiliary Hilbert space. In the recent papers by A. Dijksma and H. Langer [7,8] the compressions $C(\widetilde{A}_\tau)$ of extensions $\widetilde{A}_\tau$ are investigated in terms of the parameter τ from the Krein formula. The results of [7,8] were essentially strengthened in our paper [18]. The investigations in this paper are based on the theory of boundary triplets for symmetric operators A with equal deficiency indices $n_+(A) = n_-(A)$ and Weyl functions of these triplets (see [5,6,10,13] and references therein). By using such an approach we described in [18] the compressions $C(\widetilde{A}_\tau)$ in terms of the parameter τ. This enables us to describe, in particular, all extensions $\widetilde{A}_\tau$ with self-adjoint compressions.

In our papers [15,16] the theory of boundary triplets and their Weyl functions was extended to symmetric operators A with unequal deficiency indices $n_-(A) < n_+(A)$. In particular, we showed that in this case the Krein formula for generalized resolvents

$$P_{\mathfrak{H}}(\widetilde{A}_\tau - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)(\tau(\lambda) + M_+(\lambda))^{-1}\gamma_-^*(\overline{\lambda}), \ \lambda \in \mathbb{C}_+ \tag{1.1}$$

establishes a bijective correspondence $\widetilde{A} = \widetilde{A}_\tau$ between all Nevanlinna type functions $\tau = \tau(\lambda)$ and all extensions $\widetilde{A} = \widetilde{A}^*$ of A. In (1.1) A_0 is a fixed maximal symmetric extension of A and $\gamma_\pm(\lambda)$ (the γ-fields) and $M_+(\lambda)$ (the Weyl function) are the operator functions defined in terms of a boundary triplet for A. In the present paper we extend the results of [18] to symmetric operators A with unequal deficiency indices $n_-(A) < n_+(A)$ (clearly, in this case $n_-(A) < \infty$ and $n_+(A) \le \infty$). Our main result (see Theorem 3.5) is a description of compressions $C(\widetilde{A}_\tau)$ of extensions $\widetilde{A}_\tau = \widetilde{A}_\tau^*$ in terms of the parameter $\tau = \tau(\lambda)$ from (1.1). This description is given by means of an abstract boundary parameter θ_c, which is a certain limit value of $\tau(\lambda)$ at infinity. By using this result we describe extensions $\widetilde{A}_\tau$ with some special properties. In particular, we describe in terms of τ all extensions $\widetilde{A}_\tau$ of the second kind in the sense of M. A. Naimark (see Remark 3.7) and all extensions $\widetilde{A}_\tau$ with the maximal symmetric compression $C(\widetilde{A}_\tau)$.

2. Preliminaries

2.1 Notations

The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote separable Hilbert spaces; $\boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $\boldsymbol{B}(\mathcal{H}) := \boldsymbol{B}(\mathcal{H}, \mathcal{H})$; $A \restriction \mathcal{L}$ is a restriction of the operator $A \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ to the linear manifold $\mathcal{L} \subset \mathcal{H}_1$; $P_{\mathcal{L}}$ is the orthoprojection in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the open upper (lower) half-plane of the complex plane.

Recall that a linear manifold T in the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$ ($\mathcal{H} \oplus \mathcal{H}$) is called a linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ (resp. in $\mathcal{H}$). The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ (resp. $\widetilde{\mathcal{C}}(\mathcal{H})$). Clearly for each linear operator $T : \operatorname{dom} T \to \mathcal{H}_1$, $\operatorname{dom} T \subset \mathcal{H}_0$, its graph $\operatorname{gr} T = \{\{f, Tf\} : f \in \operatorname{dom} T\}$ is a linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$. This fact enables one to consider an operator as a linear relation.

For a linear relation T from $\mathcal{H}_0$ to $\mathcal{H}_1$ we denote by

$$\operatorname{dom} T := \{h_0 \in \mathcal{H}_0 : \exists h_1 \in \mathcal{H}_1 \;\; \{h_0, h_1\} \in T\}$$
$$\ker T := \{h_0 \in \mathcal{H}_0 : \{h_0, 0\} \in T\}$$
$$\operatorname{ran} T := \{h_1 \in \mathcal{H}_1 : \exists h_0 \in \mathcal{H}_0 \;\; \{h_0, h_1\} \in T\}$$
$$\operatorname{mul} T := \{h_1 \in \mathcal{H}_1 : \{0, h_1\} \in T\}$$

the domain, kernel, range and multivalued part of T respectively. Denote also by T^{-1} and T^* the inverse and adjoint linear relations of T respectively.

As is known a linear relation T in $\mathcal{H}$ is called symmetric (self-adjoint) if $T \subset T^*$ (resp. $T = T^*$).

2.2 Nevanlinna functions

Recall that a holomorphic operator function $M : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H})$ is called a Nevanlinna function if $\operatorname{Im} M(\lambda) \geq 0, \;\; \lambda \in \mathbb{C}_+$. The class of all Nevanlinna $\boldsymbol{B}(\mathcal{H})$-valued functions will be denoted by $R[\mathcal{H}]$. The operator-function $M \in R[\mathcal{H}]$ is referred to the class $R_c[\mathcal{H}]$, if $\operatorname{ran} \operatorname{Im} M(\lambda)$ is closed for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The following proposition is well known (see e.g. [13]).

Proposition 2.1. *If $M \in R[\mathcal{H}]$, then the equality*

$$\mathcal{B}_M = s - \lim_{y \to +\infty} \tfrac{1}{iy} M(iy) \tag{2.1}$$

defines the operator $\mathcal{B}_M \in \boldsymbol{B}(\mathcal{H})$ *such that* $\mathcal{B}_M \geq 0$*. Moreover, for each* $h \in \mathcal{H}$ *there exists the limit* $\lim\limits_{y\to+\infty} y\mathrm{Im}(M(iy)h,h) \leq \infty$ *and the equality*

$$\mathrm{dom}\,\mathcal{N}_M = \{h \in \mathcal{H} : \lim_{y\to+\infty} y\mathrm{Im}(M(iy)h,h) < \infty\} \tag{2.2}$$

defines the (not necessarily closed) linear manifold $\mathrm{dom}\,\mathcal{N}_M \subset \mathcal{H}$ *such that for each* $h \in \mathrm{dom}\,\mathcal{N}_M$ *there exists the limit*

$$\mathcal{N}_M h := \lim_{y\to+\infty} M(iy)h, \quad h \in \mathrm{dom}\,\mathcal{N}_M. \tag{2.3}$$

Hence the equalities (2.2) *and* (2.3) *define the linear operator* $\mathcal{N}_M : \mathrm{dom}\,\mathcal{N}_M \to \mathcal{H}$*.*

2.3 The classes $\mathrm{Sym}(\mathcal{H}_0,\mathcal{H}_1)$ and $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$

In the following $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$ and P_j is the orthoprojection in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1,2\}$.

Definition 2.2. [14] A linear relation θ from $\mathcal{H}_0$ to $\mathcal{H}_1$ belongs to the class $\mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$ if

$$2\mathrm{Im}(h_1,h_0)_{\mathcal{H}_0} + ||P_2h_0||^2 = 0, \quad \{h_0,h_1\} \in \theta. \tag{2.4}$$

A relation $\theta \in \mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$ belongs to the class $\mathrm{Sym}(\mathcal{H}_0,\mathcal{H}_1)$ if there is not an extension $\widetilde{\theta} \supset \theta$, $\widetilde{\theta} \neq \theta$ such that $\widetilde{\theta} \in \mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$.

Note that in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ the classes $\mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$ and $\mathrm{Sym}(\mathcal{H}_0,\mathcal{H}_1)$ coincide with the known classes of symmetric and maximal symmetric linear relations in $\mathcal{H}$ respectively.

Let $\theta \in \mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$, let $\mathcal{K} := \mathrm{mul}\,\theta$ be a closed subspace in $\mathcal{H}_1$ and let $\mathcal{H}_1' := \mathcal{H}_1 \ominus \mathcal{K}$ and $\mathcal{H}_0' := \mathcal{H}_0 \ominus \mathcal{K}$. Then $\mathcal{H}_0' = \mathcal{H}_1' \oplus \mathcal{H}_2$,

$$\mathcal{H}_1 = \mathcal{H}_1' \oplus \mathcal{K}, \qquad \mathcal{H}_0 = \mathcal{H}_0' \oplus \mathcal{K} = \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'} \oplus \mathcal{K} \tag{2.5}$$

and according to [14]

$$\theta = \mathrm{gr}\,\theta_s \oplus \widehat{\mathcal{K}} = \{\{h_0', \theta_s h_0' \oplus k\} : h_0' \in \mathrm{dom}\,\theta_s, k \in \mathcal{K}\}, \tag{2.6}$$

where $\widehat{\mathcal{K}} = \{0\} \oplus \mathcal{K}$ and $\theta_s \in \mathrm{Sym}_0(\mathcal{H}_0',\mathcal{H}_1')$ is an operator with $\mathrm{dom}\,\theta_s = \mathrm{dom}\,\theta$. Moreover, $\theta \in \mathrm{Sym}(\mathcal{H}_0,\mathcal{H}_1)$ if and only if $\theta_s \in \mathrm{Sym}(\mathcal{H}_0',\mathcal{H}_1')$. The operator θ_s in (2.6) is called the operator part of θ.

It follows from (2.5) and (2.6) that

$$P_1\mathrm{dom}\,\theta \subset \mathcal{H}_1 \ominus \mathrm{mul}\,\theta. \tag{2.7}$$

Lemma 2.3. *Let* $\dim \mathcal{H}_1 < \infty$ *and let* $\theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$*, so that decompositions* (2.5) *and* (2.6) *hold with* $\mathcal{K} = \mathrm{mul}\,\theta$*. Then there exist a subspace* $L' \subset \mathcal{H}_1'$ *and operators* $Q_1 \in \boldsymbol{B}(L', \mathcal{H}_1')$ *and* $Q_2 \in \boldsymbol{B}(L', \mathcal{H}_2)$ *such that*

$$\theta = \{\{h' \oplus Q_2 h', Q_1 h' \oplus k\} : h' \in L', k \in \mathcal{K}\}. \tag{2.8}$$

Moreover, $\theta \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ *if and only if* $L' = \mathcal{H}_1'$*.*

Proof. Since $\mathrm{dom}\,\theta_s \subset \mathcal{H}_1' \oplus \mathcal{H}_2$, it follows that $\mathrm{dom}\,\theta_s$ is a linear relation from $\mathcal{H}_1'$ to $\mathcal{H}_2$. Let $L' \subset \mathcal{H}_1'$ be the domain of this relation. Assume that $0 \oplus h_2 \in \mathrm{dom}\,\theta_s$ with some $h_2 \in \mathcal{H}_2$. Then $\{0 \oplus h_2, h_1'\} \in \theta_s$ with some $h_1' \in \mathcal{H}_1'$ and by equality (2.4) for θ_s one has $||h_2||^2 = 0$. Hence $h_2 = 0$ and consequently there exists an operator $Q_2 \in \boldsymbol{B}(L', \mathcal{H}_2)$ such that $\mathrm{dom}\,\theta_s = \{\{h' \oplus Q_2 h'\} : h' \in L'\}$. Moreover, the equality

$$Q_1 h' = \theta_s(h' \oplus Q_2 h'), \quad h' \in L'$$

correctly defines the operator $Q_1 \in \boldsymbol{B}(L', \mathcal{H}_1')$ such that

$$\mathrm{gr}\,\theta_s = \{\{h' \oplus Q_2 h', Q_1 h'\} : h' \in L'\}.$$

This and (2.6) imply (2.8).

Next according to [17, Proposition 2.7] the operator θ_s belongs to $\mathrm{Sym}\,(\mathcal{H}_0', \mathcal{H}_1')$ if and only if $\dim(\mathrm{gr}\,\theta_s) = \dim \mathcal{H}_1'$. This and the obvious equality $\dim L' = \dim(\mathrm{gr}\,\theta_s)$ yield the last statement of the theorem. □

Definition 2.4. [14,16] A function $\tau : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is referred to the class $\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ if:

(i) $2\mathrm{Im}(h_1, h_0) - ||P_2 h_0||^2 \geq 0, \ \{h_0, h_1\} \in \tau(\lambda), \ \lambda \in \mathbb{C}_+$;

(ii) $(\tau(\lambda) + iP_1)^{-1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0), \ \lambda \in \mathbb{C}_+$, and the operator-function $(\tau(\lambda) + iP_1)^{-1}$ is holomorphic on $\mathbb{C}_+$.

A function $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ is referred to the class $R(\mathcal{H}_0, \mathcal{H}_1)$ if its values are operators, i.e., if $\mathrm{mul}\,\tau(\lambda) = \{0\}, \ \lambda \in \mathbb{C}_+$

According to [14,16] the equality

$$\tau(\lambda) = \{\{K_0(\lambda)h, K_1(\lambda)h\} : h \in \mathcal{H}_1\}, \ \ \lambda \in \mathbb{C}_+$$

establishes a bijective correspondence between all functions $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and all pairs $\{K_0, K_1\}$ of holomorphic operator-functions $K_j : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_j), \ j \in \{0, 1\}$, with the block representation

$$K_0(\lambda) = (K_{01}(\lambda), K_{02}(\lambda))^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{2.9}$$

satisfying for all $\lambda \in \mathbb{C}_+$ the following relations:

$$2\,\mathrm{Im}(K_{01}^*(\lambda)K_1(\lambda)) - K_{02}^*(\lambda)K_{02}(\lambda) \geq 0, \ (K_1(\lambda) + iK_{01}(\lambda))^{-1} \in \boldsymbol{B}(\mathcal{H}_1). \tag{2.10}$$

In the following we write $\tau = \{K_0, K_1\}$ identifying a function $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and the corresponding pair $\{K_0, K_1\}$ of holomorphic operator functions satisfying (2.10)(more precisely the equivalence class of such pairs [14]).

Lemma 2.5. [14, 16] *Let $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$. Then the multivalued part $\mathcal{K} := \mathrm{mul}\,\tau(\lambda)(\subset \mathcal{H}_1)$ of $\tau(\lambda)$ does not depend on $\lambda \in \mathbb{C}_+$. Moreover, decompositions* (2.5) *and*

$$\tau(\lambda) = \mathrm{gr}\,\tau_s(\lambda) \oplus \widehat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_+ \tag{2.11}$$

hold with $\tau_s \in R(\mathcal{H}_0', \mathcal{H}_1')$ and $\widehat{\mathcal{K}} = \{0\} \oplus \mathcal{K}$.

The operator function τ_s in (2.11) is called the operator part of τ.

Remark 2.6. *In the case $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$ the class $\widetilde{R}(\mathcal{H}, \mathcal{H})$ coincides with the well-known class $\widetilde{R}(\mathcal{H})$ of Nevanlinna $\widetilde{\mathcal{C}}(\mathcal{H})$-valued functions (Nevanlinna operator pairs) $\tau = \{K_0(\lambda), K_1(\lambda)\}$, $\lambda \in \mathbb{C}_+$ (see e.g [3]). Denote by $R(\mathcal{H})$ the set of all $\tau \in \widetilde{R}(\mathcal{H})$ such that $\tau(\lambda)$ is an operator, $\lambda \in \mathbb{C}_+$. For a function $\tau \in \widetilde{R}(\mathcal{H})$ decompositions* (2.5) *and* (2.11) *take the following well known form (see e.g. [11]):*

$$\mathcal{H} = \mathcal{H}' \oplus \mathcal{K}, \qquad \tau(\lambda) = \mathrm{gr}\,\tau_s(\lambda) \oplus \widehat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_+, \tag{2.12}$$

where $\tau_s \in R(\mathcal{H}')$ is the operator part of τ.

It is clear that $R[\mathcal{H}] \subset R(\mathcal{H}) \subset \widetilde{R}(\mathcal{H})$.

Let decompositions (2.5) hold and let $Q_1(\lambda)(\in \boldsymbol{B}(\mathcal{H}_1'))$ and $Q_2(\lambda)(\in \boldsymbol{B}(\mathcal{H}_1', \mathcal{H}_2))$ be holomorphic on $\mathbb{C}_+$ operator functions.

Definition 2.7. For a function $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ we write $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\}$ if

$$\tau(\lambda) = \{\{h_1' \oplus Q_2(\lambda)h_1', Q_1(\lambda)h_1' \oplus k\} : h_1' \in \mathcal{H}_1', k \in \mathcal{K}\}, \quad \lambda \in \mathbb{C}_+ \tag{2.13}$$

If $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\}$, then $\mathcal{K} = \mathrm{mul}\,\tau(\lambda)$, $\lambda \in \mathbb{C}_+$, and in view of the inequality

$$2\mathrm{Im}Q_1(\lambda) - Q_2^*(\lambda)Q_2(\lambda) \geq 0, \quad \lambda \in \mathbb{C}_+ \tag{2.14}$$

one has $Q_1 \in R[\mathcal{H}_1']$.

Proposition 2.8. *In the case* $\dim \mathcal{H}_1 < \infty$ *each function* $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ *admits the representation* $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\}$.

Proof. Let $\tau_s = \{\widetilde{Q}_0, \widetilde{Q}_1\}$ with operator-functions $\widetilde{Q}_j : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H}_1', \mathcal{H}_j')$, $j \in \{0,1\}$, and let

$$\widetilde{Q}_0(\lambda) = (\widetilde{Q}_{01}(\lambda), \widetilde{Q}_{02}(\lambda))^\top : \mathcal{H}_1' \to \mathcal{H}_1' \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+$$

be the block representation of $\widetilde{Q}_0(\lambda)$. Since $\tau_s \in R(\mathcal{H}_0', \mathcal{H}_1')$, it follows that

$$2\mathrm{Im}(\widetilde{Q}_1(\lambda)h_1', \widetilde{Q}_{01}(\lambda)h_1') - ||\widetilde{Q}_{02}(\lambda)h_1'||^2 \geq 0, \quad \lambda \in \mathbb{C}_+, \ h_1' \in \mathcal{H}_1'.$$

Therefore for each $h_1' \in \ker \widetilde{Q}_{01}(\lambda)$ one has $h_1' \in \ker Q_{02}(\lambda)$. Hence $h_1' \in \ker \widetilde{Q}_0(\lambda)$, which implies that $\ker \widetilde{Q}_{01}(\lambda) \subset \ker \widetilde{Q}_0(\lambda)$. Since $\tau_s(\lambda)$ is an operator, it follows that $\ker \widetilde{Q}_0(\lambda) = \{0\}$ and, consequently, $\ker \widetilde{Q}_{01}(\lambda) = \{0\}$. Since $\dim \mathcal{H}_1' < \infty$, this implies that the operator $\widetilde{Q}_{01}(\lambda)$ is invertible, that is $\widetilde{Q}_{01}^{-1} : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H}_1')$ is a holomorphic operator function. Clearly, τ_s admits the representation $\tau_s = \{Q_0, Q_1\}$ with

$$Q_0(\lambda) = \widetilde{Q}_0(\lambda)\widetilde{Q}_{01}^{-1}(\lambda) = (I_{\mathcal{H}_1'}, Q_2(\lambda))^\top, \qquad Q_1(\lambda) = \widetilde{Q}_1(\lambda)\widetilde{Q}_{01}^{-1}(\lambda),$$

where $Q_2(\lambda) = \widetilde{Q}_{02}(\lambda)\widetilde{Q}_{01}^{-1}(\lambda)$. Hence

$$\mathrm{gr}\, \tau_s(\lambda) = \{\{h_1' \oplus Q_2(\lambda)h_1', Q_1(\lambda)h_1'\} : h_1' \in \mathcal{H}_1'\}, \quad \lambda \in \mathbb{C}_+,$$

which in view of (2.11) yields (2.13). □

Proposition 2.9. *Let* $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, *let* $\widetilde{L}_\infty (\subset \mathcal{H}_1')$ *be a linear manifold of all* $h \in \mathcal{H}_1'$ *such that there exists the limit* $\lim\limits_{y \to +\infty} Q_2(iy)h$ *and let* $Q_2(\infty) : \widetilde{L}_\infty \to \mathcal{H}_2$ *be the linear operator given by*

$$Q_2(\infty)h = \lim_{y \to +\infty} Q_2(iy)h, \quad h \in \widetilde{L}_\infty.$$

For $h \in \widetilde{L}_\infty$ *put*

$$\varphi_h(y) = \mathrm{Im}(Q_1(iy)h, h) - \mathrm{Re}\,(Q_2(iy)h, Q_2(\infty)h) + \tfrac{1}{2}||Q_2(\infty)h||^2, \ y \in \mathbb{R}_+. \quad (2.15)$$

Then for each $h \in \widetilde{L}_\infty$ *there exists the limit* $\lim\limits_{y \to +\infty} y\varphi_h(y) \leq \infty$ *and the equality*

$$L_\infty = \{h \in \widetilde{L}_\infty : \lim_{y \to +\infty} y\varphi_h(y) < \infty\} \quad (2.16)$$

defines the linear manifold $L_\infty \subset \mathcal{H}_1'$ *such that for each* $h \in L_\infty$ *there exists the limit*

$$Q_1(\infty)h = \lim_{y\to+\infty} Q_1(iy)h, \quad h \in L_\infty. \tag{2.17}$$

Thus the equalities (2.16) *and* (2.17) *define the linear operator* $Q_1(\infty) : L_\infty \to \mathcal{H}_1'$.

Proof. Let

$$\tau_{es}(\lambda) = \begin{pmatrix} Q_1(\lambda) & 0 \\ -iQ_2(\lambda) & \frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'} \to \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'}, \quad \lambda \in \mathbb{C}_+. \tag{2.18}$$

Then

$$\mathrm{Im}\tau_{es}(\lambda) = \begin{pmatrix} \mathrm{Im}Q_1(\lambda) & -\frac{1}{2}Q_2^*(\lambda) \\ -\frac{1}{2}Q_2(\lambda) & \frac{1}{2}I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_2 \to \mathcal{H}_1' \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+. \tag{2.19}$$

and by (2.14) $\mathrm{Im}\tau_{es}(\lambda) \geq 0$, $\lambda \in \mathbb{C}_+$. Therefore $\tau_{es} \in R[\mathcal{H}_0']$. Next, the immediate calculations show that for each $h \in \widetilde{L}_\infty$

$$\mathrm{Im}(\tau_{es}(iy)(h \oplus Q_2(\infty)h), h \oplus Q_2(\infty)h) = \varphi_h(y). \tag{2.20}$$

Therefore by Proposition 2.1 for each $h \in L_\infty$ there exists the limit $\lim\limits_{y\to+\infty} \tau_{es}(iy)(h \oplus Q_2(\infty)h)$. Since

$$\tau_{es}(iy)(h \oplus Q_2(\infty)h) = Q_1(iy)h \oplus (-iQ_2(iy)h + \tfrac{i}{2}Q_2(\infty)h),$$

this implies that there exists the limit in (2.17). □

2.4 Boundary triplets

In the following we denote by A a closed symmetric linear relation (in particular closed not necessarily densely defined symmetric operator) in a Hilbert space $\mathfrak{H}$. Let $\mathfrak{N}_\lambda(A) = \ker(A^* - \lambda)$ ($\lambda \in \mathbb{C} \setminus \mathbb{R}$) be a defect subspace of A, let $\widehat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$ and let $n_\pm(A) := \dim \mathfrak{N}_\lambda(A) \leq \infty$, $\lambda \in \mathbb{C}_\pm$, be deficiency indices of A. Denote by $\mathrm{ext}(A)$ the set of all proper extensions of A (i.e., the set of all relations $\widetilde{A}$ in $\mathfrak{H}$ such that $A \subset \widetilde{A} \subset A^*$) and by $\overline{\mathrm{ext}}(A)$ the set of closed extensions $\widetilde{A} \in \mathrm{ext}(A)$. Clearly, each symmetric extension $\widetilde{A}$ of A belongs to $\mathrm{ext}(A)$.

As before we assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$ and $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, so that $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. Moreover, denote by P_j the orthoprojections in $\mathcal{H}_0$ and $\mathcal{H}_j,\ j \in 1, 2$.

Below within this subsection we specify some definitions and results from [15, 16].

Definition 2.10. A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : A^* \to \mathcal{H}_j,\ j \in \{0, 1\}$, are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma : \widehat{f} \to \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}, \widehat{f} \in A^*$, from A^* into $\mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green's identity holds for all $\widehat{f} = \{f, f'\},\ \widehat{g} = \{g, g'\} \in A^*$:

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}_0} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}_0} + i(P_2 \Gamma_0 \widehat{f}, P_2 \Gamma_0 \widehat{g})_{\mathcal{H}_2} \tag{2.21}$$

In the following propositions some properties of boundary triplets are specified.

Proposition 2.11. *If $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, then*

$$\dim \mathcal{H}_1 = n_-(A) \leq n_+(A) = \dim \mathcal{H}_0. \tag{2.22}$$

Conversely, let A be a symmetric relation with $n_-(A) \leq n_+(A)$. Then for any Hilbert space $\mathcal{H}_0$ and a subspace $\mathcal{H}_1 \subset \mathcal{H}_0$ satisfying (2.22) *there exists a boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^*.*

Proposition 2.12. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then:*

(1) $\ker \Gamma_0 \cap \ker \Gamma_1 = A$ *and Γ_j is a bounded operator from A^* onto $\mathcal{H}_j,\ j \in \{0, 1\}$.*

(2) *The equality $A_0 := \ker \Gamma_0 = \{\hat{f} \in A^* : \Gamma_0 \hat{f} = 0\}$ define a maximal symmetric extension A_0 of A such that $n_-(A_0) = 0$.*

(3) *The equality*

$$A_\theta = \{\widehat{f} \in A^* : \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \theta\}$$

gives a bijective correspondence $\widetilde{A} = A_\theta$ between all linear relations θ from $\mathcal{H}_0$ to $\mathcal{H}_1$ and all extensions $\widetilde{A} \in \mathrm{ext}(A)$. Moreover, A_θ is symmetric (maximal symmetric) if and only if $\theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ (resp. $\theta \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$).

If $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, then the equalities

$$\gamma_+(\lambda) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \ \ \lambda \in \mathbb{C}_+;$$
$$\gamma_-(\lambda) = \pi_1(P_1\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \ \ \lambda \in \mathbb{C}_-$$
$$\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A) = M_+(\lambda)\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A), \quad \lambda \in \mathbb{C}_+$$

correctly define the holomorphic operator functions $\gamma_+ : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H}_0, \mathfrak{H})$, $\gamma_- : \mathbb{C}_- \to \boldsymbol{B}(\mathcal{H}_1, \mathfrak{H})$ and $M_+ : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H}_0, \mathcal{H}_1)$ (here π_1 is the orthoprojection in $\mathfrak{H} \oplus \mathfrak{H}$ onto $\mathfrak{H} \oplus \{0\}$). The operator-functions $\gamma_\pm$ and M_+ are called the γ-fields and the Weyl function of the triplet Π respectively.

2.5 Self-adjoint extensions and their compressions

As is known a linear relation $\widetilde{A} = \widetilde{A}^*$ in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ is called an exit space extension of A if $A \subset \widetilde{A}$ and the minimality condition $\overline{\operatorname{span}}\{\mathfrak{H}, (\widetilde{A} - \lambda)^{-1}\mathfrak{H} : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \widetilde{\mathfrak{H}}$ is satisfied. For an exit space extension $\widetilde{A} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ of A the compressed resolvent

$$R(\lambda) = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.23}$$

is called a generalized resolvent of A (here $P_{\mathfrak{H}}$ is the orthoprojection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$). If two exit space extensions $\widetilde{A}_1 \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}_1)$ and $\widetilde{A}_2 \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}_2)$ of A generates the same generalized resolvent $R(\lambda)$, then $\widetilde{A}_1$ and $\widetilde{A}_2$ are equivalent. The latter means that there exists a unitary operator $V \in \boldsymbol{B}(\widetilde{\mathfrak{H}}_1 \ominus \mathfrak{H}, \widetilde{\mathfrak{H}}_2 \ominus \mathfrak{H})$ such that $\widetilde{A}_2 = \widetilde{U}\widetilde{A}_1$ with the unitary operator $\widetilde{U} = (I_{\mathfrak{H}} \oplus V) \oplus (I_{\mathfrak{H}} \oplus V) \in B(\widetilde{\mathfrak{H}}_1^2, \widetilde{\mathfrak{H}}_2^2)$. Hence each exit space extension $\widetilde{A}$ of A is defined by the generalized resolvent (2.23) uniquely up to the equivalence.

The following proposition is well known.

Proposition 2.13. *If $n_+(A) = 0$, then there exists a unique exit space extension $\widetilde{A} = \widetilde{A}^*$ of A and*

$$P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{2.24}$$

A parametrization of all exit space self-adjoint extensions $\widetilde{A}$ of a symmetric relation A is given by the following theorem.

Theorem 2.14. [15, 16] *Assume that $n_-(A) \le n_+(A)$, $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, $A_0 = \ker \Gamma_0$ and $\gamma_\pm$ and M_+ are* *the γ-fields and the Weyl function of* Π *respectively. Then the equality (Krein formula for generalized resolvent)*

$$P_{\mathfrak{H}}(\widetilde{A}_\tau - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)(\tau(\lambda) + M_+(\lambda))^{-1}\gamma_-^*(\overline{\lambda}), \ \lambda \in \mathbb{C}_+ \quad (2.25)$$

establishes a bijective correspondence $\widetilde{A} = \widetilde{A}_\tau$ *between all relation valued functions* $\tau = \tau(\lambda) \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ *and all exit space self-adjoint extensions* $\widetilde{A}$ *of* A*. The same correspondence is given by the Shtraus formula*

$$P_{\mathfrak{H}}(\widetilde{A}_\tau - \lambda)^{-1} \upharpoonright \mathfrak{H} = (\widetilde{A}(\lambda) - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+, \quad (2.26)$$

where $\widetilde{A}(\lambda) = A_{-\tau(\lambda)}$, $\lambda \in \mathbb{C}_+$ *(see Proposition 2.12,* (3)*).*

Remark 2.15. *If* $n_-(A) < \infty$ *and* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^**, then by* (2.22) $\dim \mathcal{H}_1 < \infty$ *and according to Proposition 2.8 each function* $\tau \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ *admits the representation* $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\}$ *in the sense of Definition 2.7.*

Remark 2.16. *If* $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$*, then the triplet* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *in the sense of Definition 2.10 turns into the boundary triplet (boundary value space)* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *for* A^* *in the sense of [2, 10]. In this case:*

(i) the relation A *has equal deficiency indices* $n_+(A) = n_-(A)(= \dim \mathcal{H})$*;*

(ii) $A_0^* = A_0$ *and the* γ*-fields* $\gamma_\pm(\cdot)$ *and the Weyl function* $M_+(\cdot)$ *of* Π *turn into the* γ*-field* $\gamma : \mathbb{C} \setminus \mathbb{R} \to \boldsymbol{B}(\mathcal{H}, \mathfrak{H})$ *and the Weyl function* $M : \mathbb{C} \setminus \mathbb{R} \to \boldsymbol{B}(\mathcal{H})$ *from [5, 13]*

(iii) $M(\cdot)$ *is a* Q*-function of the pair* (A, A_0) *and formula* (2.25) *turns into the classical Krein formula for generalized resolvents of a symmetric relation* A *with equal deficiency indices [5, 11–13]. This formula gives a parametrization* $\widetilde{A} = \widetilde{A}_\tau$ *of all exit space extensions* $\widetilde{A} = \widetilde{A}^*$ *of* A *by means of functions* $\tau = \tau(\lambda) \in \widetilde{R}(\mathcal{H})$*.*

Assume that $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ is a Hilbert space, $\mathfrak{H}_r := \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$, $P_{\mathfrak{H}}$ is the orthoprojection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ and $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ is an exit space extension of A.

Definition 2.17. A linear relation $C(\widetilde{A})$ in $\mathfrak{H}$ defined by

$$C(\widetilde{A}) = P_{\mathfrak{H}}\widetilde{A} \upharpoonright \mathfrak{H} := \{\{f, f'\} \in \mathfrak{H}^2 : \{f, f' \oplus f_r'\} \in \widetilde{A} \ \text{ with some } \ f_r' \in \mathfrak{H}_r\} \quad (2.27)$$

is called the compression of $\widetilde{A}$.

Clearly, $C(\widetilde{A})$ is a (not necessarily closed) symmetric extension of A.

Theorem 2.18. [18] *Assume that* $n_+(A) = n_-(A)$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^*, $\tau \in \widetilde{R}(\mathcal{H})$, τ_s *is the operator part of* τ *(see* (2.12)*),* $\widetilde{A}_\tau = \widetilde{A}_\tau^*$ *is the corresponding exit space extension of* A *and* $C(\widetilde{A}_\tau)$ *is the compression of* $\widetilde{A}_\tau$*. Assume also that* $\tau_s \in R_c[\mathcal{H}']$ *and let* $\mathcal{B}_{\tau_s} \in \boldsymbol{B}(\mathcal{H}')$ *and* $\mathcal{N}_{\tau_s} : \operatorname{dom} \mathcal{N}_{\tau_s} \to \mathcal{H}'$ $(\operatorname{dom} \mathcal{N}_{\tau_s} \subset \mathcal{H}')$ *be operators corresponding to* τ_s *in accordance with Proposition 2.1. If* $\operatorname{ran} \mathcal{B}_{\tau_s}$ *is closed, then* $C(\widetilde{A}_\tau) = A_{\theta_c}$ *(in the triplet* Π*) with the symmetric linear relation* θ_c *in* $\mathcal{H}$ *given by*

$$\theta_c = \{\{h, -\mathcal{N}_{\tau_s} h + \mathcal{B}_{\tau_s} \psi + k\} : h \in \operatorname{dom} \mathcal{N}_{\tau_s}, \psi \in \mathcal{H}', k \in \mathcal{K}\}. \quad (2.28)$$

3. Description of compressions of exit space self-adjoint extensions

The following lemma directly follows from [16, Proposition 4.2].

Lemma 3.1. *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^**, let* A_r *be a maximal symmetric operator in a Hilbert space* $\mathfrak{H}_r$ *with* $n_+(A_r) = 0$, $n_-(A_r) = \dim \mathcal{H}_2$ *and let* $\mathfrak{H}_e := \mathfrak{H} \oplus \mathfrak{H}_r$*. Then* $A_e := A \oplus A_r$ *is a symmetric relation in* $\mathfrak{H}_e$, $A_e^* := A^* \oplus A_r^*$ *and there exists a surjective linear mapping* $\Gamma_r : A_r^* \to \mathcal{H}_2$ *such that the operators*

$$\Gamma_0^e \widehat{f}_e = P_1 \Gamma_0 \widehat{f} \oplus (P_2 \Gamma_0 \widehat{f} + \Gamma_r \widehat{f}_r) (\in \mathcal{H}_1 \oplus \mathcal{H}_2), \quad (3.1)$$

$$\Gamma_1^e \widehat{f}_e = \Gamma_1 \widehat{f} \oplus \tfrac{i}{2}(P_2 \Gamma_0 \widehat{f} - \Gamma_r \widehat{f}_r) (\in \mathcal{H}_1 \oplus \mathcal{H}_2), \quad \widehat{f}_e = \widehat{f} \oplus \widehat{f}_r \in A^* \oplus A_r^* \quad (3.2)$$

form a boundary triplet $\Pi_e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ *for* A_e^**.*

Proposition 3.2. *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^**, let* A_r, $\mathfrak{H}_r$, A_e, $\mathfrak{H}_e$ *be the same as in Lemma 3.1 and let* $\Pi_e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ *be boundary triplet* (3.1), (3.2) *for* A_e^**. Then for each linear relation* θ *from* $\mathcal{H}_0$ *to* $\mathcal{H}_1$ *the equalities*

$$\widetilde{A}_e = A_\theta \oplus A_r^*, \qquad \widetilde{A}_e' = A_\theta \oplus A_r \quad (3.3)$$

define proper extensions $\widetilde{A}_e$ *and* $\widetilde{A}_e'$ *of* A_e *and* $\widetilde{A}_e = A_{\theta_e}$, $\widetilde{A}_e' = A_{\theta_e'}$ *(in the triplet* Π_e*), where* θ_e *and* θ_e' *are linear relations in* $\mathcal{H}_0 (= \mathcal{H}_1 \oplus \mathcal{H}_2)$ *given by*

$$\theta_e = \{\{h_{01} \oplus (h_{02} + h_r), h_1 \oplus \tfrac{i}{2}(h_{02} - h_r)\} : \{h_{01} \oplus h_{02}, h_1\} \in \theta, \ h_r \in \mathcal{H}_2\} \quad (3.4)$$

$$\theta_e' = \{\{h_{01} \oplus h_{02}, h_1 \oplus \tfrac{i}{2} h_{02}\} : \{h_{01} \oplus h_{02}, h_1\} \in \theta\}. \quad (3.5)$$

Proof. The inclusions $\widetilde{A}_e, \widetilde{A}'_e \in \text{ext}(A_e)$ are obvious. Next assume that $\widehat{f}_e = \widehat{f} \oplus \widehat{f}_r \in \widetilde{A}_e$ with $\widehat{f} \in A_\theta$ and $\widehat{f}_r \in A_r^*$. Then by (3.1) and (3.2)

$$\Gamma_0^e \widehat{f}_e = h_{01} \oplus (h_{02} + h_r), \qquad \Gamma_1^e \widehat{f}_e = h_1 \oplus \tfrac{i}{2}(h_{02} - h_r),$$

where $h_{01} = P_1 \Gamma_0 \widehat{f}$, $h_{02} = P_2 \Gamma_0 \widehat{f}$, $h_1 = \Gamma_1 \widehat{f}$ and $h_r = \Gamma_r \widehat{f}_r$. Since $\{h_{01} \oplus h_{02}, h_1\} = \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \theta$ and $h_r \in \mathcal{H}_2$, it follows that $\{\Gamma_0^e \widehat{f}_e, \Gamma_1^e \widehat{f}_e\} \in \theta_e$. Conversely, let $\widehat{h} \in \theta_e$, so that

$$\widehat{h} = \{\{h_{01} \oplus (h_{02} + h_r), h_1 \oplus \tfrac{i}{2}(h_{02} - h_r)\}$$

with some $\{h_{01} \oplus h_{02}, h_1\} \in \theta$ and $h_r \in \mathcal{H}_2$. Then there exists $\widehat{f} \in A_\theta$ such that $P_1 \Gamma_0 \widehat{f} = h_{01}$, $P_2 \Gamma_0 \widehat{f} = h_{02}$ and $\Gamma_1 \widehat{f} = h_1$. Moreover, since the mapping Γ_r is surjective, there exists $\widehat{f}_r \in A_r^*$ such that $\Gamma_r \widehat{f}_r = h_r$. Clearly, $\widehat{f}_e := \widehat{f} \oplus \widehat{f}_r \in \widetilde{A}_e$ and by (3.1) and (3.2) one has $\{\Gamma_0^e \widehat{f}_e, \Gamma_1^e \widehat{f}_e\} = \widehat{h}$. This implies that $\widetilde{A}_e = A_{\theta_e}$.

Next assume that $\widehat{f}_r \in A_r$. Then $\widehat{f}_e := 0 \oplus \widehat{f}_r \in A_e$ and by (3.1) $\Gamma_0^e \widehat{f}_e = \Gamma_r \widehat{f}_r$. On the other hand, according to Proposition 2.12, (1) $\Gamma_0^e \widehat{f}_e = 0$ and, consequently, $\Gamma_r \widehat{f}_r = 0$, $\widehat{f}_r \in A_r$. This and (3.1), (3.2) yield the equality $\widetilde{A}'_e = A_{\theta'_e}$. □

Lemma 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $\dim \mathcal{H}_1 < \infty$ and let $T \in \boldsymbol{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ be an operator with the block representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & \frac{1}{2} I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Then $\text{ran}\, T$ *is closed.*

Proof. Let $\mathcal{H}_2'' = \ker T_2$ and $\mathcal{H}_2' = \mathcal{H}_2 \ominus \mathcal{H}_2''$, so that $\mathcal{H}_2 = \mathcal{H}_2' \oplus \mathcal{H}_2''$ and

$$T_2 = (T_2', \, 0) : \mathcal{H}_2' \oplus \mathcal{H}_2'' \to \mathcal{H}_1.$$

Since $\ker T_2' = \{0\}$ and $\dim \mathcal{H}_1 < \infty$, it follows that $\dim \mathcal{H}_2' < \infty$. Moreover,

$$T = \begin{pmatrix} T_1 & T_2' & 0 \\ (T_2')^* & \frac{1}{2} I_{\mathcal{H}_2'} & 0 \\ 0 & 0 & \frac{1}{2} I_{\mathcal{H}_2''} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2' \oplus \mathcal{H}_2'' \to \mathcal{H}_1 \oplus \mathcal{H}_2' \oplus \mathcal{H}_2''$$

and hence

$$T = \begin{pmatrix} \widetilde{T} & 0 \\ 0 & \frac{1}{2} I_{\mathcal{H}_2''} \end{pmatrix} : \widetilde{\mathcal{H}} \oplus \mathcal{H}_2'' \to \widetilde{\mathcal{H}} \oplus \mathcal{H}_2'',$$

where $\widetilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2'$ and $\widetilde{T} \in \boldsymbol{B}(\widetilde{\mathcal{H}})$. Since $\dim \widetilde{\mathcal{H}} < \infty$, the subspace $\text{ran}\, \widetilde{T} \subset \widetilde{\mathcal{H}}$ is closed. Moreover, $\text{ran}\, T = \text{ran}\, \widetilde{T} \oplus \mathcal{H}_2''$ and hence $\text{ran}\, T$ is closed. □

Proposition 3.4. *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*, *let* $\tau = \{K_0, K_1\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, *let* $K_0(\lambda)$ *has the block representation*

$$K_0(\lambda) = \begin{pmatrix} K_{01}(\lambda) \\ K_{02}(\lambda) \end{pmatrix} : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+$$

and let $\widetilde{A}_\tau$ *be the corresponding exit space self-adjoint extension of* A *in the Hilbert space* $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. *Assume also that* $\mathfrak{H}_r, A_r, \mathfrak{H}_e, A_e, \Gamma_r$ *are the same as in Lemma 3.1,* $\Pi_e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ *is the boundary triplet* (3.1), (3.2) *for* A_e^* *and let* $\widetilde{A}_r = \widetilde{A}_r^*$ *be a (unique) exit space extension of* A_r *in the Hilbert space* $\widetilde{\mathfrak{H}}_r \supset \mathfrak{H}_r$. *Then:*

(1) $\widetilde{A}_e := \widetilde{A}_\tau \oplus \widetilde{A}_r$ *is an exit space self-adjoint extension of* A_e *in the Hilbert space* $\widetilde{\mathfrak{H}}_e = \widetilde{\mathfrak{H}} \oplus \widetilde{\mathfrak{H}}_r$.

(2) $\widetilde{A}_e = \widetilde{A}_{\tau_e}$ *(in the triplet* Π_e*), where* $\tau_e = \{K_{0e}, K_{1e}\} \in \widetilde{R}(\mathcal{H}_0)$ *with*

$$K_{0e}(\lambda) = \begin{pmatrix} K_{01}(\lambda) & 0 \\ K_{02}(\lambda) & I_{\mathcal{H}_2} \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0}, \quad \lambda \in \mathbb{C}_+ \tag{3.6}$$

$$K_{1e}(\lambda) = \begin{pmatrix} K_1(\lambda) & 0 \\ -\frac{i}{2}K_{02}(\lambda) & \frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0}, \quad \lambda \in \mathbb{C}_+ \tag{3.7}$$

(3) *If in addition* $n_-(A) < \infty$ *and* $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\}$ *(see Remark 2.15), then*

$$\tau_e(\lambda) = \operatorname{gr} \tau_{es}(\lambda) \oplus \widehat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_+, \tag{3.8}$$

where $\tau_{es} \in R_c[\mathcal{H}_0']$ *(the operator part of* τ_e*) is given by* (2.18).

Proof. Statement (1) is obvious.

(2) Clearly, $(\widetilde{A}_e - \lambda)^{-1} = (\widetilde{A}_\tau - \lambda)^{-1} \oplus (\widetilde{A}_r - \lambda)^{-1}$. This and Proposition 2.13 give

$$P_{\mathfrak{H}_e}(\widetilde{A}_e - \lambda)^{-1} \upharpoonright \mathfrak{H}_e = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} \oplus (A_r^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+.$$

Let $\widetilde{A}_e = \widetilde{A}_{\tau_e}$ (in the triplet Π_e) with some $\tau_e \in \widetilde{R}(\mathcal{H}_0)$. Then by Shtraus formula (2.26)

$$(A_{-\tau_e(\lambda)} - \lambda)^{-1} = (A_{-\tau(\lambda)} - \lambda)^{-1} \oplus (A_r^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+$$

and, consequently,

$$A_{-\tau_e(\lambda)} = A_{-\tau(\lambda)} \oplus A_r^*. \tag{3.9}$$

Since

$$-\tau(\lambda) = \{\{K_{01}(\lambda)h \oplus K_{02}(\lambda)h, -K_1(\lambda)h\} : h \in \mathcal{H}_1\}, \quad \lambda \in \mathbb{C}_+,$$

it follows from (3.9) and Proposition 3.2 that

$$\tau_e(\lambda) = \{\{K_{01}(\lambda)h \oplus (K_{02}(\lambda)h + h_r), K_1(\lambda)h \\ \oplus(-\tfrac{i}{2}K_{02}(\lambda)h + \tfrac{i}{2}h_r)\} : h \in \mathcal{H}_1, h_r \in \mathcal{H}_2\}.$$

Therefore $\tau_e = \{K_{0e}, K_{1e}\}$ with $K_{0e}(\lambda)$ and $K_{1e}(\lambda)$ given by (3.6) and (3.7) respectively.

(3) It follows from (2.13) that

$$K_0(\lambda) = \begin{pmatrix} I_{\mathcal{H}_1'} & 0 \\ 0 & 0 \\ Q_2(\lambda) & 0 \end{pmatrix} : \underbrace{\mathcal{H}_1' \oplus \mathcal{K}}_{\mathcal{H}_1} \to \underbrace{\mathcal{H}_1' \oplus \mathcal{K}}_{\mathcal{H}_1} \oplus \mathcal{H}_2$$

$$K_1(\lambda) = \begin{pmatrix} Q_1(\lambda) & 0 \\ 0 & I_{\mathcal{K}} \end{pmatrix} : \underbrace{\mathcal{H}_1' \oplus \mathcal{K}}_{\mathcal{H}_1} \to \underbrace{\mathcal{H}_1' \oplus \mathcal{K}}_{\mathcal{H}_1}$$

and by statement (2) $\tau_e = \{K_{0e}, K_{1e}\}$, where

$$K_{0e}(\lambda) = \begin{pmatrix} I_{\mathcal{H}_1'} & 0 & 0 \\ 0 & 0 & 0 \\ Q_2(\lambda) & 0 & I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{K} \oplus \mathcal{H}_2 \to \mathcal{H}_1' \oplus \mathcal{K} \oplus \mathcal{H}_2$$

$$K_{1e}(\lambda) = \begin{pmatrix} Q_1(\lambda) & 0 & 0 \\ 0 & I_{\mathcal{K}} & 0 \\ -\frac{i}{2}Q_2(\lambda) & 0 & \frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{K} \oplus \mathcal{H}_2 \to \mathcal{H}_1' \oplus \mathcal{K} \oplus \mathcal{H}_2.$$

Let

$$K_{0s}(\lambda) = \begin{pmatrix} I_{\mathcal{H}_1'} & 0 \\ Q_2(\lambda) & I_{\mathcal{H}_2} \end{pmatrix} : \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'} \to \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'},$$

$$K_{1s}(\lambda) = \begin{pmatrix} Q_1(\lambda) & 0 \\ -\frac{i}{2}Q_2(\lambda) & \frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'} \to \underbrace{\mathcal{H}_1' \oplus \mathcal{H}_2}_{\mathcal{H}_0'}.$$

Then $\tau_e(\lambda) = \tau_{es}(\lambda) \oplus \widehat{\mathcal{K}},\ \lambda \in \mathbb{C}_+$, where $\widehat{\mathcal{K}} = \{0\} \oplus \mathcal{K}$ and

$$\tau_{es}(\lambda) = \{\{K_{0s}(\lambda)h_0, K_{1s}(\lambda)h_0\} : h_0 \in \mathcal{H}_0'\}, \quad \lambda \in \mathbb{C}_+$$

is the operator part of τ_e. Since the operator $K_{0s}(\lambda)$ is invertible, it follows that $\tau_{es} \in R[\mathcal{H}_0']$ and

$$\tau_{es}(\lambda) = K_{1s}(\lambda)K_{0s}^{-1}(\lambda) = \begin{pmatrix} Q_1(\lambda) & 0 \\ -\frac{i}{2}Q_2(\lambda) & \frac{i}{2}I \end{pmatrix} \begin{pmatrix} I & 0 \\ -Q_2(\lambda) & I \end{pmatrix},$$

which implies (2.18). Moreover, $\mathrm{Im}\tau_{es}(\lambda)$ is of the form (2.19) and by Lemma 3.3 $\operatorname{ran}\mathrm{Im}\tau_{es}(\lambda)$ is closed. Hence $\tau_{es} \in R_c[\mathcal{H}_0']$. □

In the following theorem the compression $C(\widetilde{A}_\tau)$ of the exit space extension $\widetilde{A}_\tau$ is characterized in terms of limit values of the parameter τ.

Theorem 3.5. *Assume that A is a symmetric linear relation in $\mathfrak{H}$ with $n_-(A) < \infty$, $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, $\tau = \{\mathcal{H}_1' \oplus \mathcal{K}, Q_1, Q_2\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ (see Remark 2.15), $\widetilde{A}_\tau = \widetilde{A}_\tau^*$ is the corresponding exit space extension of A and $C(\widetilde{A}_\tau)$ is the compression of $\widetilde{A}_\tau$. Then $C(\widetilde{A}_\tau) = A_{\theta_c}$ (in the triplet Π) with the linear relation $\theta_c \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ given by*

$$\theta_c = \{\{h \oplus Q_2(\infty)h, (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi) \oplus k\} : h \in L_\infty, \psi \in \mathcal{H}_1', k \in \mathcal{K}\}. \tag{3.10}$$

Here $L_\infty \subset \mathcal{H}_1'$ is the subspace and $Q_1(\infty)$ and $Q_2(\infty)$ are operators defined in Proposition 2.9; $\mathcal{B}_{Q_1} \in B(\mathcal{H}_1')$ is the operator corresponding to $Q_1 \in R[\mathcal{H}_1']$ in accordance with Proposition 2.1.

Proof. Let $A_r, \mathfrak{H}_r$ and $A_e, \mathfrak{H}_e$ be the same as in Lemma 3.1. Moreover, let $\widetilde{A}_r = \widetilde{A}_r^*$ be a (unique) exit space extension of A_r in the Hilbert space $\widetilde{\mathfrak{H}}_r$. Then according to Proposition 3.4, (1) $\widetilde{A}_e := \widetilde{A}_\tau \oplus \widetilde{A}_r$ is an exit space self-adjoint extension of A_e in $\widetilde{\mathfrak{H}}_e = \widetilde{\mathfrak{H}} \oplus \widetilde{\mathfrak{H}}_r$. Let $C(\widetilde{A}_e)$ and $C(\widetilde{A}_r)$ be compressions of $\widetilde{A}_e$ and $\widetilde{A}_r$ respectively. Clearly, $C(\widetilde{A}_e) = C(\widetilde{A}_\tau) \oplus C(\widetilde{A}_r)$. Moreover, since $C(\widetilde{A}_r)$ is a symmetric extension of the maximal symmetric operator A_r, it follows that $C(\widetilde{A}_r) = A_r$ and therefore

$$C(\widetilde{A}_e) = C(\widetilde{A}_\tau) \oplus A_r. \tag{3.11}$$

Let $C(\widetilde{A}_\tau) = A_{\theta_c}$ (in the triplet Π) with some $\theta_c \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$. Moreover, let $\Pi_e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ be boundary triplet (3.1), (3.2) for A_e^* and let $C(\widetilde{A}_e) = A_{\theta_{ce}}$ (in the triplet Π_e) with some linear relation θ_{ce} in $\mathcal{H}_0$. Then according to Proposition 3.4, (3) $\widetilde{A}_e = \widetilde{A}_{\tau_e}$ (in the triplet Π_e), where $\tau_e \in \widetilde{R}(\mathcal{H}_0)$ is of the form (3.8) with $\tau_{es} \in R_c[\mathcal{H}_0']$ given by (2.18). Let $\mathcal{B}_{\tau_{es}} \in \mathcal{B}(\mathcal{H}_0')$ be the operator corresponding to τ_{es} in accordance with Proposition 2.1. Since $\mathcal{B}_{\tau_{es}} = \mathcal{B}_{\tau_{es}}^*$, it follows from (2.18) that

$$\mathcal{B}_{\tau_{es}} = \begin{pmatrix} \mathcal{B}_{Q_1} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_2 \to \mathcal{H}_1' \oplus \mathcal{H}_2. \tag{3.12}$$

Applying Theorem 2.18 to the triplet Π_e and taking (3.12) into account one obtains

$$\theta_{ce} = \{\{h \oplus h_2, (-N_{\tau_{es}}(h \oplus h_2) + \mathcal{B}_{Q_1}\psi) \oplus k\} : h \oplus h_2 \in \operatorname{dom} N_{\tau_{es}}, \psi \in \mathcal{H}_1', k \in \mathcal{K}\}, \tag{3.13}$$

where

$$\operatorname{dom} N_{\tau_{es}} = \{h \oplus h_2 \in \mathcal{H}_1' \oplus \mathcal{H}_2 : \lim_{y \to +\infty} y \operatorname{Im}(\tau_{es}(iy)(h \oplus h_2), h \oplus h_2) < \infty\}$$

$$N_{\tau_{es}}(h \oplus h_2) = \lim_{y \to +\infty} \tau_{es}(iy)(h \oplus h_2), \quad h \oplus h_2 \in \operatorname{dom} N_{\tau_{es}}. \tag{3.14}$$

It follows from (2.19) that

$$\operatorname{dom} N_{\tau_{es}} = \{h \oplus h_2 \in \mathcal{H}_1' \oplus \mathcal{H}_2 : \lim_{y \to +\infty} y \left(\operatorname{Im}(Q_1(iy)h, h) - \operatorname{Re}\left(Q_2(iy)h, h_2\right) + \tfrac{1}{2}||h_2||^2\right) < \infty\}. \tag{3.15}$$

Moreover, in view of (2.18) for $h \oplus h_2 \in \mathcal{H}_1' \oplus \mathcal{H}_2$ one has

$$\tau_{es}(iy)(h \oplus h_2) = Q_1(iy)h \oplus (-iQ_2(iy)h + \tfrac{i}{2}h_2). \tag{3.16}$$

Therefore for each $h \oplus h_2 \in \operatorname{dom} N_{\tau_{es}}$ there exist the limits $Q_1(\infty)h := \lim_{y \to +\infty} Q_1(iy)h$, $Q_2(\infty)h := \lim_{y \to +\infty} Q_2(iy)h$ and by (3.14), (3.16)

$$N_{\tau_{es}}(h \oplus h_2) = Q_1(\infty)h \oplus (-iQ_2(\infty)h + \tfrac{i}{2}h_2), \quad h \oplus h_2 \in \operatorname{dom} N_{\tau_{es}}$$

Hence (3.13) can be written as

$$\theta_{ce} = \{\{h \oplus h_2, (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi) \oplus (iQ_2(\infty)h - \tfrac{i}{2}h_2) \oplus k\} : h \oplus h_2 \in \operatorname{dom} N_{\tau_{es}}, \psi \in \mathcal{H}_1', k \in \mathcal{K}\}. \tag{3.17}$$

On the other hand, by (3.11) and Proposition 3.2 one has

$$\theta_{ce} = \{\{h \oplus h_2, h_1 \oplus \tfrac{i}{2}h_2\} : \{h \oplus h_2, h_1\} \in \theta_c\}. \tag{3.18}$$

Now by using (3.17) and (3.18) we prove (3.10).

Let $\widehat{h} = \{h \oplus h_2, h_1\} \in \theta_c$ with $h \oplus h_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2 (= \mathcal{H}_0)$ and $h_1 \in \mathcal{H}_1$. Then by (3.18) $\{h \oplus h_2, h_1 \oplus \frac{i}{2}h_2\} \in \theta_{ce}$ and (3.17) yields $h \oplus h_2 \in \operatorname{dom} N_{\tau_{es}}$,

$$h_1 = (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi) \oplus k, \qquad \tfrac{i}{2}h_2 = iQ_2(\infty)h - \tfrac{i}{2}h_2, \tag{3.19}$$

where $\psi \in \mathcal{H}_1'$ and $k \in \mathcal{K}$. It follows from (3.15) that $h \in \mathcal{H}_1'$. Moreover, by the second equality in (3.19) $h_2 = Q_2(\infty)h$ and hence $h \in \widetilde{L}_\infty (\subset \mathcal{H}_1')$

(see Proposition 2.9). Note also that by (3.15) $\lim_{y\to+\infty} y\varphi_h(y) < \infty$, where $\varphi_h(y)$ is given by (2.15). Therefore by (2.16) $h \in L_\infty$ and the first equality in (3.19) yields

$$\widehat{h} = \{h \oplus Q_2(\infty)h, (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi) \oplus k\}. \tag{3.20}$$

Conversely, assume that $h \in L_\infty$, $\psi \in \mathcal{H}_1'$, $k \in \mathcal{K}$ and $\widehat{h} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ is given by (3.20). Let us put

$$\begin{aligned}\widehat{m} &:= \{h \oplus Q_2(\infty)h, (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi) \oplus \tfrac{i}{2}Q_2(\infty)h \oplus k\} \qquad (3.21)\\ &= \{h \oplus Q_2(\infty)h, (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi) \oplus (iQ_2(\infty)h - \tfrac{i}{2}Q_2(\infty)h) \oplus k\}\end{aligned}$$

Since $h \in L_\infty$, it follows from (3.15), (2.16) and (2.15) that $h\oplus Q_2(\infty)h \in \operatorname{dom} N_{\tau_{es}}$. Therefore by (3.17) $\widehat{m} \in \theta_{ce}$ and in view of (3.18) there exists $\{h'\oplus h_2', h_1'\} \in \theta_c$ such that $\widehat{m} = \{h'\oplus h_2', h_1' \oplus \frac{i}{2}h_2'\}$ (here $h', h_1' \in \mathcal{H}_1$ and $h_2' \in \mathcal{H}_2$). Comparing this equality with (3.21) one gets $h' = h$, $h_2' = Q_2(\infty)h$ and $h_1' = (-Q_1(\infty)h + \mathcal{B}_{Q_1}\psi)\oplus k$. Hence $\widehat{h} = \{h'\oplus h_2', h_1'\}$, that is $\widehat{h} \in \theta_c$. This proves (3.10). □

Corollary 3.6. *Let the assumptions of Theorem 3.5 be satisfied and let* $A_0 = \ker \Gamma_0$. *Then:*

(1) $C(\widetilde{A}_\tau) \subset A_0$ *if and only if*

$$\lim_{y\to+\infty} y\varphi_h(y) = \infty, \quad h \in \widetilde{L}_\infty, \ \ h \neq 0 \tag{3.22}$$

(for $\widetilde{L}_\infty$ *and* $\varphi_h(y)$ *see Proposition 2.9). In this case*

$$C(\widetilde{A}_\tau) = \{\widehat{f} \in A^* : \Gamma_0\widehat{f} = 0, \Gamma_1\widehat{f} = \mathcal{B}_{Q_1}\psi \oplus k \text{ with some } \psi \in \mathcal{H}_1' \text{ and } k \in \mathcal{K}\}. \tag{3.23}$$

(2) $C(\widetilde{A}_\tau) = A_0$ *if and only if* $\ker \mathcal{B}_{Q_1} = \{0\}$.

(3) $C(\widetilde{A}_\tau) = A$ *if and only if* $\tau \in R(\mathcal{H}_0, \mathcal{H}_1)$ *(that is* $\mathcal{K} = \{0\}$*),* $\mathcal{B}_{Q_1} = 0$ *and* (3.22) *is satisfied.*

Proof. (1) According to Theorem 3.5 $C(\widetilde{A}_\tau) = A_{\theta_c}$ with $\theta_c \in \operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ given by (3.10). In the following we need the relations

$$\operatorname{mul}\theta_c = \operatorname{ran}\mathcal{B}_{Q_1} \oplus \mathcal{K}, \qquad \mathcal{H}_1 \ominus \operatorname{mul}\theta_c = \ker \mathcal{B}_{Q_1} \tag{3.24}$$

$$L_\infty \subset \ker \mathcal{B}_{Q_1} \tag{3.25}$$

$$\operatorname{dom}\theta_c = \{0\} \iff L_\infty = \{0\}. \tag{3.26}$$

The first equality in (3.24) directly follows from (3.10). Next,

$$\mathcal{H}_1 \ominus \operatorname{mul}\theta_c = \mathcal{H}_1' \ominus \operatorname{ran}\mathcal{B}_{Q_1} = \ker \mathcal{B}_{Q_1},$$

that is the second equality in (3.24) holds. The inclusion (3.25) is implied by (3.24), (2.7) and the obvious equality $P_1 \mathrm{dom}\, \theta_c = L_\infty$. Finally, (3.26) directly follows from (3.10).

Clearly, $C(\widetilde{A}_\tau) \subset A_0$ if and only if $\mathrm{dom}\, \theta_c = \{0\}$. Therefore by (3.26) $C(\widetilde{A}_\tau) \subset A_0$ if and only if $L_\infty = \{0\}$, which is equivalent to (3.22). Moreover, in this case the first equality in (3.24) gives

$$\theta_c = \{0\} \oplus \mathrm{mul}\, \theta_c = \{\{0, \mathcal{B}_{Q_1}\psi \oplus k\}; \psi \in \mathcal{H}_1', k \in \mathcal{K}\},$$

which implies (3.23).

Next, the equality $C(\widetilde{A}_\tau) = A_0$ holds if and only if $\mathrm{dom}\, \theta_c = \{0\}$ and $\mathrm{mul}\, \theta_c = \mathcal{H}_1$. Moreover, by the second equality in (3.24) $\mathrm{mul}\, \theta_c = \mathcal{H}_1$ if and only if $\ker \mathcal{B}_{Q_1} = \{0\}$. Therefore by (3.26) $C(\widetilde{A}_\tau) = A_0$ if and only if $L_\infty = \{0\}$ and $\ker \mathcal{B}_{Q_1} = \{0\}$, which in view of (3.25) yields statement (2).

Finally, by Proposition 2.12, (1) $C(\widetilde{A}_\tau) = A$ if and only if $\theta_c = \{0\}$, i.e., $\mathrm{dom}\, \theta_c = \{0\}$ and $\mathrm{mul}\, \theta_c = \{0\}$. Therefore by (3.24) and (3.26) $C(\widetilde{A}_\tau) = A$ if and only if $\mathcal{K} = \{0\}$, $\mathcal{B}_{Q_1} = 0$ and $L_\infty = \{0\}$. This yields statement (3). □

Remark 3.7. *Assume that A is a closed densely defined symmetric operator in $\mathfrak{H}$. Then each exit space extension $\widetilde{A} = \widetilde{A}^*$ of A is a densely defined operator and according to M. A. Naimark [19] (see also [1, ch. 9]) an extension $\widetilde{A}$ of A is said to be of the second kind if* $\mathrm{dom}\, \widetilde{A} \cap \mathfrak{H} = \mathrm{dom}\, A$ *or equivalently if $C(\widetilde{A}) = A$. Clearly, Corollary 3.6, (3) gives a parametrization of all extensions $\widetilde{A}$ of the second kind of an operator A with unequal deficiency indices $n_-(A) < n_+(A)$ in terms of the parameter τ from Krein resolvent formula* (2.25)*. Note that for an operator A with equal deficiency indices $n_-(A) = n_+(A) \le \infty$ the criterion for an extension $\widetilde{A}_\tau$ of A with $\tau \in R[\mathcal{H}]$ to be of the second kind was obtained in [4]. This criterion is of the form*

$$\mathcal{B}_\tau = 0 \quad \text{and} \quad \lim_{y \to +\infty} y \,\mathrm{Im}(\tau(iy)h, h) = \infty, \;\; h \in \mathcal{H}, \;\; h \ne 0. \qquad (3.27)$$

Later on the sufficiency of conditions (3.27) *was rediscovered in [8] for a more restrictive case $n_-(A) = n_+(A) < \infty$. In the case $n_-(A) = n_+(A) \le \infty$ a description of all extensions $\widetilde{A}_\tau$ of the second kind with the closed relation $T(\widetilde{A}_\tau) := \{\{P_{\mathfrak{H}} f, P_{\mathfrak{H}} \widetilde{A} f\} : f \in \mathrm{dom}\, \widetilde{A}_\tau\}$ was obtained in our paper [18]. Observe also that a somewhat other parametrization of the second kind extensions can be found in [20].*

In the following theorem we describe all exit space extensions $\widetilde{A}_\tau$ of A such that the compression of $\widetilde{A}_\tau$ is a maximal symmetric relation.

Theorem 3.8. *Let the assumptions of Theorem 3.5 be satisfied. Then* $C(\widetilde{A}_\tau)$ *is maximal symmetric if and only if* $\ker \mathcal{B}_{Q_1} \subset \widetilde{L}_\infty$ *and*

$$\lim_{y\to+\infty} y\varphi_h(y) < \infty, \quad h \in \ker \mathcal{B}_{Q_1}$$

(here $\varphi_h(y)$ *is given by* (2.15)*).*

Proof. It follows from Theorem 3.5 and Proposition 2.12, (3) that $C(\widetilde{A}_\tau)$ is maximal symmetric if and only if $\theta_c \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$, where θ_c is given by (3.10). Moreover, by Lemma 2.3 $\theta_c \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if $L_\infty = \mathcal{H}_1 \ominus \mathrm{mul}\, \theta_c$. Therefore by the second equality in (3.24) $\theta_c \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if $L_\infty = \ker \mathcal{B}_{Q_1}$. This and (3.25) yield the equivalence $\theta_c \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1) \iff \ker \mathcal{B}_{\mathrm{Q}_1} \subset \mathrm{L}_\infty$, which implies the statement of the theorem. □

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