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Isotone extensions and complete lattices

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Abstract. A set of necessary and sufficient conditions under which an isotone mapping from a subset of a poset X to a poset Y has an isotone extension to an isotone mapping from X to Y is found.

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1. Introduction

Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be partially ordered sets (posets). A mapping $f\colon X \to Y$ is *isotone* if the statement

$$(x \leqslant_X y) \Rightarrow (f(x) \leqslant_Y f(y))$$

holds for all $x, y \in X$. In particular, if f is an isotone bijection and f^{-1} is isotone, then f is an isomorphism between $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$. Let $A \subseteq X$ and $f\colon A \to Y$ be isotone as a mapping of the poset $(A, \leqslant_A)$ with the order $\leqslant_A := \leqslant_X \cap (A \times A)$. A mapping $g\colon X \to Y$ is said to be an *isotone extension* of f if g is isotone and $f(x) = g(x)$ holds for every $x \in A$.

The problem of extension of a mapping $f\colon A \to Y$ to an isotone mapping $g\colon X \to Y$ is usually considered under additional restrictions on f, g, X, A and Y. For example, in [3] an isotone extension is constructed in the case if X and Y are closed cones in topological vector spaces, A is the interior of the cone X, and g is continuous or semicontinuous.

Another typical situation is an extension of an isotone continuous mapping defined on an ordered or preordered topological space to an isotone continuous mapping on the compactification of this space. As it is noted in [16], such a kind of investigations is motivated, in particular,

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by attempts to transfer the casual relations on the ideal bondaries of Lorentz manifolds.

The problem of the monotone interpolation of monotone data which arises in numerical analysis (see, e.g., [23]), and the theorem on extension of a measure from a Boolean algebra to the corresponding σ-algebra (see, e.g., [17, §1.5]) are also should be pointed out. Moreover, it is necessary to note that an isotone extension of origin isotone data is carried out by interpolation of cubic splines [23], while an extension of the measure is naturally required its subadditivity.

In the present paper the problem of an isotone extension of a mapping $f\colon A \to Y$, $A \subseteq X$ to an isotone mapping from X to Y is investigated without algebraic or topological limitations.

It is proved that this problem is solvable for:

- every A, every $X \supseteq A$ and each isotone f if and only if Y is a complete lattice (Theorem 2.5);
- all Y and X, and each isotone f defined on a given $A \subseteq X$ if and only if A is a complete lattice (Theorem 2.8);
- all subsets A of a given X, every Y and each isotone f if and only if X is isomorphic to a cardinal sum of subsets of the set $\mathbb{Z}$ (Theorem 3.6);
- every A with cardinality less than a given cardinal number α, every $X \supseteq A$ and each isotone $f\colon A \to Y$ if and only if Y is a $(< \alpha)$-quasilattice (Definition 4.7 and Theorem 4.11).

It is also shown that for every X and every bounded $A \subseteq X$, each isotone mapping $f\colon A \to Y$ can be extended to an isotone mapping $g : X \to Y$ such that $g(A) \subseteq [f(0_A), f(1_A)]$ if and only if Y is a complete local lattice (Definition 5.1 and Theorem 5.2).

Some facts described in the paper can be formulated using the language of category theory. Fore example, Theorem 2.5 claims that the complete lattices coincide with the injective objects of the category **Pos**. (See also [1] for similar results in categories other than **Pos**.) Nevertheless, we tried to make the presentation as simple as possible and do not use the Category Theory language in what follows.

2. Complete lattices and isotone extensions

Recall that a poset $(Y, \leqslant_Y)$ is a *complete lattice* if its nonempty subsets have both the supremum and the infimum. If the existence of the supremum and the infimum is required only for finite nonempty sets $B \subseteq Y$, then, by definition, $(Y, \leqslant_Y)$ is a *lattice*.

Let $(X, \leqslant_X)$ be a nonempty poset. In what follows we denote by 1_X and by 0_X the greatest element of X and, respectively, the least element of X if these elements exist. If A is the empty subset of X which has 0_X and 1_X, then we write

$$\inf\nolimits_X A := 1_X \quad \text{and} \quad \sup\nolimits_X A := 0_X. \tag{2.1}$$

Let $(Y, \leqslant_Y)$ be a poset and let $A \subseteq Y$. The *upper cone* of the set A is the subset A^{Δ} of the set Y, such that

$$y \in A^{\Delta} \Leftrightarrow a \leqslant_Y y$$

for every $a \in A$ (see, e.g., [22, p. 6]). The *lower cone* A^{∇} can be defined by duality. If A is a singleton, $A = \{a\}$, then

$$a^{\Delta} := A^{\Delta} \ \text{ and } \ a^{\nabla} := A^{\nabla}.$$

The elements of A^{Δ} are called the *majorants* of the set A, and, respectively, the elements of A^{∇} are the *minorants* of A.

It is known, that every poset $(X, \leqslant_X)$ is isomorphically embeddable in the boolean $(\mathfrak{B}(X), \subseteq)$ of X via the mapping

$$\nabla_X \colon X \to \mathfrak{B}(X), \quad \nabla_X(t) = t^{\nabla}, \quad t \in X.$$

Thus, the following lemma holds.

Lemma 2.1. *For every poset $(Y, \leqslant_Y)$ there is a complete lattice $(M, \leqslant_M\ )$ and $A \subseteq M$ such that A is isomorphic to Y. If the poset Y is finite, then the lattice M also is finite.*

Remark 2.2. The existence of the Dedekind–MacNeille completion (see., e.g., [18, p. 86]) is a much more deep result than Lemma 2.1 but this result is not used in the paper .

Lemma 2.3. *Let $(X, \leqslant_X)$ be a complete lattice and $A \subseteq X$. If the identical mapping $\mathrm{id}_A \colon A \to A$ has an isotone extension $g \colon X \to A$, then $(A, \leqslant_A)$ is a complete lattice with respect to the order $\leqslant_A = \leqslant_X \cap (A \times A)$.*

Proof. Let $g\colon X \to A$ be an isotone extension of $\mathrm{id}_A\colon A \to A$. We prove that there are $\sup_A B$ and $\inf_A B$ for every nonvoid subset B of A. Since X is a complete lattice, there is $\bar{b} := \sup_X B$. Let us show that $g(\bar{b})$ is $\sup_A B$. Indeed, since the inequality $x \leqslant_X \bar{b}$ holds for every $x \in B$ and g is an isotone extension of the mapping id_A, we have

$$x = g(x) \leqslant_A g(\bar{b})$$

for every $x \in B$. Consequently, $g(\bar{b})$ is a majorant of B in $(A, \leqslant_A)$. Let t be an arbitrary majorant of B in $(A, \leqslant_A)$. Then t is a majorant of B in $(X, \leqslant_X)$. Since $\bar{b} = \sup_X B$, we have $\bar{b} \leqslant_X t$. Hence,

$$g(\bar{b}) \leqslant_A g(t) = \mathrm{id}_A(t) = t.$$

It follows that $g(\bar{b}) = \sup_A B$. The existence of the infimum can be obtained by duality. □

Definition 2.4. Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be posets, $A \subseteq X$. Let $f\colon A \to Y$ be an isotone mapping and let $\Psi\colon X \to Y$ be an isotone extension of f. We shall say that Ψ is an *upper (lower) isotone extension* of f if for every isotone extension $\mathcal{X}\colon X \to Y$ of f and every $x \in X$ the inequality

$$\mathcal{X}(x) \leqslant_Y \Psi(x) \quad (\Psi(x) \leqslant_Y \mathcal{X}(x))$$

holds.

The construction of the upper isotone extension which is used in the proof of the following theorem, was taken from [5].

Let $(A, \leqslant_A)$ be a poset. Denote by $\mathbf{S}_A$ the class of all posets $(X, \leqslant_X)$, for which $A \subseteq X$ and $\leqslant_A = \leqslant_X \cap (A \times A)$.

Theorem 2.5. *Let $(Y, \leqslant_Y)$ be a nonempty poset. The following statements are equivalent.*

(i) *$(Y, \leqslant_Y)$ is a complete lattice.*

(ii) *Each isotone mapping $f\colon A \to Y$ has a lower isotone extension on X for every poset $(X, \leqslant_X) \in \boldsymbol{S}_A$.*

(iii) *Each isotone mapping $f\colon A \to Y$ has an upper isotone extension on X for every poset $(X, \leqslant_X) \in \boldsymbol{S}_A$.*

(iv) *Each isotone mapping $f\colon A \to Y$ has an isotone extension on X for every poset $(X, \leqslant_X) \in \boldsymbol{S}_A$.*

Proof. **(i)⇒(ii)**. Suppose (i) holds. Let $(X, \leqslant_X)$ be a poset and let $f\colon A \to Y$ be an isotone mapping. For every $x \in X$ define

$$f_*(x) := \sup_Y\{f(t)\colon t \in A \cap x^\nabla\}. \tag{2.2}$$

By (2.1), we have

$$f_*(x) = 0_Y, \tag{2.3}$$

if $A \cap x^\nabla = \varnothing$. The mapping

$$X \ni x \mapsto f_*(x) \in Y$$

is isotone. Indeed, if $x_1 \leqslant_X x_2$, then $x_1^\nabla \subseteq x_2^\nabla$ holds. The last inclusion and (2.2) imply $f_*(x_1) \leqslant_Y f_*(x_2)$. We must prove that f_* is an extension of f. Let $x \in A$. Then $t \leqslant_A x$ holds for every $t \in A \cap x^\nabla$. Hence, from the isotonicity of f, it follows that $f(t) \leqslant_Y f(x)$ for every $t \in A \cap x^\nabla$. Now, using (2.2), we obtain

$$f_*(x) \leqslant_Y f(x).$$

To prove the equality $f_*(x) = f(x)$ it is sufficient to note that

$$f_*(x) = \sup_Y\{f(t)\colon t \in A \cap x^\nabla\} \geqslant_Y f(x)$$

because $x \in A \cap x^\nabla$ for $x \in A$. Thus, f_* is an isotone extension of f.

Let $\mathcal{X}\colon X \to Y$ be an arbitrary isotone extension of f. To prove that f_* is the lower isotone extension of f we show that

$$f_*(x) \leqslant_Y \mathcal{X}(x) \tag{2.4}$$

for every $x \in X$. If $A \cap x^\nabla = \varnothing$, then (2.4) follows from (2.3). Let $A \cap x^\nabla \neq \varnothing$ and let $t \in A \cap x^\nabla$. Since $\mathcal{X}$ is an isotone mapping, the inequality $\mathcal{X}(t) \leqslant_Y \mathcal{X}(x)$ holds. Moreover, since $\mathcal{X}$ extends f and $t \in X$, we have

$$f(t) \leqslant_Y \mathcal{X}(x)$$

for every $t \in A \cap x^\nabla$. Inequality (2.4) follows from the last inequality and (2.2).

(i)⇒(iii). The proof of this implication can be obtained by duality. Note, in particular, that the upper isotone extension f^* of an isotone mapping $f\colon A \to Y$ satisfies the equality

$$f^*(x) = \inf_Y \{f(t)\colon t \in A \cap x^\Delta\} \tag{2.5}$$

for every $x \in X$ if $(Y, \leqslant_Y)$ is a complete lattice.

The implications (iii)⇒(iv) and (ii)⇒(iv) are obvious.

(iv)⇒(i). Let (iv) hold. We must show that $(Y, \leqslant_Y)$ is a complete lattice. By Lemma 2.1, there exists a complete lattice $(X, \leqslant_X) \in \mathbf{S}_Y$. It follows from (iv) that the identical mapping $\mathrm{id}_Y \colon Y \to Y$ can be extended to an isotone mapping $g\colon X \to Y$. Now, by Lemma 2.3, the poset $(Y, \leqslant_Y)$ is a complete lattice. □

Let $(Y, \leqslant_Y)$ be a complete lattice, $(X, \leqslant_X)$ be a poset, $A \subseteq X$, and let $f\colon A \to Y$ be an isotone mapping. Denote by $\mathbf{C}_f^X$ the set of all isotone extensions $g\colon X \to Y$ of the mapping f. Then, the equalities

$$f_*(x) = \inf_Y\{g(x)\colon g \in \mathbf{C}_f^X\} \quad \text{and} \quad f^*(x) = \sup_Y\{g(x)\colon g \in \mathbf{C}_f^X\} \tag{2.6}$$

hold. Indeed, if f_* is defined by the first formula of (2.6), then

$$f_*(x_1) \leqslant_Y g(x_1) \leqslant_Y g(x_2)$$

holds for all $x_1, x_2 \in X$ with $x_1 \leqslant_X x_2$, and every $g \in \mathbf{C}_f^X$. Now, from the inequality $f_*(x_1) \leqslant_Y g(x_2)$ it follows that $f_*(x_1) \leqslant_Y f_*(x_2)$, i.e. $f_*\colon X \to Y$ is an isotone mapping. The equality $f_*(x) = f(x)$ is obvious for $x \in A$. Hence, $f_*(x)$ is the lower isotone extension of f. The dual statement shows that the f^* is the upper isotone extension of f.

We define a partial order on $\mathbf{C}_f^X$ by setting

$$(g \leqslant_{\mathbf{C}_f^X} \psi) \Leftrightarrow (g(y) \leqslant_Y \psi(y) \text{ for every } y \in Y).$$

Corollary 2.6. *Let $(Y, \leqslant_Y)$ be a complete lattice. Then for every isotone mapping $f\colon A \to Y$ and every $(X, \leqslant_X) \in \boldsymbol{S}_A$ the poset $(\boldsymbol{C}_f^X, \leqslant_{\boldsymbol{C}_f^X})$ is a complete lattice with*

$$0_{\boldsymbol{C}_f^X} = f_* \quad \textit{and} \quad 1_{\boldsymbol{C}_f^X} = f^*. \tag{2.7}$$

Remark 2.7. There are several results which are closely connected with Theorem 2.5. The Sikorski Theorem, saying that every homomorphism from a subalgebra of a boolean algebra to a complete boolean algebra can be extended to a homomorphism of whole algebra (see [20] or [21, p. 228]). A theorem of Fofanova, which says that every isotone mapping from a poset to a complete lattice is extendable to an isotone mapping defined on the lattice which is freely generated by this poset (see [9]). By G. M. Bergman and G. Grätzer [2] every isotone mapping from arbitrary partial lattice P to a lattice M has an isotone extension on its free lattice Free (P) if and only if M is complete.

The necessary and sufficient conditions under which there exists an isotone extension of an isotone mapping $f\colon A \to Y$ with finite A will be presented in the fourth section of the paper (see Theorem 4.6). Note only that there is a poset $(Y, \leqslant_Y)$ which is not a lattice but admits such extensions for every X and every finite $A \subseteq X$.

Theorem 2.8. *A poset $(A, \leqslant_A)$ is a complete lattice, if and only if for every isotone mapping $f\colon A \to Y$ and every $(X, \leqslant_X) \in \boldsymbol{S}_A$ there is an isotone mapping $\Psi\colon X \to Y$ such that $\Psi_{|A} = f$.*

Proof. Let $(A, \leqslant_A)$ be a complete lattice, $(Y, \leqslant_Y)$ be a poset, $f\colon A \to Y$ be an isotone mapping and let $(X, \leqslant_X) \in \mathbf{S}_A$. By Theorem 2.5, the identical mapping $\mathrm{id_A}\colon A \to A$ has an isotone extension $g\colon X \to A$. Then the mapping $X \xrightarrow{g} A \xrightarrow{f} Y$ is an isotone extension of f.

Now suppose that every isotone mapping $f\colon A \to Y$ has an isotone extension $g\colon X \to Y$ for every $(X, \leqslant_X) \in \mathbf{S}_A$. By Lemma 2.1, there is a complete lattice $(X, \leqslant_X) \in \mathbf{S}_A$. Let $Y = A$ with $\leqslant_Y = \leqslant_A$ and let $f = \mathrm{id}_A$. Then f has an isotone extension $g\colon X \to A$. By Lemma 2.3, A is a complete lattice. □

3. Chains and isotone extensions

Let $(X, \leqslant_X)$ be a poset. Elements $a, b \in X$ are *comparable* if $a \leqslant_X b$ or $b \leqslant_X a$. We define the binary relation ρ on X by the rule: $a \rho b$ holds if and only if there is a finite sequence $a_1, \dots, a_m$ such that $a_1 = a$, $a_m = b$ and the elements a_i and a_{i+1} are comparable for $i = 1, \dots, m-1$. Then ρ is an equivalence relation on X. The equivalence classes of the relation ρ are called the *connected components* of the poset $(X, \leqslant_X)$. In particular, if $x \rho y$ holds for all $x, y \in X$, then we say that X is connected.

Recall that a poset X is called a *linearly ordered set* or a *chain* if every two distinct elements of X are comparable.

Lemma 3.1. *Let $(X, \leqslant_X)$ be a connected poset. If $(X, \leqslant_X)$ is not a chain, then there is $A \subseteq X$ such that the identical mapping* $\mathrm{id_A}\colon A \to A$ *cannot be extended to an isotone mapping from X to A.*

Proof. Assume that X is not a chain. Then there are incomparable $a, b \in X$. We set $A := \{a, b\}$ and show $\mathrm{id_A}\colon A \to A$ does not have any isotone extension on X.

Suppose that, on the contrary, there is an isotone mapping $g\colon X \to A$ with $g_{|A} = \mathrm{id_A}$. Since X is connected, there is a finite sequence $a_1, \dots, a_m$ such that $a_1 = a$, $a_m = b$ and the elements a_i and a_{i+1} are comparable for $i = 1, \dots, m-1$. Let us consider the finite sequence $g(a_1), \dots, g(a_m)$. Since g is an extension of $\mathrm{id_A}$ and $a_1 = a$, we have $g(a_1) = a$. The definition of the set A implies $g(a_2) = a$ or $g(a_2) = b$. If $g(a_2) = b$, then b is comparable with a which is a contradiction. Consequently, $g(a_2) = a$. The equalities

$$a = g(a_3) = \ldots = g(a_{m-1}) = g(a_m).$$

can be proved similarly, i.e., $a = g(a_m) = b$, which contradicts $a \neq b$. □

Let $(X, \leqslant_X)$ be a chain and let $x_1, x_2 \in X$. The element x_2 *covers* x_1, denote by $x_1 \sqsubset_X x_2$, if $x_1 \leqslant_X x_2$ and no element $x \in X$ lies strictly between x_1 and x_2, i.e., such that $x_1 <_X x <_X x_2$.

Lemma 3.2. *Let $(X, \leqslant_X)$ be a chain, $x_1 \in X$ and let $A := \{x \in X : x <_X x_1\}$. Suppose that x_1 is not 0_X and there are no element y in X such that $y \sqsubset_X x_1$. Then the identical mapping $\mathrm{id_A} \colon A \to A$ cannot be extended to an isotone mapping from X to A.*

Proof. Suppose that $g \colon X \to A$ is an isotone extension of $\mathrm{id_A} \colon A \to A$. Write

$$x_{-1} := g(x_1). \tag{3.8}$$

From the definition of the set A it follows that $x_{-1} <_X x_1$. Since $x_{-1} \sqsubset_X x_1$ does not hold, we have

$$x_{-1} <_X x_0 <_X x_1. \tag{3.9}$$

for some $x_0 \in A$. Relations (3.9) and (3.8) imply

$$g(x_{-1}) \leqslant_A g(x_0) \leqslant_A g(x_1) \quad \text{and} \quad x_{-1} \leqslant_X x_0 \leqslant_X x_{-1}$$

because g is an isotone extension of id_A. Thus, $x_{-1} = x_0$. The last equality contradicts (3.9). □

The next lemma is the dual statement to Lemma 3.2.

Lemma 3.3. *Let $(X, \leqslant_X)$ be a chain, $x_1 \in X$ and let $A := \{x \in X \colon x >_X x_1\}$. Suppose that x_1 is not 1_X, and there are no the element y in X such that $x_1 \sqsubset_X y$. Then the identical mapping $\mathrm{id_A} \colon A \to A$ cannot be extended to an isotone mapping from X to A.*

Denote by $\mathbb{Z}$ the set of all integer numbers with the standard order

$$\ldots - 2 \leqslant -1 \leqslant 0 \leqslant 1 \leqslant 2 \ldots.$$

The following lemma gives a characteristic property of the order types of subsets of the poset $(\mathbb{Z}, \leqslant)$.

Lemma 3.4. *Let $(X, \leqslant_X)$ be a linearly ordered set. The set $(X, \leqslant_X)$ is isomorphic to a subset of $(\mathbb{Z}, \leqslant)$ if and only if the next statement holds for every $B \subseteq X$ and every $b_1 \in B$.*

(i) If b_1 is not 0_B, then there is $b_0 \in B$ such that $b_0 \sqsubset_B b_1$, moreover, if b_1 is not 1_B, then there is $b_2 \in B$ such that $b_1 \sqsubset_B b_2$.

Proof. If $(X, \leqslant_X)$ is isomorphic to a subset of $(\mathbb{Z}, \leqslant)$, then (i) is trivial.

Conversely, let (i) hold. For every $x \in X$ denote by $x + 1$ the unique element of X which covers x in $(X, \leqslant_X)$, if $x \neq 1_X$. If $x = 1_X$, then write $x + 1 := x$. Let $a \in X$. Consider the set $A^+ = \{a + n : n \in \mathbb{N}\}$, where $a+0 = a$ and $a+(k+1) = (a+k)+1$. We claim that $A^+ = a^\Delta$. Otherwise, there is an element $b \in X$ such that $a \leqslant_X b$ and $b \notin A^+$. Then b has no $y \in A^+$ such that $y \sqsubset_{A^+} b$. Since $b \neq 0_B$, it contradicts condition (i). The set $A^- = \{a - n : n \in \mathbb{N}\}$ can be defined by duality. Hence, $A^- = a^\nabla$ holds. Consequently, we have $X = a^\nabla \cup a^\Delta = \{a + m \colon m \in \mathbb{Z}\}$ and the existence of an embedding of $(X, \leqslant_X)$ in $(\mathbb{Z}, \leqslant)$ follows. □

Remark 3.5. Statement (i) of Lemma 3.4 is equivalent to the fact that B contains 1_B if $B^\Delta \neq \varnothing$ and contains 0_B if $B^\nabla \neq \varnothing$.

The order types of subsets of the poset $(\mathbb{Z}, \leqslant)$ play an important role under investigations of the scattered sets [8, 12]. Lemma 3.4 can be derived from Theorem 5.37 [19], where the scattered sets are characterized through the so-called F-rank. However, we prefer the more simple proof given above.

Theorem 3.6. *Let $(X, \leqslant_X)$ be a poset. The following statements are equivalent.*

(i) Every connected component B of $(X, \leqslant_X)$ is isomorphic to a subset of $(\mathbb{Z}, \leqslant)$.

(ii) Every isotone mapping $f \colon A \to Y$ can be extended to an isotone mapping $g \colon X \to Y$ for every $A \subseteq X$ and every poset $(Y, \leqslant_Y)$.

Proof. **(i)⇒(ii).** Let (i) hold and let $\varnothing \neq A \subseteq X$. To prove (ii) it is sufficient to show that the identical mapping $\mathrm{id_A} \colon A \to A$ has an isotone extension $g \colon X \to A$. Indeed, in this case, as it was noted in the proof of Theorem 2.8, for every isotone mapping $f \colon A \to Y$ the mapping $X \xrightarrow{g} A \xrightarrow{f} Y$ is an isotone extension of f.

At first, we consider the case when $(X, \leqslant_X)$ is connected. By statement (i), we can assume that $X \subseteq \mathbb{Z}$. The next alternatives hold for every $A \subseteq X$:

(i_1) Neither 1_A nor 0_A exists in A;

(i_2) A contains 1_A but 0_A does not exist;

(i_3) A contains 0_A but 1_A does not exist;

(i_4) There exist 0_A and 1_A.

If (i_1) or (i_2) holds, then the required isotone mapping $g\colon X \to A$ can be given as

$$g(x) := \sup_A(x^\nabla \cap A), \ x \in X. \tag{3.10}$$

Since $A \subseteq X \subseteq \mathbb{Z}$, the mapping g is an isotone extension of $\mathrm{id_A}$. In the case where (i_3) or (i_4) holds, mapping (3.10) will be an isotone extension of $f\colon A \to A$ if we set $\sup_A(\varnothing) = 0_A$. Thus, (ii) holds if X is connected.

If X is disconnected, then X can be presented as the cardinal sum of its connected components X_α, $\alpha \in I$, where I is a set of indexes with $|I| \geqslant 2$ (see, e.g., [22, p. 9]).

Let x_0 be an arbitrary point of the set A and let $A_\alpha := A \cap X_\alpha$ for every $\alpha \in I$. We define $g\colon X \to A$ by the rule:

$$g(x) = \begin{cases} \sup_{A_\alpha}(x^\nabla \cap A_\alpha) & \text{if } x \in X_\alpha \text{ and } A_\alpha \neq \varnothing, \\ x_0 & \text{if } x \in X_\alpha \text{ and } A_\alpha = \varnothing. \end{cases} \tag{3.11}$$

Since $\{X_\alpha\colon \alpha \in I\}$ is a partition of the set X, $g(x)$ is defined for every $x \in X$. The mapping $g|_{X_\alpha}$ is an isotone extension of id_{A_α} for every $\alpha \in I$. (If $A_\alpha = \varnothing$, then the mapping id_{A_α} is empty as the mapping from the empty set. Consequently, every isotone mapping $X_\alpha \to A$ is an isotone extension of id_{A_α}.) The mapping g is isotone because if $x \leqslant_X y$, then there is $\alpha \in I$ such that $x, y \in X_\alpha$ and the mapping $g|_{X_\alpha}$ is isotone.

(ii)⇒(i). Let (ii) hold and let X_α be an arbitrary connected component of X. Then, for every $A \subseteq X_\alpha$, the mapping $\mathrm{id_A}\colon A \to A$ can be extended to an isotone mapping on X, and, consequently, on X_α. It follows from Lemma 3.1 that X_α is a chain. Using Lemma 3.2 and Lemma 3.3 with $X = X_\alpha$, we see that for every $B \subseteq X_\alpha$ and every $b_1 \in B$ statement (i) of Lemma 3.4 holds. Now, Lemma 3.4 implies statement (i) of the theorem. □

4. Isotone extension of mappings from subsets of bounded cardinality

In the present section the problem of isotone extension of mappings $f\colon A \to Y$ is considered under the condition

$$|A| < \alpha, \tag{4.12}$$

where $|A| = \operatorname{card} A$ and α is a given infinite cardinal number. In particular, for $\alpha = \aleph_0$, where as usually $\aleph_0$ is the first infinite cardinal, condition (4.12) is equivalent to the finiteness of A. The problem of isotone extension with $\alpha > \aleph_0$ is, in a sense, less elementary. Thus, we begin with the case $\alpha = \aleph_0$.

To formulate a criterion of the solvability of the problem it is necessary to give a suitable generalization of lattices.

Definition 4.1. A poset $(Y, \leqslant_Y)$ is a *quasilattice* if for all finite $A, B \subseteq Y$ satisfying the conditions

$$A \subseteq B^{\nabla} \ \text{ and } \ B \subseteq A^{\Delta}, \tag{4.13}$$

there is $y^* \in Y$ such that

$$a \leqslant_Y y^* \leqslant_Y b \tag{4.14}$$

holds for all $a \in A$ and $b \in B$.

Remark 4.2. If $A = \varnothing$, then from (4.13) we obtain that for every finite $B \subseteq Y$ there is a minorat. The existence of majorants for every finite $A \subseteq Y$ follows by duality.

Remark 4.3. Double inequality (4.14) holds for all $a \in A$ and $b \in B$ if and only if it holds for all maximal elements a of the set A and all minimal elements b of the set B. Consequently, instead of finite $A, B \subseteq Y$, satisfying condition (4.13) it is sufficient to consider finite antichains $A, B \subseteq Y$ which satisfy (4.13).

It is evident that every nonempty chain is a quasilattice. The converse is also true for finite quasilattices.

Proposition 4.4. *Every finite quasilattice is a chain.*

Proof. Let $(Y, \leqslant_Y)$ be a finite quasilattice and let $A \subseteq Y$. Write $B := A^{\Delta}$. Since the inclusion $A \subseteq A^{\Delta\nabla}$ holds, we have

$$A \subseteq B^{\nabla} \ \text{ and } \ B \subseteq A^{\Delta}.$$

Consequently, there is $y^* \in Y$ such that (4.14) holds for all $a \in A$ and $b \in A^{\Delta}$. Hence we have $y^* = \sup_Y A$. Using the duality principle we obtain the existence of $\inf_Y A$. $\square$

An example of a countable quasilattice which is not a lattice, will be given below after the proof of Theorem 4.6.

Lemma 4.5. *Let $(Y, \leqslant_Y)$ be a nonempty poset. Then $(Y, \leqslant_Y)$ is a quasilattice if and only if every isotone mapping $f\colon A \to Y$ can be extended to an isotone mapping from X to Y for every finite poset $(X, \leqslant_X) \in \boldsymbol{S}_A$.*

Proof. **Necessity.** Let $(Y, \leqslant_Y)$ be a quasilattice. We must show that for every finite poset $(X, \leqslant_X)$ and every $A \subseteq X$ every isotone mapping $f\colon A \to Y$ has an isotone extension on X. The proof is by induction on $|X|$. The existence of the desired extension is evident under $|X| = 1$. Let $m \in \mathbb{N}$. Assume that such an extension exists for $|X| \leqslant m$. Let $|X| = m + 1$, $A \subseteq X$ and let $f\colon A \to Y$ be an isotone mapping. If $A = X$, then there is nothing to prove. Suppose there is $x_1 \in X \setminus A$. Write $A_1 := X \setminus \{x_1\}$. By induction assumption, there exists an isotone extension $g_1\colon A_1 \to Y$ of f. We set $g(x) = g_1(x)$ for all $x \in A_1$. It remains to define $g(x_1)$. Let $\underline{X}_1 := x_1^{\nabla} \setminus \{x_1\}$ and let $\overline{X}_1 := x_1^{\Delta} \setminus \{x_1\}$. Write

$$\underline{B} := g_1(\underline{X}_1) \text{ and } \overline{B} := g_1(\overline{X}_1).$$

In accordance with the isotonicity of the mapping g_1, the inequality $\underline{b} \leqslant_Y \overline{b}$ holds for all $\underline{b} \in \underline{B}$ and $\overline{b} \in \overline{B}$. Now, using that $(Y, \leqslant_Y)$ is a quasilattice, we can find $y^* \in Y$ such that

$$\underline{b} \leqslant_Y y^* \leqslant_Y \overline{b} \tag{4.15}$$

holds for all $\underline{b} \in \underline{B}$ and $\overline{b} \in \overline{B}$. Let us define the mapping g at the point x_1 as

$$g(x_1) := y^*.$$

It is evident that g is an extension of f on X. Moreover, g is isotone if

$$((x <_X x_1) \Rightarrow (g(x) \leqslant_Y y^*)) \text{ and } ((x_1 <_X x) \Rightarrow (y^* \leqslant_Y g(x))). \tag{4.16}$$

Let $x <_X x_1$. Then $x \in \underline{X}_1$, so that $g(x) \in \underline{B}$ holds. Since (4.15) holds for every $\underline{b} \in \underline{B}$, we have $g(x) \leqslant_Y y^*$. The second implication in (4.16) can be proved similarly. Thus if $(Y, \leqslant_Y)$ is a quasilattice, then there exists a required isotone extension.

Sufficiency. Suppose that every isotone mapping, defined on an arbitrary subset of an finite poset X and taking the values in $(Y, \leqslant_Y)$, can be extended to an isotone mapping from X to Y. We must show that $(Y, \leqslant_Y)$ is a quasilattice. Suppose A and B are two finite subsets of Y for which (4.13) holds. It is sufficient to find y^* such that (4.14) holds for all $a \in A$ and $b \in B$. Let $C = A \cup B$ and let $(L, \leqslant_L)$ be a finite lattice such that $C \subseteq L$ and

$$(C \times C) \cap \leqslant_L = (C \times C) \cap \leqslant_Y . \tag{4.17}$$

Note that the existence of a finite lattice $(L, \leqslant_L)$ satisfying (4.17) follows from Lemma 2.1. By Theorem 2.5, the mapping $\mathrm{in}\colon C \to Y$, $\mathrm{in}(c) = c$ for every $c \in C$, can be extended to an isotone mapping $g\colon L \to Y$. Write $\bar{a} = \sup_L A$. Then

$$a \leqslant_L \bar{a} \leqslant_L b \tag{4.18}$$

for all $a \in A$ and $b \in B$. The isotonicity of g and (4.18) imply

$$a = g(a) \leqslant_Y g(\bar{a}) \leqslant_Y g(b) = b.$$

Consequently (4.14) holds with $y^* = g(\bar{a})$. □

Denote by FS the set of all finite subsets of the set $\mathbb{N}$.

Theorem 4.6. *Let $(Y, \leqslant_Y)$ be a nonempty poset. The following statements are equivalent.*

(i) $(Y, \leqslant_Y)$ is a quasilattice.

(ii) Every isotone mapping $f\colon A \to Y$ has an isotone extension on X for every finite poset $(X, \leqslant_X) \in \boldsymbol{S}_A$.

(iii) Every isotone mapping $f\colon A \to Y$ has an isotone extension on X for every poset $(X, \leqslant_X)$ and every finite $A \subseteq X$.

(iv) Every isotone mapping $f\colon A \to Y$ has an isotone extension on FS for every finite subset A of the poset $(FS, \subseteq)$.

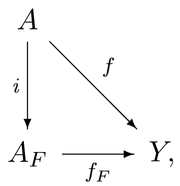
Proof. The logical equivalence (i)⇔(ii) is proved in Lemma 4.5. The implications (iii)⇒(ii) and (iii)⇒(iv) are trivial.

(ii)⇒(iii). Suppose that (ii) holds. Let us consider an arbitrary poset $(X, \leqslant_X)$, a finite set $A \subseteq X$ and an isotone mapping $f\colon A \to X$. Let $LX \in \mathbf{S}_X$ be a complete lattice and let LA be a sublattice of the lattice LX generated by the set A. Since LA is finite, by Lemma 4.5 there is an isotone mapping $g\colon LA \to Y$ with $g|_A = f$. Every finite lattice is complete. Consequently, by Theorem 2.5 there is an isotone extension $\Psi\colon LX \to LA$ of the identical mapping $\mathrm{id}_L\colon LA \to LA$. Let $\mathrm{in}\colon X \to LX$ be the embedding of X in LX with $\mathrm{in}(x) = x$ for every $x \in X$. It is easy to show that the mapping

$$X \xrightarrow{\mathrm{in}} LX \xrightarrow{\Psi} LA \xrightarrow{g} Y$$

is a required isotone extension of the mapping f.

(iv)⇒(ii). Let (iv) hold, let $(X, \leqslant_X)$ be a finite poset, $A \subseteq X$ and let $f\colon A \to Y$ be an isotone mapping. Denote by X_F a subset of FS which is isomorphic to the poset $(X, \leqslant_X)$. Then there is $A_F \subseteq X_F$ and an isotone mapping $f_F\colon A_F \to Y$ such that A_F and A are isomorphic and the diagram



is commutative, where i is an isomorphism between A and A_F. The existence of an isotone extension of the mapping f_F on X_F follows from (iv). □

Example. Let $\mathbb{Q}$ be the set of all rational numbers with the usual order $\leqslant$. Write

$$Q = (\mathbb{Q} \setminus \{0\}) \cup \{0_1, 0_2\},$$

where $\mathbb{Q} \cap \{0_1, 0_2\} = \varnothing$. Define on Q an order $\leqslant_Q$ by the rule:

if $x, y \in \mathbb{Q} \setminus \{0\}$, then $(x \leqslant_Q y) \Leftrightarrow (x \leqslant y)$;

if $x \in \mathbb{Q} \setminus \{0\}$, $y \in \{0_1, 0_2\}$, then

$$(x \leqslant_Q y) \Leftrightarrow (x \leqslant 0) \ \text{ and } \ (y \leqslant_Q x) \Leftrightarrow (0 \leqslant x);$$

if $x, y \in \{0_1, 0_2\}$, then $(x \leqslant_Q y) \Leftrightarrow (x = y)$.

The poset $(Q, \leqslant_Q)$ is not a lattice because the set $\{0_1, 0_2\}$ has no the supremum in Q. It is easy to prove that $(Q, \leqslant_Q)$ is a quasilattice. Consequently, from Theorem 4.6 it follows that every isotone mapping $f\colon A \to Q$ can be extended to an arbitrary $(X, \leqslant_X) \in \mathbf{S}_A$ if A is finite.

Let us consider now the problem of the isotone extension of isotone mappings defined on A which satisfies (4.13) with $\alpha > \aleph_0$.

Definition 4.7. Let α be an infinite cardinal. A poset $(Y, \leqslant_Y)$ is a $(< \alpha)$-*quasillatice*, if for all $A, B \subseteq Y$, satisfying (4.13) and the inequalities $|A| < \alpha$, $|B| < \alpha$, there is $y^* \in Y$, such that (4.14) holds for all $a \in A$ and $b \in B$.

It is clear that the quasilattices are precisely the $(< \aleph_0)$-quasilattices. The quasilattice from Example 4 is a $(< \aleph_0)$-quasilattice, but it is not a $(< \mathfrak{c})$-quasilattice (here, as usual, $\mathfrak{c}$ is the cardinality of the continuum).

The next definition is a counterpart of Definition 2.1 in [11, Section 5].

Definition 4.8. Let β be an infinite cardinal. A poset P is $(<\beta)$-universal, if for every poset X with $|X| < \beta$ there is $T \subseteq P$ such that T is isomorphic to X.

Remark 4.9. The poset $(\mathbb{N}, \subseteq)$ is $(<\chi_0)$-universal. Some $(<\chi_0)$-universal posets other than $(FS, \subseteq)$ can be found in [13]. It is easy to see that we can use any $(<\chi_0)$-universal poset instead of $(FS, \subseteq)$ in statement (iv) of Theorem 4.6.

Remark 4.10. In Theorem 4.11, that is the main result of the section, we assume the following:

- every infinite cardinal β is identified with the smallest ordinal which has the cardinality β, i.e., $\beta = \aleph_\alpha$, where α is an ordinal number;
- $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α, i.e., the generalized continuum hypothesis (GCH) holds;
- the ordinal numbers are transitive sets which are well-ordered by the relation $\in$. In particular, $\beta = \aleph_\alpha$ is the well-ordered set of all ordinals which are strictly less than β (see, e.g., [14, Chapter 2]).

To find isotone extensions of mappings $f\colon A \to Y$ we shall use the following construction. Let I be a chain. Suppose for every $i \in I$ there exist a set $A_i \supseteq A$ and a mapping $g_i\colon A_i \to Y$, such that $g_i|_A = f$, $A_i \subseteq A_j$ and $g_j|_{A_i} = g_i$ for $i \leqslant_I j$. Then, for the set $X := \bigcup_{i\in I} A_i$, there exists a unique mapping $g\colon X \to Y$ such that $g|_{A_i} = g_i$ for every $i \in I$. We denote this mapping by the symbol $\operatorname*{Lim}_{i\in I} g_i$. It is easy to see, that if X is equipped with an order $\leqslant_X$, and every A_i has an order $\leqslant_{A_i}$ such that

$$\leqslant_{A_i} = (A_i \times A_i) \cap \leqslant_X,$$

and the mappings g_i are isotone, then $\operatorname*{Lim}_{i\in I} g_i$ is an isotone extension of f on X.

Theorem 4.11. *(GCH) Let $(Y, \leqslant_Y)$ be a nonempty poset and let α be an infinite cardinal. The following statements are equivalent.*

(i) $(Y, \leqslant_Y)$ is a $(<\alpha)$-quasilattice.

(ii) For every poset $(X, \leqslant_X)$ with $|X| < \alpha$ and every $A \subseteq X$, each isotone mapping $f\colon A \to Y$ has an isotone extension on X.

(iii) For every poset $(X, \leqslant_X)$ and every $A \subseteq X$ with $|A| < \alpha$, each isotone mapping $f\colon A \to Y$ has an isotone extension on X.

(iv) Let $(P, \leqslant_P)$ be $(<\alpha)$-universal. Then for every $A \subseteq P$ with $|A| < \alpha$, each isotone mapping $f\colon A \to Y$ has an isotone extension on P

Proof. **(i)⇒(ii).** Let $(Y, \leqslant_Y)$ be a $(<\alpha)$-quasilattice, $(X, \leqslant_X)$ be a poset with $|X| < \alpha$, $A \subseteq X$, $X \setminus A \neq \varnothing$ and let a mapping $f\colon A \to Y$ be isotone. To prove that f has an isotone extension on X, we define on $X \setminus A$ a well-ordering $\preccurlyeq$. For every $t \in X \setminus A$ denote by A_t the set $\{x \in X \setminus A : x \preccurlyeq t\}$. There is $x_i \in X \setminus A$ such that for every $x_j \prec x_i$ the mapping f has an isotone extension $g_{x_j}\colon A \cup A_{x_j} \to Y$ with $g_{x_j}|_{A_{x_k}} = g_{x_k}$ if $x_k \preccurlyeq x_j$. The set of such x_i is nonempty. In particular, it contains the least element of $(X \setminus A, \preccurlyeq)$. Write

$$g^{\circ}_{x_i} := \operatorname*{Lim}_{x_j \in A^{\circ}_{x_i}} g_{x_j}, \tag{4.19}$$

where $A^{\circ}_{x_i} := \{x \in X \setminus A \colon x \prec x_i\}$. Then $g^{\circ}_{x_i}\colon A \cup A^{\circ}_{x_i} \to Y$ is an isotone extension of f. Let us extend $g^{\circ}_{x_i}$ to an isotone mapping $g_{x_i}\colon A \cup A_{x_i} \to Y$. Write

$$\underline{X}_{x_i} := x_i^{\nabla} \cap (A \cup A^{\circ}_{x_i}) \ \text{ and } \ \overline{X}_{x_i} := x_i^{\Delta} \cap (A \cup A^{\circ}_{x_i}),$$

where x_i^{∇} and x_i^{Δ} are, respectively, the lower cone and the upper cone of the element x_i in $(X, \leqslant_X)$. It is evident that $\underline{x} \leqslant_X x_i \leqslant_X \overline{x}$ for all $\overline{x} \in \overline{X}_{x_i}$ and $\underline{x} \in \underline{X}_{x_i}$. For simplicity of notation, we write $\underline{B} := g^{\circ}_{x_i}(\underline{X}_{x_i})$ and $\overline{B} := g^{\circ}_{x_i}(\overline{X}_{x_i})$. Then the inequality $\underline{b} \leqslant_Y \overline{b}$ holds for all $\underline{b} \in \underline{B}$ and $\overline{b} \in \overline{B}$ and, moreover,

$$|\underline{B} \cup \overline{B}| \leqslant |\underline{X}_{x_i} \cup \overline{X}_{x_i}| \leqslant |X| < \alpha.$$

Since $(Y, \leqslant_Y)$ is a $(<\alpha)$-quasilattice, there is $y_* \in Y$ such that

$$\underline{b} \leqslant_Y y_* \leqslant_Y \overline{b}$$

holds for all $\underline{b} \in \underline{B}$ and $\overline{b} \in \overline{B}$. Let us define a mapping g_{x_i} on $A \cup A_{x_i}$ as

$$g_{x_i}(x) := \begin{cases} g^{\circ}_{x_i}(x) & \text{if } x \in A \cup A^{\circ}_{x_i} \\ y^* & \text{if } x = x_i. \end{cases} \tag{4.20}$$

Equalities (4.19) and (4.20) imply that the mapping g_{x_i} is isotone and

$$g_{x_k} = g_{x_i}|_{A \cup A_{x_k}} \tag{4.21}$$

holds for every $x_k \preccurlyeq x_i$. Using the transfinite induction, we see that for every $x_i \in X \setminus A_{x_j}$ there is $g_{x_i}\colon A \cup A_{x_i} \to Y$ which extends f and satisfies (4.21) for $x_k \preccurlyeq x_i$. It remains to set

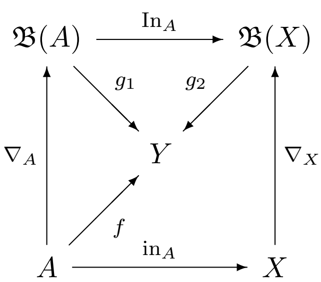
$$g := \operatorname*{Lim}_{x_i \in X \setminus A} g_{x_i} \tag{4.22}$$

and the required isotone extension of f on X is found.

(ii)⇒(iii). Let $(X, \leqslant_X)$ be a poset, $A \subseteq X$, $|A| < \alpha$ and let $f\colon A \to Y$ be an isotone mapping. Suppose that (ii) holds. We must find an isotone extension of f on X. If $|A| = |X|$, then (iii) is trivial. Assume that $|X|$ is the first cardinal, which is strictly greater than $|A|$. Let β be the smallest ordinal with $|\beta| = |X|$. We can define the well-ordering $\preccurlyeq$ such that $(X \setminus A, \preccurlyeq)$ is isomorphic to β. As in the proof of the implication (i)⇒(ii), we can extend f on $A^{\circ}_{x_i}$, where $x_i \in X \setminus A$, as in (4.19). (The designations from the first part of the proof are used.) In accordance with Remark 4.10, $|A^{\circ}_{x_i}| < \alpha$ holds. Thus,

$$|A \cup A^{\circ}_{i}| = |A| + |A^{\circ}_{x_i}| < \alpha.$$

Consequently, by (ii), there is an isotone extension $g_{x_i}\colon A \cup A_{x_i} \to Y$ of f. The required isotone extension $g\colon X \to Y$ of the mapping f we obtain as in (4.22). Let us prove the existence of an isotone extension of f in the case where X has an arbitrary cardinality. Due to the generalized continuum hypothesis, the cardinality of the boolean $\mathfrak{B}(A)$ is the first ordinal which is strictly greater than $|A|$. Using the previous case, we can embed A in $\mathfrak{B}(A)$ through $\nabla_A\colon A \to \mathfrak{B}(A)$, where $\nabla_A(x) = x^{\nabla}$ for every $x \in A$, and construct a mapping $g_1\colon \mathfrak{B}(A) \to Y$ for which $g_1 \circ \nabla_A = f$. Considering $\mathfrak{B}(A)$ as a subset of $\mathfrak{B}(X)$ and using Theorem 2.8, we find an isotone mapping $g_2\colon \mathfrak{B}(X) \to Y$ which is an isotone extension of g_1. The mapping $X \xrightarrow{\nabla_X} \mathfrak{B}(X) \xrightarrow{g_2} Y$ is the desired isotone extension of f,



where In_A and in_A are the corresponding embeddings.

(iii)⇒(i). Let (iii) hold, let A and B be subsets of Y with

$$\max(|A|, |B|) < \alpha,$$

and let $a \leqslant_Y b$ be valid for all $a \in A$ and $b \in B$. It is sufficient to prove that there is $y^* \in Y$ satisfying the double inequality

$$a \leqslant_Y y^* \leqslant_Y b$$

for all $a \in A$ and $b \in B$. Assume that such y^* does not exist. Then, using the embedding $\nabla_Y : Y \to \mathfrak{B}(Y)$ we can find $e^* \notin Y$ and a poset $(\overline{Y}, \leqslant_{\overline{Y}})$ such that

$$\overline{Y} = Y \cup e^* \quad \text{and} \quad \leqslant_Y = \leqslant_{\overline{Y}} \cap (Y \times Y)$$

and

$$a \leqslant_{\overline{Y}} e^* \leqslant_{\overline{Y}} b \tag{4.23}$$

for all $a \in A$ and $b \in B$. Write

$$X := A \cup B \cup \{e^*\} \text{ and } \leqslant_X := \leqslant_{\overline{Y}} \cap (X \times X).$$

By property (iii), the embedding $\text{in}\colon A \cup B \to Y$ with $\text{in}(t) = t$ for every $t \in A \cup B$ has an isotone extension $g\colon X \to Y$. It follows from (4.23) that

$$a \leqslant_Y g(e^*) \leqslant_Y b \tag{4.24}$$

for all $a \in A$, $b \in B$. Since $g(e^*) \in Y$, double inequality (4.24) contradicts the above assumption. Consequently, $(Y, \leqslant_Y)$ is a $(< \alpha)$-quasilattice.

The implication (iii)⇒(iv) is evident, and the verification of (iv)⇒(ii) can be done as in the proof of Theorem 4.6. □

Remark 4.12. In Definition 2.1 [11] of the β-universal sets $(P, \leqslant_P)$ it is required that every $(X, \leqslant_X)$ with $|X| = \beta$ is embeddable in P. If α is an ordinal number, for which $\beta = \aleph_\alpha$, then

$$(|X| \leqslant \beta) \Leftrightarrow (|X| < \aleph_{\alpha+1}).$$

Consequently, every poset $(P, \leqslant_P)$ which is β-universal in the sense of [11] with $\beta = \aleph_\alpha$ also is $(< \aleph_{\alpha+1})$-universal in the sense of Definition 4.8.

If a poset $(P, \leqslant_P)$ is $(< \beta)$-universal in the sense of Definition 4.8, then using Theorem 4.11, we can prove that $(P, \leqslant_P)$ is a $(< \beta)$-quasilattice. Thus, the universal posets investigated earlier in [4, 6, 7, 10, 15] are $(< \beta)$-quasilattices.

5. Complete local lattices and isotone extensions

Recall that a poset $(A, \leqslant_A)$ is bounded if it has both 0_A and 1_A [18, p. 7]. Let $(X, \leqslant_X)$ be a poset and let A be a bounded subset of X. Then

$$f(0_A) \leqslant_Y f(a) \leqslant_Y f(1_A) \tag{5.25}$$

holds for all isotone mappings $f\colon A \to Y$ and every $a \in A$. If $g\colon X \to Y$ is an isotone extension of $f : A \to Y$, then we say that g *preserves the extremal values* if

$$f(0_A) \leqslant_Y g(x) \leqslant_Y f(1_A) \tag{5.26}$$

holds for every $x \in X$.

Definition 5.1. A poset $(Y, \leqslant_Y)$ is a *complete local lattice* if the interval

$$[y_*, y^*]_Y = \{y \in Y : y_* \leqslant_Y y \leqslant_Y y^*\}$$

is a complete lattice for all $y_*, y^* \in Y$ satisfying $y_* \leqslant_Y y^*$.

It is evident that every complete lattice is a complete local lattice. Note also, that every antichain is a complete local lattice. Thus, a complete local lattice can be not a lattice.

The lexicographical sum of two antichains gives an example of the complete local lattice which is no even a quasilattice. The quasilattice Q from Example 4, is not a complete local lattice because the interval $[-1, 1]_Q$ is not a lattice.

Theorem 5.2. *Let $(Y, \leqslant_Y)$ be a nonempty poset. The following statements are equivalent.*

(i) $(Y, \leqslant_Y)$ *is a complete local lattice.*

(ii) For every poset $(X, \leqslant_X)$ and every bounded set $A \subseteq X$, each isotone mapping $f\colon A \to Y$ has an isotone extension $g : X \to Y$ which preserves the extremal values.

Proof. **(i)⇒(ii).** Let $(Y, \leqslant_Y)$ be a complete local lattice, $(X, \leqslant_X)$ be a poset, and let $A \subseteq X$ be bounded. For an arbitrary isotone mapping $f\colon A \to Y$ and we must find an isotone extension $g\colon X \to Y$ such that (5.26) holds for every $x \in X$. It is clear that (5.25) holds for every $a \in A$, i.e.,

$$f(A) \subseteq [f(0_A), f(1_A)]_Y.$$

Since $(Y, \leqslant_Y)$ is a complete local lattice, the interval $I = [f(0_A), f(1_A)]_Y$ is a complete lattice. By Theorem 2.5, there is an isotone extension

$$g_1\colon X \to I$$

of the mapping $A \ni a \mapsto f(a) \in I_Y$. Now, we can define $g\colon X \to Y$ as

$$X \xrightarrow{g_1} I_Y \xrightarrow{\text{in}} Y,$$

where $\mathrm{in}(y) = y$ for every $y \in I_Y$.

(ii)⇒(i). Let (ii) hold. Let us consider an arbitrary interval $I = [y_*, y^*]_Y \subseteq Y$, $y_* \leqslant_Y y^*$. It is evident that $y_* = 0_I$ and $y^* = 1_I$. Let $(L, \leqslant_L)$ be a complete lattice, such that $L \supseteq I$ and $\leqslant_I = (I \times I) \cap \leqslant_L$. Property (ii) implies the existence of an isotone extension of the identical mapping $\mathrm{id_I} \colon I \to I$ on L. Using Lemma 2.3, we see that I is a complete lattice. □

Let us consider now the problem of isotone extension for isotone mappings defined on bounded A with $|A| < \alpha$.

Definition 5.3. Let α be an infinite cardinal number. The poset $(Y, \leqslant_Y)$ is a *local* $(< \alpha)$*-quasilattice* if for all $A, B \subseteq Y$ which satisfy (4.13) and the condition $A^{\nabla} \neq \varnothing \neq B^{\Delta}$ and the inequality $\max(|A|, |B|) < \alpha$, there is $y^* \in Y$ such that (4.14) holds for all $a \in A$ and $b \in B$.

It is evident that every $(< \alpha)$-quasilattice is a local $(< \alpha)$-quasilattice. Note also that every complete local lattice $(Y, \leqslant_Y)$ is a local $(< \alpha)$-lattice for every α. Indeed, let $A, B \subseteq Y$ satisfy (4.13) and the inequality $\max(|A|, |B|) < \alpha$. Let b^* be a majorant of B and let a_* be a minorant of A. Then A and B are subsets of the complete lattice $I = [a_*, b^*]_Y$ and (4.14) holds for all $a \in A$ and $b \in B$ with $y^* = \inf_I B$.

The following lemma directly follows from Definition 4.7 and Definition 5.3.

Lemma 5.4. *Let α be an infinite cardinal, $(Y, \leqslant_Y)$ be a local $(< \alpha)$-quasilattice and let $y_*, y^* \in Y$ with $y_* \leqslant_Y y^*$. Then $I = [y^*, y_*]_Y$ is an $(< \alpha)$-quasilattice with respect to the order $\leqslant_I = (I \times I) \cap \leqslant_Y$.*

Theorem 5.5. *Let α be an infinite cardinal and let $(Y, \leqslant_Y)$ be a nonempty poset. The following statements are equivalent.*

(i) $(Y, \leqslant_Y)$ is a local $(< \alpha)$-quasilattice.

(ii) For every poset $(X, \leqslant_X)$ and every bounded $A \subseteq X$ which satisfies the inequality $|A| < \alpha$, each isotone mapping $f \colon A \to Y$ has an isotone, preserving the extremal values extension on X.

(iii) For every poset $(X, \leqslant_X)$ with $|X| < \alpha$ and every bounded $A \subseteq X$, each isotone mapping $f \colon A \to Y$ has an isotone, preserving the extremal values extension on X.

(iv) Let $(P, \leqslant_P)$ be $(< \alpha)$-universal. Then for every bounded $A \subseteq P$ with $|A| < \alpha$ each isotone mapping $f \colon A \to Y$ has an isotone preserving the extremal values extension on P.

Proof. **(i)⇒(ii)**. Let (i) hold. Let $(X, \leqslant_X)$ be a poset and let $A \subseteq X$ be bounded and satisfy $|A| < \alpha$. Let us consider an arbitrary isotone mapping $f\colon A \to Y$. Write $I := [f(0_A), f(1_A)]_Y$. By Lemma 5.4, the poset $(I, \leqslant_I)$ is a $(< \alpha)$-quasilattice. Consequently, from Theorem 4.11 it follows that the mapping

$$A \ni a \mapsto f(a) \in I$$

has an isotone extension $g\colon X \to I$. It is evident that g preserves the extremal values,

$$f(0_A) \leqslant_Y g(x) \leqslant_Y f(1_A)$$

for every $x \in X$. The mapping $X \xrightarrow{g} I \xrightarrow{\text{in}} Y$ gives an isotone extension of f which preservs the extremal values.

The implication (ii)⇒(iii) is evident. The implication (iii)⇒(i) can be obtained by simple modification of the proof of Theorem 4.11. □

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