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Minmax bornologies

Taras Banakh, Igor Protasov

Abstract. A bornology $\mathcal{B}$ on a set X is called minmax if the smallest and the largest coarse structures on X compatible with $\mathcal{B}$ coincide. We prove that $\mathcal{B}$ is minmax if and only if the family $\mathcal{B}^\sharp = \{p \in \beta X : \{X \setminus B : B \in \mathcal{B}\} \subset p\}$ consists of ultrafilters which are pairwise non-isomorphic via $\mathcal{B}$-preserving bijections of X. Also we construct a minmax bornology $\mathcal{B}$ on ω such that the set $\mathcal{B}^\sharp$ is infinite. We deduce this result from the existence of a closed infinite subset in $\beta\omega$ that consists of pairwise non-isomorphic ultrafilters.

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1. Introduction

Let X be a set. A family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X := \{(x,x) : x \in X\}$ of X;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x,y) : \exists z\ ((x,z) \in E,\ \ (z,y) \in E')\}$ and $E^{-1} = \{(y,x) : (x,y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\bigcup \mathcal{E} = X \times X$.

A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote $E[x] = \{y \in X : (x,y) \in E\}$, $E[A] = \bigcup_{a \in A} E[a]$ and say that $E[x]$ and $E[A]$ are *balls of radius E around x* and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [9] or a *ballean* [7], [8].

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For a coarse space $(X, \mathcal{E})$, a subset $B \subset X$ is called *bounded* if $B \subset E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. A coarse space $(X, \mathcal{E})$ is called *unbounded* if X is unbounded. In what follows, all balleans under consideration are **supposed to be unbounded**.

A family $\mathcal{B}$ of subsets of an infinite set X is called a *bornology* on X if $\cup\mathcal{B} = X \notin \mathcal{B}$ and $\mathcal{B}$ is closed under taking subsets and finite unions. For a coarse space $(X, \mathcal{E})$, we denote by $\mathcal{B}_{(X,\mathcal{E})}$ the bornology of all bounded subsets of $(X, \mathcal{E})$.

A coarse structure $(X, \mathcal{E})$ is called

- *discrete* if for every $E \in \mathcal{E}$ there exists $B \in \mathcal{B}_{(X,\mathcal{E})}$ such that $E[x] = \{x\}$ for each $x \in X \backslash B$;
- *ultradiscrete* if $(X, \mathcal{E})$ is discrete and the family $\{X \backslash B : B \in \mathcal{B}_{(X,\mathcal{E})}\}$ is an ultrafilter;
- *maximal* if $(X, \mathcal{E})$ is bounded in every strictly stronger coarse structure on X.

Let $\mathcal{B}$ be a bornology on X. Following [2], we say that a coarse structure $\mathcal{E}$ on X is *compatible* with $\mathcal{B}$ if $\mathcal{B} = \mathcal{B}_{(X,\mathcal{E})}$.

By [2, §6], the family of all coarse structures on X, compatible with $\mathcal{B}$, has the smallest and the largest elements $\Downarrow\mathcal{B}$ and $\Uparrow\mathcal{B}$, respectively.

The smallest coarse structure $\Downarrow\mathcal{B}$, compatible with the bornology $\mathcal{B}$, is generated by the base consisting of the entourages $(B \times B) \cup \triangle_X$, where $B \in \mathcal{B}$. A coarse structure $\mathcal{E}$ on X is discrete if and only if $(X, \mathcal{B}) = \Downarrow\mathcal{B}_{(X,\mathcal{E})}$. A discrete coarse structure is maximal if and only if $\mathcal{E}$ is ultradiscrete [8, Example 10.1.2].

The largset coarse structure $\Uparrow\mathcal{B}$, compatible with the bornology $\mathcal{B}$, consists of all entourages $E \subset X \times X$ such that for any set $B \in \mathcal{B}$ the set $E[B] \cup E^{-1}[B]$ belongs to $\mathcal{B}$, see [2, §6].

2. Characterizing minmax bornologies

A bornology $\mathcal{B}$ on a set X is called *minmax* if $\Downarrow\mathcal{B} = \Uparrow\mathcal{B}$. Equivalently, $\mathcal{B}$ is minmax if $\mathcal{B}$ is compatible with a unique coarse structure on X. It is clear that each ultradiscrete bornology is minmax. In this section we show that the converse is not true.

We recall that two ultrafilters p, q on a set X are *isomorphic* if there exists a bijection $f : X \to X$ such that the ultrafilter $\bar{f}(p) := \{f(P) : P \in p\}$ is equal to q.

Let $\mathcal{B}$ be a bornology on a set X. We say that two ultrafilters p, q on X are *$\mathcal{B}$-isomorphic* if there is a bijection $f : X \to X$ such that $\bar{f}(p) = q$ and $\{f(B) : B \in \mathcal{B}\} = \mathcal{B}$.

We denote by $\mathcal{B}^\sharp$ the set of all ultrafilters p on X such that $\{X \setminus B : B \in \mathcal{B}\} \subset p$. Observe that a bornology $\mathcal{B}$ is ultradiscrete if and only if $\mathcal{B}^\sharp$ is a singleton.

Theorem 1. *A bornology $\mathcal{B}$ on a set X is minmax if and only if every two distinct ultrafilters $p, q \in \mathcal{B}^\sharp$ are not $\mathcal{B}$-isomorphic.*

Proof. To prove the "only if" part, assume that there exist two distinct $\mathcal{B}$-isomorphic ultrafilters p, q in $\mathcal{B}$ and take a bijection $f : X \to X$ witnessing this fact. Since $p \neq q$, the set $\{x \in X : f(x) \neq x\}$ does not belong to the bornology $\mathcal{B}$. Then the entourage $E = \{(x, y) \in X \times X : y \in \{x, f(x)\}\}$ belongs to the coarse structure $\Uparrow\!\mathcal{B}$ and witnesses that it is not discrete and hence not equal to $\Downarrow\!\mathcal{B}$. This means that $\mathcal{B}$ is not minmax.

To prove the "if" part, assume that $\mathcal{B}$ is not minmax. Then the coarse structure $\Uparrow\!\mathcal{B}$ is not discrete and there exists $E \in \Uparrow\!\mathcal{B}$ such that the set $\{x \in X : |E[x]| > 1\}$ does not belong to the bornology $\mathcal{B}$. We take a maximal by inclusion subset $Y \subseteq \{x \in X : |E[x]| > 1\}$ such that $E[y] \cap E[z] = \emptyset$ for all distinct $y, z \in Y$. We note that Y does not belong to $\mathcal{B}$ and take an arbitrary ultrafilter $p \in \mathcal{B}^\sharp$ such that $Y \in p$. For each $y \in Y$ choose $z_y \in E[y] \setminus y$ and consider the bijection $f : X \to X$ acting as the transposition on each pair y, z_y and identical on all other elements of X. Observe that $\{f(B) : B \in \mathcal{B}\} = \mathcal{B}$ and hence $p \neq \bar{f}(p) \in \mathcal{B}^\sharp$. So p and $\bar{f}(p)$ are two distinct $\mathcal{B}$-isomorphic ultrafilters in $\mathcal{B}^\sharp$. □

The following example was first presented in [8, Example 1].

Example 1. *There exists a minmax bornology $\mathcal{B}$ on ω which is not ultradiscrete.*

Proof. Choose any two non-isomorphic ultrafilters p, q on ω and consider the bornology $\mathcal{B} = \{B \subset \omega : \omega \setminus B \in p \cap q\}$. Since $\mathcal{B}^\sharp = \{p, q\}$, the bornology $\mathcal{B}$ is minmax (by Theorem 1) and not ultradiscrete. □

Now we shall construct a minmax bornology $\mathcal{B}$ on ω for which the set $\mathcal{B}^\sharp$ has cardinality $2^{\mathfrak{c}}$. For this we need the following fact, which can have an independent value.

Theorem 2. *The Stone-Čech compactification $\beta\omega$ of ω contains a closed infinite subset consisting of pairwise non-isomorphic ultrafilters.*

Proof. Let us recall that a point x of a topological space X is called a *weak P-point* if x does not belong to the closure of any countable subset $C \subset X \setminus \{x\}$. By Corollaries 4.5.2 and 4.3.2 in [6], the space $\beta\omega \setminus \omega$ contains $2^{\mathfrak{c}}$ weak P-points. Consequently, we can choose a sequence of free ultrafilters $(p_n)_{n\in\omega}$ consisting of pairwise non-isomorphic weak P-points in $\beta\omega \setminus \omega$. The definition of a weak P-point implies that the subspace $D = \{p_n\}_{n\in\omega}$ of $\beta\omega \setminus \omega$ is discrete. Now the regularity of $\beta\omega$ implies that there exists a family $\{P_n\}_{n\in\omega}$ of pairwise disjoint sets in ω such that $P_n \in p_n$ for every $n \in \omega$.

We claim that the closure $\bar{D}$ of D consists of pairwise non-isomorphic ultrafilters. To derive a contradiction, assume that $\bar{D}$ contains two distinct isomorphic ultrafilters p, q. Then $p \in \bar{P}$ and $q \in \bar{Q}$ for some disjoint sets $P, Q \subset D$. Find a bijection $f : \omega \to \omega$ such that $\bar{f}(p) = q$. The bijection f extends to a homeomorphism $\bar{f} : \beta\omega \to \beta\omega$. Since the set D consists of pairwise non-isomorphic ultrafilters, $\bar{f}(P)$ is disjoint with Q. So, $\bar{f}(P)$ and Q are two disjoint countable discrete subspaces of $\beta\omega \setminus \omega$ consisting of weak P-points. Then $\bar{f}(P)$ is disjoint with the closure of Q and Q is disjoint with the closure of $\bar{f}(P)$ (which is equal to $\bar{f}(\bar{P})$). By Lemma 1 of Frolík [5] (see also Theorem 1.5.2 in [6]), $\bar{Q} \cap \bar{f}(\bar{P}) = \emptyset$. On the other hand, $q = \bar{f}(p) \in \bar{Q} \cap \bar{f}(\bar{P})$. □

Remark 1. Theorem 2 has also a "near-coherent" version. Let us recall [3] that two ultrafilters p, q on ω are *near-coherent* if there exists a finite-to-one function $f : \omega \to \omega$ such that $\bar{f}(p) = \bar{f}(q)$. It is clear that any two isomorphic ultrafilters on ω are near-coherent. By [1], the space $\beta\omega$ contains an infinite closed set consisting of pairwise non-near-coherent ultrafilters if and only if $\beta\omega$ contains infinitely many non-near-coherent ultrafilters. The latter happens if $\mathfrak{u} \geq \mathfrak{d}$. On the other hand, by [4, 9.18], the strict inequality $\mathfrak{u} < \mathfrak{g}$ (which is consistent with ZFC by [4, 11.2]) implies that all free ultrafilters on ω are near-coherent.

Example 2. *There exist a minmax bornology $\mathcal{B}$ on ω such that $|\mathcal{B}^\sharp| = 2^{\mathfrak{c}}$.*

Proof. By Theorem 2, the space $\beta\omega$ contains an infinite closed subset F consisting of pairwise non-isomorphic ultrafilters. By Lemma 3.1.2(c) in [6], $|F| = 2^{\mathfrak{c}}$. Consider the bornology $\mathcal{B} = \{B \subset \omega : \omega \setminus B \in \bigcap_{p\in F} p\}$ and observe that $\mathcal{B}^\sharp = F$. By Theorem 1, the bornology $\mathcal{B}$ is minmax. □

3. Characterizing bornologies with maximal coarse structure $\Uparrow\mathcal{B}$

By [8, Theorem 10.2.1], any unbounded set L in a maximal coarse space $(X, \mathcal{E})$ is *large* (which means that $X = E[L]$ for some $E \in \mathcal{E}$). The converse is not true:

Example 3. *There exists a coarse structure $\mathcal{E}$ on a countable set X such that the coarse space $(X, \mathcal{E})$ is not maximal but each unbounded subset of $(X, \mathcal{E})$ is large.*

Proof. Let $X = \omega$, G be the group of all finitely supported permutations of X, and $[G]^{<\omega}$ be the family of all finite subsets of G. The action of the group G induces the coarse structure

$$\mathcal{E} = \{E \subset X \times X : \exists F \in [G]^{<\omega},\ \triangle_X \subset E \subset \{(x, y) : y \in \{x\} \cup Fx\}\}$$

on X, whose bornology coincides with the bornology $\mathcal{B}$ of all finite subsets of X.

The coarse structure $\mathcal{E}$ is not maximal since $\mathcal{E} \subset \Uparrow\mathcal{B}$ and $\mathcal{E} \neq \Uparrow\mathcal{B}$. Indeed, the coarse structure $\Uparrow\mathcal{B}$ contains the entourage

$$E = \bigcup_{n\in\omega} [n^2, (n+1)^2) \times [n^2, (n+1)^2)$$

that does not belong to the (finitary) coarse structure $\mathcal{E}$.

On the other hand, each unbounded set $L \subset X$ is large since we can find a bijection $f : X \to X$ such that $X \setminus L \subset f(L)$. This bijection determines the entourage $E := \{(x, y) \in X \times X : y \in \{x, f(x)\}\} \in \mathcal{E}$ such that $E[L] = X$. □

Theorem 3. *For a bornology $\mathcal{B}$ on a set X, the coarse space $(X, \Uparrow\mathcal{B})$ is maximal if and only if each unbounded subset of $(X, \Uparrow\mathcal{B})$ is large.*

Proof. The "only if" part follows from Theorem 10.2.1 in [8].

To prove the "if" part, assume that each unbounded subset of $(X, \Uparrow\mathcal{B})$ is large, but $(X, \Uparrow\mathcal{B})$ is not maximal. Then there is an unbounded coarse structure $\mathcal{E}$ on X such that $\Uparrow\mathcal{B} \subsetneq \mathcal{E}$. By the definition of $\Uparrow\mathcal{B}$, the coarse structure $\mathcal{E}$ is not compatible with the bornology $\mathcal{B}$. Consequently, there exists a set $L \subset X$ which is bounded in $(X, \mathcal{E})$ but does not belong to $\mathcal{B}$. Then L is unbounded in $(X, \Uparrow\mathcal{B})$ and hence $X = E[L]$ for some $E \in \Uparrow\mathcal{B} \subset \mathcal{E}$, which implies that X is bounded in $(X, \mathcal{E})$. But this contradicts the choice of $\mathcal{E}$. □

Example 4. Theorem 3 implies that for any infinite set X and the bornology $\mathcal{B} := \{A \subset X : |A| < |X|\}$ the coarse structure $\Uparrow\mathcal{B}$ is maximal. Indeed, for any subset $L \subset X$ of cardinality $|L| = |X|$, we can find a bijection f of X such that $X \setminus L \subset f(L)$. Then $E = \{(x, y) \in X \times X : y \in \{x, f(x)\}\}$ is an entourage in $\Uparrow\mathcal{B}$ such that $X = E[L]$, which means that the set L in large in $(X, \Uparrow\mathcal{B})$.

Following [2], we say that a coarse structure $\mathcal{E}$ on X is *relatively maximal* if $\mathcal{E} = \Uparrow\mathcal{B}_{(X,\mathcal{E})}$. Clearly, $\mathcal{E}$ is relatively maximal if either $\mathcal{E}$ is maximal or $\mathcal{B}_{(X,\mathcal{E})}$ is minmax.

Question 1. *Given a coarse structure $\mathcal{E}$, how can one detect whether $\mathcal{E}$ is relatively maximal?*

Remark 2. In light of Theorem 1, it is a very rare case when the coarse structure $\mathcal{E}$ on X is uniquely defined by the bornology $\mathcal{B}_{(X,\mathcal{E})}$.

We denote by $so(X, \mathcal{E})$ the set of all slowly oscillating functions of $(X, \mathcal{E})$. By Theorem 7.3.1 [8], if the coarse structures $\mathcal{E}$ and $\mathcal{E}'$ on X have linearly ordered bases and $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E})'}$, $so(X, \mathcal{E}) = so(X, \mathcal{E}')$, then $\mathcal{E} = \mathcal{E}'$. We denote by $\delta_{(X,\mathcal{E})}$ the binary relation on the power-set 2^X of X defined by $A\delta_{(X,\mathcal{E})}B$ if and only if there exists $E \in \mathcal{E}$ such that $A \subseteq E[B]$ and $B \subseteq E[A]$. By Theorem 7.5.3 from [8], if coarse structures $\mathcal{E}, \mathcal{E}'$ on X have linearly ordered bases and $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ then $\mathcal{E} = \mathcal{E}'$.

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Contact information

Taras Banakh Ivan Franko National University of Lviv, Lviv, Ukraine
E-Mail: t.o.banakh@gmail.com

Igor Protasov Faculty of Computer Science and Cybernetics, Kyiv University, Kyiv, Ukraine
E-Mail: i.v.protasov@gmail.com