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Pointwise estimates of solutions to weighted porous medium and fast diffusion equations via weighted Riesz potentials

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Abstract. For the weighted parabolic equation

$$v\,(x)\,u_t - div(\omega(x)u^{m-1}\nabla u) = f(x,t)\,,\ u \geq 0\,,\ m \neq 1$$

we prove the local boundedness for weak solutions in terms of the weighted Riesz potential of the right-hand side of equation.

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1. Introduction

The purpose of this paper is to study the local properties of solutions to weighted parabolic equations whose simplest example is

$$v\,(x)\,u_t - div(\omega(x)u^{m-1}\nabla u) = f(x,t)\,,\ u \geq 0\,,\ m \neq 1 \tag{1.1}$$

in $\Omega_T = \Omega \times (0,T)$, where $v(x)$, $w(x)$ belong to the corresponding Muckenhoupt classes, Ω is a bounded domain in R^n, $n \geq 2$, $0 < T < +\infty$, and $f \in L^1(\Omega_T)$.

In the non weighted case, i. e. $v(x) \equiv w(x) = const$, this class of equations has numerous applicantions and has been attracting attention for several decades (see, e. g. the monographs [4, 28, 49, 50, 68] and references therein).

More generally we deal with the parabolic equations of the type

$$v\,(x)\,u_t - divA(x,t,u,\nabla u) = f(x,t) \text{ in } \Omega_T\,, \tag{1.2}$$

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where the vector-field $A = (a_1, a_2, ..., a_n) : \Omega_T \times R^1 \times R^n \to R^n$ is Lebesgue measurable with respect to $(x,t) \in \Omega_T$ for all $(u,\xi) \in R^1 \times R^n$, and continuous with respect to (u,ξ) for a. a. $(x,t) \in \Omega_T$. We also assume that the following structure conditions are satisfied with some positive constants K_1, K_2

$$A(x,t,u,\xi)\xi \geq K_1 w(x) u^{m-1} |\xi|^2 \ , \ u \geq 0$$

$$|A(x,t,u,\xi)| \leq K_2 w(x) u^{m-1} |\xi| \ , \ m \neq 1 \tag{1.3}$$

For the function $f(x,t)$ we assume that $f \in L^1(\Omega_T)$ and $v(x), w(x) \geq 0$, $v(x) \in A_\infty$, $w(x) \in A_2$ where A_p, $1 < p < \infty$ denotes the Muckenhoupt class. This means that

$$sup \frac{w(B)}{|B|} \left(\frac{1}{|B|} \int_B \omega^{-\frac{1}{p-1}}(x) \ dx \right)^{p-1} = K_{p,w} < +\infty \ , \ w(B) = \int_B \omega \ dx \ ,$$

where the supremum is taken over all balls $B \subset R^n$. And we say that $v(x) \in A_\infty$ if there exists $p_0 > 1$ such that $v(x) \in A_{p_0}$.

As an immediate consequence of the definition of A_p class one has

$$\frac{w(B_\rho(y))}{w(B_r(y))} \leq K_{p,w} \left(\frac{\rho}{r} \right)^{np}$$

for any ball $B_r(y) \subset B_\rho(y)$.

Moreover (see [43] for the details), there exist constants $K_3 > 0$, $0 < \xi_1 \leq 1$, $0 < \xi_2 < p$ depending only on $n, p, K_{p,w}$ such that

$$K_3^{-1} \left(\frac{|B_\rho(y)|}{|E|} \right)^{\xi_1} \leq \frac{w(B_\rho(y))}{w(E)} \leq K_3 \left(\frac{|B_\rho(y)|}{|E|} \right)^{\xi_2}$$

for any ball $B_\rho(y)$ and $E \subset B_\rho(y)$

Further we will assume that

$$K_4^{-1} \left(\frac{\rho}{r} \right)^{n\nu_1} \leq \frac{v(B_\rho(y))}{v(B_r(y))} \leq K_4 \left(\frac{\rho}{r} \right)^{n\nu} \tag{1.4}$$

$$K_4^{-1} \left(\frac{\rho}{r} \right)^{n\mu_1} \leq \frac{w(B_\rho(y))}{w(B_r(y))} \leq K_4 \left(\frac{\rho}{r} \right)^{n\mu} , \tag{1.5}$$

with some positive $K_4, \nu_1, \mu_1, \nu, \mu$ and for balls $B_r(y) \subset B_\rho(y)$. Our next assumption is a relation between v and w. Fix $y \in \Omega$ and R such that $B_{8R}(y) \subset \Omega$ and set $\psi_y(r) := r^2 \frac{v(B_r(y))}{w(B_r(y))}$, $0 < r \leq R$.

We suppose that $\psi_y(r)$ is strictly increasing for $r \in (0, R]$ and there exist two positive constants α, K_5 such that

$$\frac{\psi_y(r)}{\psi_y(\rho)} \leq K_5 \left(\frac{r}{\rho} \right)^{\alpha} . \tag{1.6}$$

One can easily check that by (1.4) the condition (1.6) implies

$$\left(\frac{r}{\rho}\right)^2 \left(\frac{v(B_r(y))}{v(B_\rho(y))}\right)^{\frac{2}{q}} \frac{w(B_\rho(y))}{w(B_r(y))} \leq K_6,\ 0<r<R,\ q=\frac{2n\nu}{n\nu-\alpha} \qquad (1.7)$$

with $K_6 = K_4^{1-\frac{2}{q}} K_5$, and vice versa, inequality (1.7) with some $q>2$ implies (1.6) with $\alpha = n\nu_1(1-\frac{2}{q})$.

We note that condition (1.7) is essentially necessary and sufficient to have the Sobolev–Poincaré inequality (for details we refer the reader to [16]).

Remark 1.1. It seems that conditions on $\psi_y(r)$ are rather natural. Particularly in the case $v=w$ they are obvious. In the case $v \equiv 1$, following [23] we introduce the function $\tilde{\psi}_y(r) := \left(\int_{B_r(y)} w^{-\frac{n}{2}} dx\right)^{\frac{2}{n}} \asymp \psi_y(r)$, where the symbol $\asymp$ means that there exists $c>0$ such that $c^{-1}\psi_y(r) \leq \tilde{\psi}_y(r) \leq c\psi_y(r)$. And the previous assumptions will be fulfilled for the function $\tilde{\psi}_y(r)$. We also note that condition (1.6) is evident consequence of (1.4), (1.5) if $\mu < \nu_1 + \frac{2}{n}$, then $\alpha = 2 + n(\nu_1 - \mu) > 0$.

To formulate our results we also need the definition of the weighted parabolic Riesz potential. Let $(x_0,t_0) \in \Omega_T$ for any $\rho, \theta > 0$ we define $Q_{\rho,\theta}(x_0,t_0) := B_\rho(x_0) \times (t_0-\theta, t_0)$. We set

$$I_{v,w,f}(x_0,t_0,R,\theta) := \int_0^R \frac{1}{v(B_\rho(x_0))} \iint_{Q_{\rho,\theta\psi_{x_0}(\rho)}(x_0,t_0)} |f(x,t)| dx dt \frac{d\rho}{\rho} \qquad (1.8)$$

In the case $v \equiv w \equiv 1$ and $\theta = 1$ the potential $I_{1,1,f}(x_0,t_0,R,1)$ reduces to the standard truncated parabolic Riesz potential

$$I_f(x_0,t_0,R) := \int_0^R \rho^{-n} \iint_{Q_{\rho,\rho^2}(x_0,t_0)} |f(x,t)| dx dt \frac{d\rho}{\rho}.$$

In the case when f depends on the spatial variable only and $\theta = 1$ the potential $I_{v,w,f}(x_0,R,1)$ reduces to the elliptic weighted Riesz potential

$$I_{w,f}(x_0,R) := \int_0^R \frac{\rho^2}{w(B_\rho(x_0))} \int_{B_\rho(x_0)} |f(x)| dx \frac{d\rho}{\rho}$$

which coincides with the weighted Wolf potential $W^f_{w,2}(x_0,R)$, where $W^f_{w,p}(x_0,R) = \int_0^R \left(\frac{\rho^p}{w(B_\rho(x_0))} \int_{B_\rho(x_0)} |f| dx\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$ for the details we refer the reader to [43].

We briefly recall the definition of weighted Sobolev space for $v \in A_\infty$ and $w \in A_2$. By $L^p(\Omega, v)$ we denote the set of measurable functions $u : \Omega \to R^1$ such that $\left(\int_\Omega v|u|^p dx\right)^{\frac{1}{p}} < \infty$. By $W^{1,2}(\Omega, v, w)$ we denote the space $\left\{u \in L^2(\Omega, v) \cap W^{1,1}(\Omega) : |\bigtriangledown u| \in L^2(\Omega, w)\right\}$ endowed with the norm $||u||_{L^2(\Omega,v)} + ||\nabla u||_{L^2(\Omega,w)}$. The space $W_0^{1,2}(\Omega, v, w)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega, v, w)$.

Before formulating the main results, let us recall the definition of a weak solution to equation (1.2). Set $m^- = min(1, m)$, we say that u is a nonnegative weak solution to equation (1.2) if $u \in C_{loc}(0, T; L_{loc}^{1+m^-}(\Omega, v))$ and $u^{\frac{m+m^-}{2}-1}|\nabla u| \in L^2_{loc}(0, T; W_{loc}^{1,2}(\Omega, v, w))$ and for every compact set $E \subset \Omega$ and for every subinterval $(t_1, t_2) \subset (0, T)$ the following identity

$$\int_E vu\varphi dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_E (-vu\varphi_t + A(x, t, u, \nabla u)\nabla\varphi)\, dxdt$$
$$= \int_{t_1}^{t_2} \int_E f\varphi dxdt \tag{1.9}$$

holds true for any testing function $\varphi \in L^2(0, T; W_0^{1,2}(E, v, w))$, $\varphi, \varphi_t \in L^\infty(E \times (0, T))$.

The constants $m, n, K_{2,w}, K_{p_0,v}, K_1, ..., K_6$ are further reffered to as the data. In what follows γ stands for a constant depending only on the data which may vary from line to line.

Theorem 1.1. *Let u be a nonnegative weak solution to the equation* (1.2)*, let the conditions* (1.3)–(1.6) *be fulfilled and let also $m > 1$. Then there exist positive constants $\lambda_0 \in (0, 1)$, c_1 depending only on the data such that for all $\lambda \in (0, \lambda_0)$, for almost all $(x_0, t_0) \in \Omega_T$ and any cylinder $Q_{R,\theta}(x_0, t_0) \subset \Omega_T$ following inequalities holds*

$$u(x_0, t_0) \le c_1 \left(\frac{1}{v(B_R(x_0))\psi_{x_0}(R)} \iint_{Q_{R,\theta}(x_0,t_0)} vu^{m+\lambda} dxdt\right)^{\frac{1}{1+\lambda}}$$
$$+ c_1 \left(\frac{1}{w(B_R(x_0))\psi_{x_0}(R)} \iint_{Q_{R,\theta}(x_0,t_0)} wu^{m+\lambda} dxdt\right)^{\frac{1}{1+\lambda}}$$
$$+ c_1 \left(\frac{\psi_{x_0}(R)}{\theta}\right)^{\frac{1}{m-1}} + c_1 I_{v,w,f}\left(x_0, t_0, c_1 R, \frac{\theta}{\psi_{x_0}(R)}\right). \tag{1.10}$$

Theorem 1.2. *Let u be a nonnegative weak solution to the equation* (1.2)*, let the conditions* (1.3)–(1.6) *be fulfilled and let also*

$$0 < 1 - \frac{\alpha}{n} min\left(\frac{1}{\nu}, \frac{1}{\mu}\right) < m < 1. \tag{1.11}$$

Then there exist positive constants $\lambda_0 \in (0,1)$, c_1 depending only on the data such that for all $\lambda \in (0,\lambda_0)$, for almost all $(x_0,t_0) \in \Omega_T$ and any cylinder $Q_{R,\theta}(x_0,t_0) \subset \Omega_T$ following inequalities holds

$$u(x_0,t_0) \le c_1 \left(\frac{\psi_{x_0}(R)}{\theta}\right)^{\frac{n\nu}{\alpha+(m-1)n\nu+\alpha\lambda}} \left(\frac{1}{v(B_R(x_0))\theta}\right.$$

$$\left.\times \iint_{Q_{R,\theta}(x_0,t_0)} v u^{1+\lambda} dxdt\right)^{\frac{\alpha}{\alpha+(m-1)n\nu+\alpha\lambda}} + c_1 \left(\frac{\psi_{x_0}(R)}{\theta}\right)^{\frac{n\mu}{\alpha+(m-1)n\mu+\alpha\lambda}}$$

$$\times \left(\frac{1}{w(B_R(x_0))\theta} \iint_{Q_{R,\theta}(x_0,t_0)} w u^{1+\lambda} dxdt\right)^{\frac{\alpha}{\alpha+(m-1)n\mu+\alpha\lambda}}$$

$$+ c_1 \left(\frac{\theta}{\psi_{x_0}(R)}\right)^{\frac{1}{1-m}}$$

$$+ c_1 \left(\frac{\psi_{x_0}(R)}{\theta}\right)^{\frac{n\nu}{\alpha+(m-1)n\nu}} I_{v,w,f}\left(x_0,t_0,c_1R,\frac{\theta}{\psi_{x_0}(R)}\right)^{\frac{\alpha}{\alpha+(m-1)n\nu}} \tag{1.12}$$

Before describing the method of proof, few words concerning the history of the problem. The basic qualitative properties such as local boundedness, Hölder continuity and Harnack's inequality to homogeneous linear divergence type second-order elliptic equations with measurable coefficients wthout lower order terms are known since the famous results by De Giorgi [27], Nash [57] and Moser [54,55]. These results were extended, by Serrin [61], Ladyzhenskaya and Uraltseva [49, 50], Aronson and Serrin [3], and Trudinger [66] to a wide class of elliptic and parabolic equations ($m = 1$) with lower order terms from the corresponding L^q-classes. Analogous results for the porous medium or p-Laplace evolution equations appeared much later. For the continuity of solutions to the porous medium equations we refer the reader to the famous papers by Caffarelli, Friedman [14], Caffarelli, Evans [13] and Di Benedetto, Friedman [29]. Di Benedetto developed an innovative intrinsic scaling method (see [28] and references to the original papers there) and proved the Hölder continuity of weak solutions to the evolution p-Laplace equations with lower order terms from L^q-classes.

As for elliptic and parabolic equations with singular lower order terms we refer the reader to the pioneering paper by Aizenman and Simon [2], where the Harnack's inequality and continuity of weak solutions to the equation $-\Delta u + V(x)u = 0$ were proved under the local Kato class condition on the potential V. Moreover, they showed that the Kato-type condition on the potential V is necessary for the validity of the

Harnack's inequality. This result was extended to linear elliptic and parabolic equations with lower order terms by Chiarenza, Fabes and Garofalo [20], Kurata [47] and Zhang [70, 71]. First substantial moves to the p-Laplacian equation with measure were achieved by Kilpeläinen and Malý [46], where pointwise estimates were established for such type equations in terms of the nonlinear Wolf potential $W^{\mu}_{1,p}(x,r)$ of the measure μ : $W^{\mu}_{\beta,p}(x,r) := \int_0^r \left(\frac{\mu(B_\rho(x))}{\rho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$. These results were subsequently extended to fully nonlinear and subelliptic quasilinear equations by Trudinger and Wang [67] and Labutin [48]. For the parabolic equations the corresponding results were recently given in [51, 52] for the p-Laplace evolution equation and in [5–7, 12, 52, 63] for the porous medium equation. The nonuniformly elliptic and the nonuniformly parabolic equations without/or with singular lower order terms have been studied for a long time. The first results in this direction were obtained by Fabes, Kenig and Separioni [31] and Gutierrez [38] for weighted linear elliptic equation with weight from the correspondent A_2 Muckenhoupt class. We refer the reader to [1,8–11,15–26,30–45,53,56,58–60,64,65, 70] for an account of the results.

We note that in the proof of Theorem 1.1 we do not distinguish the case $m > 1$ or $m < 1$, the difference will be only in the choice of r_0 in the first step of the iteration (see Section 3). We also note that the weighted Riesz potential $I_{v,w,f}$ plays the same role as the linear Riesz potential I_f in the linear and quasilinear setting, we refer the reader to [43].

The difficulties arising here are related not only to the presence of the function $f \in L^1_{loc}(\Omega_T)$ but also to the presence of weights v and w. Our strategy of the proof is similar to that of [28]. However, due to the different structure conditions the De Giorgi-type iteration cannot be used. Instead, we adapt the Kilpeläinen–Malý iteration [46], which was developed in [51, 52] and [5–7,12,52, 63] for the porous medium and evolution p-Laplace equations.

Finally, we test our result against the fundamental solution to the equation

$$|x|^{\alpha_1} u_t - div\left(|x|^{\alpha_2} u^{m-1} \nabla u\right) = \delta_{(0,0)} \tag{1.16}$$

where $\alpha_1 > -n$, $-n < \alpha_2 < n$, $\alpha_2 < 2 + \alpha_1$ and $\delta_{(0,0)}$ is the delta-function at the origin. In this setting $\nu_1 = \nu = 1 + \frac{\alpha_1}{n}$, $\mu_1 = \mu = 1 + \frac{\alpha_2}{n}$, $\alpha = 2 + \alpha_1 - \alpha_2 > 0$. Consider a point $(0, t_0)$ with $u_0 = u(0, t_0) > 0$ and let $\theta = u_0^{1-m} R^{\alpha}$. In order for the cylinder $Q_{\rho, \frac{\theta}{R^\alpha}\rho^\alpha}(0, t_0)$ to contain the origin, we need to have $\rho \geq \left(t_0 u_0^{m-1}\right)^{\frac{1}{\alpha}}$. Then we get

$$I_{v,w,\delta}\left(0,t_0,R,\frac{\theta}{R^\alpha}\right) \leq \int_{\left(t_0 u_0^{m-1}\right)^{\frac{1}{\alpha}}}^{\infty} \frac{\delta\left(Q_{\rho,\frac{\theta}{R^\alpha}\rho^\alpha}(0,t_0)\right)}{\rho^{n+\alpha_1}}\frac{d\rho}{\rho}$$
$$= (n+\alpha_1)^{-1}\left(t_0 u_0^{m-1}\right)^{-\frac{n+\alpha_1}{\alpha}}.$$

Now, if we limit ourselves to consider only the bound from above that comes from the Riesz potential, namely

$$u_0 \leq c_1 I_{v,w,\delta}\left(0,t_0,R,\frac{\theta}{R^\alpha}\right), \; if\; m>1,$$

$$u_0 \leq c_1\left(\frac{R^\alpha}{\theta}\right)^{\frac{n+\alpha_1}{\alpha+(m-1)(n+\alpha_1)}}\left(I_{v,w,\delta}\left(0,t_0,R,\frac{\theta}{R^\alpha}\right)\right)^{\frac{\alpha}{\alpha+(m-1)(n+\alpha_1)}},$$
$$if\;\; 1-\frac{\alpha}{n} min\left(\frac{1}{1+\frac{\alpha_1}{n}},\frac{1}{1+\frac{\alpha_2}{n}}\right) < m < 1,$$

which implies in both cases $m>1$ and $m<1$ an estimate

$$u_0 = u(0,t_0) \leq c_4 t_0^{-\frac{n+\alpha_1}{\beta}},$$

where $\beta = 2+\alpha_1-\alpha_2+(m-1)(n+\alpha_1)$, and constant $c_4>0$ depends only on the data. This corresponds exactly to the time decay of the fundamental solutions and shows that (1.11)and (1.12) are optimal under this point of view (see [11]).

The rest of the paper contains the proof of the above theorems.

2. Auxilary material and integral estimates of solutions

2.1 Auxilary propositions

The next lemmas will be used in the sequel. The first one is the weighted Sobolev and Poincaré inequalities (see [16]).

Lemma 2.1. *Let* $p>1$, $w_1 \in A_p$, $w_2 \in A_\infty$ *and let*

$$\left(\frac{r}{\rho}\right)^p\left(\frac{w_2(B_r(y))}{w_2(B_\rho(y))}\right)^{\frac{p}{q}}\frac{w_1(B_\rho(y))}{w_1(B_r(y))} \leq c \tag{2.1}$$

for every ball $B_r(y)\subset B_\rho(y)$, *some* $c>0$ *and some* $q>p$. *Then there exists* $\gamma>0$ *depending only on the data,* q *and* c *such that*

$$\left(\frac{1}{w_2(B_R(\overline{x}))}\int_{B_R(\overline{x})} w_2|\varphi|^q dx\right)^{\frac{1}{q}} \leq \gamma R\left(\frac{1}{w_1(B_R(\overline{x}))}\int_{B_R(\overline{x})} w_1|\nabla\varphi|^p dx\right)^{\frac{1}{p}} \tag{2.2}$$

for every ball $B_R(\overline{x})$ *and every* $\varphi \in C_0^\infty(B_R(\overline{x}))$,

$$\left(\frac{1}{w_2(B_R(\overline{x}))}\int_{B_R(\overline{x})} w_2|\varphi-\varphi_R|^q dx\right)^{\frac{1}{q}} \leq$$

$$\leq \gamma R\left(\frac{1}{w_1(B_R(\overline{x}))}\int_{B_R(\overline{x})} w_1|\nabla\varphi|^p dx\right)^{\frac{1}{p}} \tag{2.3}$$

for every ball $B_R(\overline{x})$ *and every* $\varphi \in C^\infty(B_R(\overline{x}))$. *Here*

$$\varphi_R = \frac{1}{w_2(B_R(\overline{x}))}\int_{B_R(\overline{x})} w_2\varphi dx \ \ or \ \ \varphi_R = \frac{1}{|B_R(\overline{x})|}\int_{B_R(\overline{x})}\varphi dx.$$

The next lemma is the parabolic version of the Sobolev imbedding theorem (see [22, 25]).

Lemma 2.2. *Let* $w \in A_2$, $v \in A_\infty$ *and let inequality* (2.1) *holds true with* $w_2 = v$, $w_1 = w$, $p = 2$ *and some* $q > 2$. *Then there exist* h, $\gamma > 0$ *depending only on the data such that*

$$\iint_{Q_{R,\tau}(\overline{x},\overline{t})} w|\varphi|^{2+p_1 h} dxdt \leq \gamma\left(\frac{1}{v(B_R(\overline{x}))}\sup_{\overline{t}-\tau<t<\overline{t}}\int_{B_R(\overline{x})} v|\varphi|^{p_1}dx\right)^h$$

$$\times R^2\iint_{Q_{R,\tau}(\overline{x},\overline{t})} w|\nabla\varphi|^2 dxdt, \tag{2.4}$$

$$\frac{1}{v(B_R(\overline{x}))}\iint_{Q_{R,\tau}(\overline{x},\overline{t})} v|\varphi|^{2+p_1 h}dxdt$$

$$\leq \gamma\left(\frac{1}{v(B_R(\overline{x}))}\sup_{\overline{t}-\tau<t<\overline{t}}\int_{B_R(\overline{x})} v|\varphi|^{p_1}dx\right)^h$$

$$\times\frac{R^2}{w(B_R(\overline{x}))}\iint_{Q_{R,\tau}(\overline{x},\overline{t})} w|\nabla\varphi|^2 dxdt, \tag{2.5}$$

for every cylinder $Q_{R,\tau}(\overline{x},\overline{t})$ *and every*

$$\varphi \in C\left(\overline{t}-\tau,\overline{t}; L^{p_1}(B_R(\overline{x}),v)\right) \cap L^2(\overline{t}-\tau,\overline{t}; W_0^{1,2}(B_R(\overline{x}),v,w)).$$

The following lemma is the weighted analogue of the well-known De Giorgi–Poincaré lemma.

Lemma 2.3. *Let $p>1$, $w\in A_p$, $v\in A_\infty$ and let inequality (2.1) holds true with $w_1=v$, let k and l be real numbers such that $l>k$. Then there exists a positive constant γ depending only on the data such that*

$$(l-k)^p v(A_{k,R})(v(B_R(\overline{x}))-v(A_{l,R}))$$
$$\le \gamma R^p \frac{v^2(B_R(\overline{x}))}{w(B_R(\overline{x}))}\int_{A_{l,R}\setminus A_{k,R}} w|\nabla\varphi|^p dx, \tag{2.6}$$

for every ball $B_R(\overline{x})$ and every $u\in C^\infty(B_R(\overline{x}))$. Here $A_{k,R}=\{x\in B_R(\overline{x}):u(x)<k\}$, $v(A_{k,R})=\int_{A_{k,R}} v dx$.

Proof. We use inequality (2.3) for the function $\varphi=(l-max(u,k))_+$ and $\varphi_R=\frac{1}{v(B_R(\overline{x}))}\int_{B_R(\overline{x})} v\varphi dx$. Since $\varphi_R\le (l-k)\frac{v(A_{l,R})}{v(B_R(\overline{x}))}\le l-k$, by the Hölder inequality we obtain $(l-k)^p\frac{v(A_{l,R})}{v(B_R(\overline{x}))}\left(1-\frac{v(A_{l,R})}{v(B_R(\overline{x}))}\right)\le$ $\le \frac{1}{v(B_R(\overline{x}))}\int_{A_{k,R}} v(\varphi-\varphi_R)^p dx \le \left(\frac{1}{v(B_R(\overline{x}))}\int_{B_R(\overline{x})} v|\varphi-\varphi_R|^q dx\right)^{\frac{p}{q}}\le$ $\le \gamma\frac{R^p}{w(B_R(\overline{x}))}\int_{A_{l,R}\setminus A_{k,R}} w|\nabla u|^p dx$, which proves the lemma. □

2.2 Local energy estimates

Lemma 2.4. *Let u be a bounded nonnegative weak solution to (1.2) in Ω_T. Then there exist $\gamma>0$ depending only on the data such that for every cylinder $Q_{\rho,\tau}(\overline{x},\overline{t})\subset\Omega_T$ any $l>0, k\ge 2$ and any smooth $\zeta(x,t)$ which is zero for $(x,t)\in\partial B_\rho(\overline{x})\times(\overline{t}-\tau,\overline{t})$ one has*

$$\sup_{\bar{t}-\tau<t<\bar{t}}\int_{B_\rho(\overline{x})} v\int_l^u (s^{m^-}-l^{m^-})ds\zeta^k dx+\iint_L wu^{m+m^--2}|\nabla u|^2\zeta^k dxdt$$
$$\le \int_{B_\rho(\overline{x})} v\int_l^u (s^{m^-}-l^{m^-})ds\zeta^k(x,\bar{t}-\tau)dx$$
$$+\gamma\iint_L vu^{m^-}(u-l)|\zeta_t|\zeta^{k-1}dxdt$$
$$+\gamma\iint_L wu^{m-m^-}(u^{m^-}-l^{m^-})^2|\nabla\zeta|^2\zeta^{k-2}dxdt$$
$$+\gamma\iint_L |f|(u^{m^-}-l^{m^-})\zeta^k dxdt, \tag{2.7}$$

$$\sup_{\bar{t}-\tau<t<\bar{t}}\int_{B_\rho(\overline{x})} v\int_u^l (l^{m^-}-s^{m^-})ds\zeta^k dx+\iint_{\tilde{L}} wu^{m+m^--2}|\nabla u|^2\zeta^k dxdt$$
$$\le \int_{B_\rho(\overline{x})} v\int_u^l (l^{m^-}-s^{m^-})ds\zeta^k(x,\bar{t}-\tau)dx$$

$$+\gamma l^{m^-}\iint_{\tilde{L}} v(l-u)|\zeta_t|\zeta^{k-1}dxdt$$

$$+\gamma\iint_{\tilde{L}} wu^{m-m^-}(l^{m^-}-u^{m^-})^2|\nabla\zeta|^2\zeta^{k-2}dxdt$$

$$+\gamma\iint_{\tilde{L}}|f|(l^{m^-}-u^{m^-})\zeta^k dxdt, \tag{2.8}$$

where $L=Q_{\rho,\tau}(\overline{x},\overline{t})\cap\{u>l\}$, $\tilde{L}=Q_{\rho,\tau}(\overline{x},\overline{t})\cap\{u<l\}$.

Proof. Test (1.9) by $\varphi=(u^{m^-}-l^{m^-})_\pm\zeta^k$, use conditions (1.3) and the Young inequality.

Lemma 2.5. *Let u be a nonnegative weak solution to* (1.2) *in* Ω_T. *Then there exists $\gamma>0$ depending only on the data such that for every cylinder $Q_{\rho,\tau}(\overline{x},\overline{t})\subset\Omega_T$, any $\lambda\in(0,1), a, l, \delta>0$, $k\geq 2$ and any smooth $\zeta(x,t)$ which is zero on the parabolic boundary of $Q_{\rho,\tau}(\overline{x},\overline{t})$ one has*

$$\frac{1}{\delta}\sup_{\overline{t}-\tau<t<\overline{t}}\int_{B_\rho(\overline{x})} v\int_l^u\left(1-\left(1+a\frac{s^{m^-}-l^{m^-}}{\delta^{m^-}}\right)^{-\lambda}\right)ds\zeta^k dx$$

$$+\frac{a}{\delta^{m^-+1}}\iint_L\frac{wu^{m-m^-}|\nabla u^{m^-}|^2\zeta^k}{\left(1+a\frac{u^{m^-}-l^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}}dxdt\leq\gamma\iint_L v\frac{u-l}{\delta}|\zeta_t|\zeta^{k-1}dxdt$$

$$+\gamma\frac{\delta^{m^--1}}{a}\iint_L wu^{m-m^-}\left(a\frac{u^{m^-}-l^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}|\nabla\zeta|^2\zeta^{k-2}dxdt$$

$$+\frac{\gamma}{\delta}\iint_{Q_{\rho,\tau}(\overline{x},\overline{t})}|f|dxdt, \tag{2.9}$$

$$\frac{1}{\delta}\sup_{\overline{t}-\tau<t<\overline{t}}\int_{B_\rho(\overline{x})} v\int_u^l\left(1-\left(1+a\frac{l^{m^-}-s^{m^-}}{\delta^{m^-}}\right)^{-\lambda}\right)ds\zeta^k dx$$

$$+\frac{a}{\delta^{m^-+1}}\iint_{\tilde{L}}\frac{wu^{m-m^-}|\nabla u^{m^-}|^2\zeta^k}{\left(1+a\frac{l^{m^-}-u^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}}dxdt\leq\gamma\iint_{\tilde{L}} v\frac{l-u}{\delta}|\zeta_t|\zeta^{k-1}dxdt$$

$$+\gamma l^{m-m^-}\frac{\delta^{m^--1}}{a}\iint_{\tilde{L}} wu^{m-m^-}\left(a\frac{l^{m^-}-u^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}|\nabla\zeta|^2\zeta^{k-2}dxdt$$

$$+\frac{\gamma}{\delta}\iint_{Q_{\rho,\tau}(\overline{x},\overline{t})}|f|dxdt, \tag{2.10}$$

Proof. First note that

$$1-(1+u)^{-\lambda}\asymp\frac{u}{1+u}=1-(1+u)^{-1},\ \ u>0. \tag{2.11}$$

Testing (1.9) by $\varphi=\left(1-\left(1+a\frac{(u^{m^-}-l^{m^-})_+}{\delta^{m^-}}\right)^{-\lambda}\right)\zeta^k$, using conditions (1.3), (2.11) and the Young inequality we arrive at (2.9).

Testing (1.9) by $\varphi=\left(1-\left(1+a\frac{(l^{m^-}-u^{m^-})_+}{\delta^{m^-}}\right)^{-\lambda}\right)\zeta^k$, using conditions (1.3), (2.11) and the Young inequality we arrive at (2.10). □

3. Boundedness of solutions. Proof of Theorems 1.1, 1.2

Let $\psi_{x_0}^{-1}(r)$ be an inverse function to the function $\psi_{x_0}(r)$, where $\psi_{x_0}(r)$ was defined in (1.6). The inverse function $\psi_{x_0}^{-1}(r)$ exist by our assumptions on the function $\psi_{x_0}(r)$. Moreover by (1.4)–(1.6) we have

$$K_6^{-1}\left(\frac{r}{\rho}\right)^{\frac{1}{\alpha}}\le\frac{\psi_{x_0}^{-1}(r)}{\psi_{x_0}^{-1}(\rho)}\le K_5\left(\frac{r}{\rho}\right)^{\frac{1}{2+n\nu_1}},\ 0<r\le\rho\le R \tag{3.1}$$

We set $m^+=max(1,m)$ and let $r_0:=\psi_{x_0}(R)$ if $m>1$ and $r_0:=\theta$ if $m<1$. Fix $\sigma\in(0,1)$ so that $K_5\sigma^{\frac{1}{2+n\nu_1}}=\frac{1}{2}$ and for $j=0,1,2,...$ we define the sequences $r_j:=\sigma^j r_0,\ \ \rho_j:=\psi_{x_0}^{-1}\left(r_j l_j^{m-m_+}\right),\ \ \tau_j:=\frac{r_j}{l_j^{m_+-1}}$, $B_j:=B_{\rho_j}(x_0),\ \ Q_j:=B_j\times(t_0-\tau_j,t_0),\ \ L_j:=Q_j\cap\{u>l_j\}$. And let $\zeta_j\in C^\infty(Q_j)$ be such that $\zeta_j=1$ in $B_{\frac{1}{2}\rho_j}(x_0)\times(t_0-\frac{1}{2}\tau_j,t_0),\ \ \zeta_j=0$ for $t\le t_0-\tau_j,\ \ \zeta_j=0$ for $|x-x_0|\ge\rho_j,\ \ 0\le\zeta_j\le1,$ and $|\nabla\zeta|\le\frac{2}{\rho_j}$, $\left|\frac{\partial\zeta_j}{\partial t}\right|\le\frac{2}{\tau_j}$.

The sequences of positive numbers $l_j,\ \ j=0,1,2,...$ and $\delta_j,\ \ j=-1,0,1,2,...$ are defined inductively as follows. Fix a positive number $\kappa\in(0,1)$ depending only on the data and λ, which will be spesified later. Set $l_{-1}=0$ and

$$l_0=\delta_{-1}:=\left(\frac{\psi_{x_0}(R)}{r_0}\right)^{\frac{n\nu}{\beta\nu(m^+)+\alpha\lambda}}\Bigg(\frac{\kappa^{-1}}{v(B_R(x_0))r_0}$$
$$\times\iint_{Q_{R,\theta}(x_0,t_0)}vu^{m^++\lambda}dxdt\Bigg)^{\frac{\alpha}{\beta\nu(m^+)+\alpha\lambda}}+\left(\frac{\psi_{x_0}(R)}{r_0}\right)^{\frac{n\nu}{\beta\mu(m^+)+\alpha\lambda}}$$
$$\times\left(\frac{\kappa^{-1}}{v(B_R(x_0))r_0}\iint_{Q_{R,\theta}(x_0,t_0)}wu^{m^++\lambda}dxdt\right)^{\frac{\alpha}{\beta\mu(m^+)+\alpha\lambda}}+\left(\frac{\psi_{x_0}(R)}{r_0}\right)^{\frac{1}{m-1}}, \tag{3.2}$$

where $\beta_\nu(m^+)=\alpha+(m-m^+)n\nu>0,\ \beta_\mu(m^+)=\alpha+(m-m^+)n\mu>0.$ Assume that $l_1,...l_j$ have been already chosen. For $l>l_j$ and $\delta_j(l)=l-l_j$ set

$$A_j(l)=\frac{1}{v(B_j)r_j}\iint_{L_j} vu^{m^+-1}\left(\frac{u-l_j}{\delta_j(l)}\right)^{1+\lambda}\zeta_j^{k-2}dxdt$$

$$+\frac{1}{w(B_j)r_j}\iint_{L_j} wu^{m^+-1}\left(\left(\frac{l_j}{\delta_j(l)}\right)^{1-m^-}\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}(l)}\right)^{1+\lambda}\zeta_j^{k-2}dxdt. \tag{3.3}$$

If $A_j(l_j+\frac{1}{2}\delta_{j-1})\le\kappa$ we set $l_j=l_j+\frac{1}{2}\delta_{j-1}$ where $\delta_{j-1}=\delta_{j-1}(l_j)=l_j-l_{j-1}$. Note that $A_j(l)\searrow 0$ as $l\to\infty$, so if $A_j(l_j+\frac{1}{2}\delta_{j-1})>\kappa$, there exists $\tilde l>l_j+\frac{1}{2}\delta_{j-1}$ such that $A_j(\tilde l)=\kappa$. In this case we set $l_{j+1}=\tilde l$ and in both cases we set $\delta_j=\delta_j(l_{j+1})=l_{j+1}-l_j$. By our choices we have an inclusion $Q_{j+1}\subset B_{\frac{1}{2}}\rho_j(x_0)\times(t_0-\frac{1}{2}\tau_j,t_0)\subset Q_j$, $j\in N$, and moreover

$$A_j(l_{j+1})\le\kappa,\ j\in N. \tag{3.4}$$

Claim. Set $c_5=\left(K_4K_5\sigma^{-1-\frac{n\mu}{\alpha}-\frac{n\nu}{\alpha}}2^{1+\frac{\lambda}{m^-}}\right)^{\frac{1}{m^-(1+\lambda)}}$, then for any $j\in N$

$$l_{j+1}\le c_5 l_j. \tag{3.5}$$

We establish the claim by induction. By our choice of l_0, (1.4), (1.5), (3.1) and by the inequalities

$$v(B_0)\ge v\left(K_6^{-1}\left(\frac{r_0}{\psi_R(x_0)l_0^{m^+-m}}\right)^{\frac{1}{\alpha}}B_R(x_0)\right)$$

$$\ge K_5^{-1}K_6^{-1}\left(\frac{r_0}{\psi_R(x_0)l_0^{m^+-m}}\right)^{\frac{n\nu}{\alpha}}v(B_R(x_0)),$$

$$w(B_0)\ge K_5^{-1}K_6^{-1}\left(\frac{r_0}{\psi_R(x_0)l_0^{m^+-m}}\right)^{\frac{n\mu}{\alpha}}w(B_R(x_0)),$$

and since $l_0^{1-m^-}\left(u^{m^-}-l_0^{m^-}\right)_+\le(u-l_0)_+$, we have for $j=0$

$$A_0(c_5l_0)\le 2^{1+\lambda}c_5^{-1-\lambda}\frac{l_0^{-1-\lambda}}{r_0}\iint_{Q_0}\left(\frac{v}{v(B_0)}+\frac{w}{w(B_0)}\right)u^{m^++\lambda}dxdt$$

$$\le 2^{1+\lambda}K_5K_6c_5^{-1-\lambda}\left(\frac{\psi_R(x_0)}{r_0}\right)^{\frac{n\nu}{\alpha}}\frac{l_0^{-\frac{\beta_\nu(m^+)}{\alpha}-\lambda}}{v(B_R(x_0))r_0}\iint_{Q_{R,\theta}(x_0,t_0)}vu^{m^++\lambda}dxdt$$

$$+2^{1+\lambda}K_5K_6c_5^{-1-\lambda}\left(\frac{\psi_R(x_0)}{r_0}\right)^{\frac{n\mu}{\alpha}}\frac{l_0^{-\frac{\beta\mu(m^+)}{\alpha}-\lambda}}{w(B_R(x_0))r_0}\iint_{Q_{R,\theta}(x_0,t_0)}wu^{m^++\lambda}dxdt$$
$$\le 2^{1+\lambda}K_5K_6c_5\kappa\le\kappa.$$

Now if $l_1 = l_0 + \frac{1}{2}\delta_{-1} = \frac{3}{2}l_0$, then $l_1 \le c_5 l_0$ and if $A_0(l_0) = \kappa \ge A_0(c_5l_0)$ and since $A_0(l)$ is decreasing, then $l_1 \le c_5l_0$ and in both cases we obtain $l_1 \le c_5l_0$. Assume that (3.5) holds for $i = 1, ..., j-1$, then by (1.4), (1.5)

$$A_j(c_5l_j)\le 2^{1+\lambda}c_5^{-1-\lambda}r_j^{-1}\Bigg[\frac{1}{v(B_j)}\iint_{L_j}vu^{m^+-1}\left(\frac{u-l_{j-1}}{l_j}\right)^{1+\lambda}\zeta_{j-1}^{k-2}dxdt$$
$$+\frac{1}{w(B_j)}\iint_{L_j}wu^{m^+-1}\left(\frac{u^{m^-}-l_{j-1}^{m^-}}{l_j^{m^-}}\right)^{1+\lambda}\zeta_{j-1}^{k-2}dxdt\Bigg]$$
$$\le\frac{K_5K_62^{1+\lambda}c_5^{-m^-(1+\lambda)}}{\sigma^{1+\frac{n\mu}{\alpha}+\frac{n\nu}{\alpha}}r_{j-1}}\Bigg[\frac{1}{v(B_{j-1})}$$
$$\times\iint_{L_{j-1}}vu^{m^+-1}\left(\frac{u-l_{j-1}}{\delta_j}\right)^{1+\lambda}\zeta_{j-1}^{k-2}dxdt$$
$$\frac{1}{w(B_{j-1})}\iint_{L_{j-1}}wu^{m^+-1}\left(\left(\frac{l_{j-1}}{\delta_{j-1}}\right)^{1-m^-}\frac{u^{m^-}-l_{j-1}^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_{j-1}^{k-2}dxdt\Bigg]$$
$$\le K_5K_6\sigma^{-1-\frac{n\mu}{\alpha}-\frac{n\nu}{\alpha}}2^{1+\lambda}c_5^{-m^-(1+\lambda)}A_{j-1}(l_j)\le A_{j-1}(l_j)\le\kappa.$$

Now again if $l_{j+1} = l_j + \frac{1}{2}\delta_{j-1} \le \frac{3}{2}l_j$, and if $A_j(l_{j+1}) = \kappa \ge A_j(c_5l_j)$, then since $A_j(l)$ is decreasing then $l_{j+1} \le c_5l_j$. This proves the claim.

The following lemma is a key in the Kilpeläinen–Malý technique.

Lemma 3.1. *For every $j \ge 1$ the following inequality holds*

$$\delta_j\le\frac{1}{2}\delta_{j-1}+\frac{\gamma}{v(B_j)}\iint_{Q_{j-1}}|f|dxdt \tag{3.6}$$

Proof. Without loss of generality we assume that

$$\delta_j>\frac{1}{2}\delta_{j-1}, \tag{3.7}$$

since otherwise (3.6) is evident. This inequality quarantees that $A_j(l_{j+1}) = \kappa$. First note the inequality

$$\frac{1}{r_j}\iint_{L_j}\left(\frac{v}{v(B_j)}+\frac{w}{w(B_j)}\right)u^{m^+-1}dxdt\le\gamma\kappa. \tag{3.8}$$

Indeed by our choices and by the inequality

$$l_j - l_{j-1} \le l_j^{1-m^-} \int_{l_{j-1}}^{l_j} s^{m^- -1} ds = \frac{l_j^{1-m^-}}{m^-}(l_j^{m^-} - l_{j-1}^{m^-})$$

$$\le \frac{l_j^{1-m^-}}{m^-}(u_j^{m^-} - l_{j-1}^{m^-})$$

on L_j, we have

$$\frac{1}{r_j} \iint_{L_j} \left(\frac{v}{v(B_j)} + \frac{w}{w(B_j)} \right) u^{m^+-1} dxdt$$

$$= \frac{1}{r_j} \iint_{L_j} \left(\frac{v}{v(B_j)} + \frac{w}{w(B_j)} \right) u^{m^+-1} \left(\frac{l_j - l_{j-1}}{\delta_j} \right)^{1+\lambda} \zeta_{j-1}^{k-2} dxdt$$

$$\le \frac{\gamma}{v(B_j) r_j} \iint_{L_j} v u^{m^+-1} \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right)^{1+\lambda} \zeta_{j-1}^{k-2} dxdt$$

$$+ \frac{\gamma}{w(B_j) r_j} \iint_{L_j} w u^{m^+-1} \left(\left(\frac{l_{j-1}}{\delta_{j-1}} \right)^{1+\lambda} \frac{u^{m^-} - l_{j-1}^{m^-}}{\delta_{j-1}^{m^-}} \right)^{1+\lambda} \zeta_{j-1}^{k-2} dxdt$$

$$\le \gamma A_{j-1}(l_j) \le \gamma \kappa,$$

which proves (3.8).

Let us estimate the terms on the right-hand side of (3.3) for $l = l_{j+1}$. By the inequality

$$\left(\frac{u - l_j}{\delta_j} \right)_+ \le \gamma \left(\frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)_+^{\frac{1}{m^-}} + \gamma a_j \frac{(u^{m^-} - l_j^{m^-})_+}{\delta_j^{m^-}}, \; a_j = \left(\frac{l_j}{\delta_j} \right)^{1-m^-}$$

(see for example [6], Lemma 2.5) we obtain

$$\kappa = \frac{\gamma}{v(B_j) r_j} \iint_{L_j} v u^{m^+-1} \left(\frac{u - l_j}{\delta_j} \right)^{1+\lambda} \zeta_j^{k-2} dxdt$$

$$+ \frac{\gamma}{w(B_j) r_j} \iint_{L_j} w u^{m^+-1} \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{1+\lambda} \zeta_j^{k-2} dxdt$$

$$\le \frac{\gamma r_j^{-1}}{v(B_j)} \iint_{L_j} \frac{v}{u} \left(\frac{u - l_j}{\delta_j} \right)^{1+\lambda} \left(\delta_j^{m^+} \left(\frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{\frac{m^+}{m^-}} + l_j^{m^+} \right) \zeta_j^{k-2} dxdt$$

$$+\frac{\gamma}{w(B_j)r_j}\iint_{L_j} w\left(\delta_j^{m^+-1}\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{m^+-1}{m^-}}+l_j^{m^+-1}\right)$$

$$\times\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_j^{k-2}dxdt$$

$$\leq\frac{\gamma\delta_j^{m^+-1}}{v(B_j)r_j}\iint_{L_j} v\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{m^++\lambda}{m^-}}\zeta_j^{k-2}dxdt$$

$$+\frac{\gamma\delta_j^{m^+-1}}{v(B_j)r_j}\iint_{L_j} v\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{m^+}{m^-}}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\lambda}\zeta_j^{k-2}dxdt$$

$$+\frac{\gamma l_j^{m^+-1}}{v(B_j)r_j}\iint_{L_j} v\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{1+\lambda}{m^-}}\zeta_j^{k-2}dxdt$$

$$+\frac{\gamma l_j^{m^+-1}}{v(B_j)r_j}\iint_{L_j} v\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_j^{k-2}dxdt$$

$$+\frac{\gamma\delta_j^{m^+-1}}{w(B_j)r_j}\iint_{L_j} w\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{m+\lambda m^-}{m^-}}\zeta_j^{k-2}dxdt$$

$$+\frac{\gamma l_j^{m^+-1}}{w(B_j)r_j}\iint_{L_j} w\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_j^{k-2}dxdt=\sum_{i=1}^{6}I_i, \qquad (3.9)$$

here we also used the evident equalities $m^+-1+m^-=m$ and $(1-m^-)(m^+-1)=0$.

Now we estimate the terms on the right hand side of (3.9), for this we set

$$G(u):=\int_0^u(1+s)^{-\frac{1+\lambda}{2}}ds,\ H(u):=\int_0^u s^{\frac{m-m^-}{2m^-}}(1+s)^{-\frac{1+\lambda}{2}}ds,\ ,u>0$$

$$F(u,a):=\frac{1}{\delta}\int_l^u\left(1-\left(1+a\frac{s^{m^-}-l^{m^-}}{\delta^{m^-}}\right)\right)^{-\lambda}ds,\ a,l,\delta>0,\ u>l.$$

Claim. For any $u,l,\delta>0$, $\varepsilon\in(0,1)$ there exists $\gamma>0$ depending only on the data, λ and ε such that

$$u\leq\varepsilon+\gamma(\varepsilon)G^{\frac{2}{1-\lambda}}(u), \qquad (3.10)$$

$$u \le \varepsilon + \gamma(\varepsilon) H^{\frac{2m^-}{m-\lambda m^-}}(u), \tag{3.11}$$

$$\left(\frac{l}{\delta}\right)^{1-m^-} \frac{u^{m^-} - l^{m^-}}{\delta^{m^-}} \le \varepsilon + \frac{\gamma(\varepsilon)}{\delta} F\left(u, \left(\frac{l}{\delta}\right)^{1-m^-}\right), \tag{3.12}$$

$$\left(\frac{u^{m^-} - l^{m^-}}{\delta^{m^-}}\right)^{\frac{1}{m^-}} \le \varepsilon + \frac{\gamma(\varepsilon)}{\delta} F(u, 1). \tag{3.13}$$

Proof. If $u \ge \varepsilon$ then $G(u) \ge \int_0^u \left(\frac{u}{\varepsilon} + s\right)^{-\frac{1+\lambda}{2}} ds \ge \left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{1+\lambda}{2}} u^{\frac{1-\lambda}{2}}$,

$$H(u) \ge \int_0^u s^{\frac{m-m^-}{2m^-}} \left(\frac{u}{\varepsilon} + s\right)^{-\frac{1+\lambda}{2}} ds \ge \frac{2m^-}{m^- + m} \left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{1+\lambda}{2}} u^{\frac{m-\lambda m^-}{2m^-}},$$

from which (3.10), (3.11) follow. To prove (3.12) we note that

$$\int_0^u (1 - (1+s)^{-\lambda}) ds = \lambda \int_0^u ds \int_0^s \frac{dz}{(1+z)^{1+\lambda}} = \lambda \int_0^u \frac{u-z}{(1+z)^{1+\lambda}} dz$$

$$\ge \frac{\lambda u}{2} \int_0^{\frac{u}{2}} \frac{dz}{(1+z)^{1+\lambda}} \ge \frac{\lambda u}{2} \int_0^{\frac{\varepsilon}{2}} \frac{dz}{(1+z)^{1+\lambda}} \ge \frac{\lambda 2^{\lambda-1} \varepsilon u}{(2+\varepsilon)^{1+\lambda}},$$

if $u \ge \varepsilon$. From this $F\left(u, \left(\frac{l}{\delta}\right)^{1-m^-}\right)$

$$\ge \frac{l^{1-m^-}}{\delta} \int_l^u \left(1 - \left(1 + \left(\frac{l}{\delta}\right)^{1-m^-} \frac{s^{m^-} - l^{m^-}}{\delta^{m^-}}\right)^{-\lambda}\right) s^{m^- - 1} ds$$

$$\ge \frac{m^- \lambda 2^{\lambda-1} \varepsilon}{(2+\varepsilon)^{1+\lambda}} \left(\frac{l}{\delta}\right)^{1-m^-} \frac{u^{m^-} - l^{m^-}}{\delta^{m^-}}, \quad \text{if} \left(\frac{l}{\delta}\right)^{1-m^-} \frac{u^{m^-} - l^{m^-}}{\delta^{m^-}} \ge \varepsilon,$$

which proves (3.12). Similarly

$$F(u,1) = \frac{\lambda m^-}{\delta^{m^-+1}} \int_l^u ds \int_l^s \frac{z^{m^- - 1} dz}{\left(1 + \frac{z^{m^-} - l^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}}$$

$$= \frac{\lambda m^-}{\delta^{m^-+1}} \int_l^u \frac{z^{m^- - 1}(u - z) dz}{\left(1 + \frac{z^{m^-} - l^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}}$$

$$\ge \frac{\lambda m^-}{\delta^{m^-+1}} \int_l^u \frac{z^{m^- - 1} \left(u^{m^-} - z^{m^-}\right)^{\frac{1}{m^-}} dz}{\left(1 + \frac{z^{m^-} - l^{m^-}}{\delta^{m^-}}\right)^{1+\lambda}}$$

$$= \frac{\lambda}{\delta}\int_0^{\frac{u^{m^-}-l^{m^-}}{\delta^{m^-}}} \frac{(u^{m^-}-l^{m^-}-\delta^{m^-}z)^{\frac{1}{m^-}}dz}{(1+z)^{1+\lambda}}$$

$$\geq \lambda\left(\frac{u^{m^-}-l^{m^-}}{2\delta^{m^-}}\right)^{\frac{1}{m^-}}\int_0^{\frac{u^{m^-}-l^{m^-}}{\delta^{m^-}}}\frac{dz}{(1+z)^{1+\lambda}}$$

$$\geq \frac{\lambda 2^{\lambda-1}\varepsilon}{(2+\varepsilon)^{1+\lambda}}\left(\frac{u^{m^-}-l^{m^-}}{\delta^{m^-}}\right)^{\frac{1}{m^-}}, \quad \text{if} \quad \frac{u^{m^-}-l^{m^-}}{\delta^{m^-}}\geq\varepsilon,$$

which proves the claim.

To estimate the right hand side of (3.9) we rewrite inequality (2.9) for $a=1$ and $a=a_j$ in terms of G,H and F. For simplicity let us set
$G_j := G\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)$, $\quad G_j(a_j) := G\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)$,
$H_j := H\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)$, $\quad H_j(a_j) := H\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)$,
$F_j := F(u,1)$, $F_j(a_j) := F(u,a_j)$. By Lemma 2.5 we obtain

$$\sup_{t_0-\tau_j<t<t_0}\int_{B_j} vF_j\zeta_j^k dx + \sup_{t_0-\tau_j<t<t_0}\int_{B_j} vF_j(a_j)\zeta_j^k dx$$

$$+l_j^{m-m^-}\delta_j^{m^- -1}\iint_{L_j} w|\nabla G_j|^2\zeta_j^k dxdt + l_j^{m-1}\iint_{L_j} w|\nabla G_j(a_j)|^2\zeta_j^k dxdt$$

$$+\delta_j^{m-1}\iint_{L_j} w|\nabla H_j|^2\zeta_j^k dxdt + a_j^{-\frac{m}{m^-}}\delta_j^{m-1}\iint_{L_j} w|\nabla H_j(a_j)|^2\zeta_j^k dxdt$$

$$\leq \frac{\gamma}{\tau_j}\iint_{L_j} v\frac{u-l_j}{\delta_j}\zeta_j^{k-1}dxdt$$

$$+\frac{\gamma l_j^{m^- -1}}{\rho_j^2}\iint_{L_j} wu^{m-m^-}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_j^{k-2}dxdt$$

$$+\frac{\gamma}{\tau_j}\iint_{Q_j}|f|dxdt, \tag{3.14}$$

here we also used the evident inequality

$$\delta_j^{m-1}\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda} = l_j^{m^- -1}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\lambda}$$

$$\leq \gamma l_j^{m^- -1}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}.$$

By (3.5), (3.8), (3.10), (3.11) we obtain for every $\varepsilon \in (0,1)$

$$I_1 + I_2 + I_5 \le \gamma \frac{\delta_j^{m^+-1}}{v(B_j) r_j} \left(\varepsilon^{\frac{m^+ + \lambda}{m^-}} + \varepsilon^{\frac{m^+}{m^-}} \right) \iint_{L_j} v \zeta_j^{k-2} dxdt$$

$$+ \gamma \delta_j^{m^+-1} \varepsilon^{\frac{m^+ + \lambda m^-}{m^-}} \iint_{L_j} w \zeta_j^{k-2} dxdt$$

$$+ \gamma(\varepsilon) \frac{\delta_j^{m^+-1}}{v(B_j) r_j} \iint_{L_j} v H_j^{2 + 2\frac{m^+ - m + \lambda}{m - \lambda m^-}} G_j(a_j)^{\frac{2\lambda}{1-\lambda}} \zeta_j^{k-2} dxdt$$

$$+ \gamma(\varepsilon) \frac{\delta_j^{m^+-1}}{w(B_j) r_j} \iint_{L_j} w H_j(a_j)^{2 + \frac{4\lambda m^-}{m - \lambda m^-}} \zeta_j^{k-2} dxdt$$

$$\le \gamma \varepsilon^{\frac{m^+}{m^-}} \kappa + \frac{\delta_j^{m^+-1}}{v(B_j) r_j} \iint_{L_j} v H_j^{2 + 2\frac{m^+ - m + \lambda}{m - \lambda m^-}} G_j(a_j)^{\frac{2\lambda}{1-\lambda}} \zeta_j^{k-2} dxdt$$

$$+ \gamma(\varepsilon) \frac{\delta_j^{m^+-1}}{w(B_j) r_j} \iint_{L_j} w H_j(a_j)^{2 + \frac{4\lambda m^-}{m - \lambda m^-}} \zeta_j^{k-2} dxdt = \gamma \varepsilon^{\frac{m^+}{m^-}} \kappa + I_7 + I_8. \tag{3.15}$$

Similarly

$$I_3 + I_4 + I_6 \le \gamma \varepsilon^{1+\lambda} \kappa + \frac{\gamma(\varepsilon) l_j^{m^+-1}}{v(B_j) r_j} \iint_{L_j} v G_j^{2 + \frac{2(m^+ - m + \lambda(1+m^-))}{m^-(1-\lambda)}} \zeta_j^{k-2} dxdt$$

$$+ \frac{\gamma(\varepsilon) l_j^{m^+-1}}{r_j} \iint_{L_j} \left(\frac{v}{v(B_j)} + \frac{w}{w(B_j)} \right) G_j(a_j)^{2 + \frac{4\lambda}{(1-\lambda)}} \zeta_j^{k-2} dxdt$$

$$= \gamma \varepsilon^{1+\lambda} \kappa + I_9 + I_{10}. \tag{3.16}$$

Further we will assume that λ satisfies the condition $0 < \lambda < min\left(\frac{h}{2}, \frac{\beta\nu(m^+)}{n\nu(1+m^-)}\right)$, where $h > 0$ was defined in Lemma 2.2. By the Hölder inequality, Lemma 2.1 with $p = 2$, $w_1 = w$, $w_2 = v$, $q = \frac{2n\nu}{n\nu - \alpha}$, Lemma 2.2, (3.12), (3.13) and the evident inequalities $G(u) \le \frac{2}{1-\lambda} u^{\frac{1-\lambda}{2}}, H(u) \le \frac{2m^-}{m - \lambda m^-} u^{\frac{m - \lambda m^-}{2m^-}}$, $u \ge 0$, we obtain for any $\varepsilon_1 \in (0,1)$

$$I_7 + I_8 \le \frac{\gamma(\varepsilon)}{r_j} \delta_j^{m^+-1} \left(\varepsilon_1 + \frac{\gamma(\varepsilon_1)}{\delta_j v(B_j)} \sup_{t_0 - \tau_j < t < t_0} \int_{B_j} v F_j \zeta_j^{k_1} dx \right)^{m^+ - m + \lambda}$$

$$\times \left(\varepsilon_1 + \frac{\gamma(\varepsilon_1)}{\delta_j v(B_j)} \sup_{t_0 - \tau_j < t < t_0} \int_{B_j} v F_j(a_j) \zeta_j^{k_1} dx \right)^{\frac{\beta\nu(m^+) - \lambda\nu}{n\nu}}$$

$$\times\frac{\rho_j^2}{w(B_j)}\iint_{L_j} w|\nabla(H_j\zeta_j^{k_1})|^2dxdt$$

$$+\frac{\gamma(\varepsilon)}{r_j}\delta_j^{m^+-1}\left(\varepsilon_1+\frac{\gamma(\varepsilon_1)}{v(B_j)}\sup_{t_0-\tau_j<t<t_0}\int_{B_j} vF_j(a_j)\zeta_j^{k_1}dx\right)^h$$

$$\times\frac{\rho_j^2}{w(B_j)}\iint_{L_j} w|\nabla(H_j(a_j)\zeta_j^{\frac{k_1}{2}})|^2dxdt,$$

where $k_1=(k-2)min\left\{\frac{n\nu-\alpha}{n\nu},\left(2+\frac{4\lambda m^-}{h(m-\lambda m^-)}\right)^{-1}\right\}$.

Note that by our choices $\delta_j^{m^+-1}\le\gamma\delta_j^{m-1}l_j^{m^+-m}a_j^{-\frac{m}{m^-}}$ and

$$\frac{\rho_j^2}{w(B_j)}=\frac{r_jl^{m-m^+}}{v(B_j)}. \tag{3.17}$$

Hence from the previous and (3.14) we obtain

$$I_7+I_8\le\varepsilon_1^{\frac{\alpha}{n\nu}}\gamma(\varepsilon)I_{11} + \gamma(\varepsilon,\varepsilon_1)I_{11}^{1+m^+-m+\lambda}$$

$$+\gamma(\varepsilon,\varepsilon_1)I_{11}^{1+\frac{\beta_\nu(m^+-\lambda n\nu)}{n\nu}} + \gamma(\varepsilon,\varepsilon_1)I_{11}^{1+\frac{\alpha}{n\nu}}, \tag{3.18}$$

where

$$I_{11}=\frac{1}{\tau_jv(B_j)}\iint_{L_j} v\frac{u-l_j}{\delta_j}\zeta_j^{k-2}dxdt$$

$$+\frac{l_j^{m^--1}}{\rho_j^2v(B_j)}\iint_{L_j} wu^{m-m^-}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_j^{k\alpha_1-2}dxdt$$

$$+\frac{\delta_j^{m^+-1}}{r_jw(B_j)}\iint_{L_j} wu^{m-m^-}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{m-\lambda m^-}{m^-}}\zeta_j^{k\alpha_1-2}dxdt$$

$$+\frac{\delta_j^{m^+-1}}{r_jw(B_j)}\iint_{L_j} wu^{m-m^-}\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{\frac{m-\lambda m^-}{m^-}}\zeta_j^{k\alpha_1-2}dxdt$$

$$+\frac{1}{\delta_jv(B_j)}\iint_{Q_j}|f|dxdt.$$

Similarly, for any $\varepsilon_1\in(0,1)$ we obtain

$$I_9+I_{10}\le\frac{\gamma(\varepsilon)}{r_j}l_j^{m^+-1}\left(\varepsilon_1+\frac{\gamma(\varepsilon_1)}{\delta_jv(B_j)}\sup_{t_0-\tau_j<t<t_0}\int_{B_j} vF_j\zeta_j^{k_1}dx\right)^{\frac{\alpha}{n\nu}}$$

$$\times\frac{\rho_j^2}{w(B_j)}\iint_{L_j} w|\nabla(G_j\zeta_j^{\frac{k_1}{2}})|^2dxdt$$

$$+\frac{\gamma(\varepsilon)}{r_j}l_j^{m^+-1}\Bigg[\left(\varepsilon_1+\frac{\gamma(\varepsilon_1)}{\delta_j v(B_j)}\sup_{t_0-\tau_j<t<t_0}\int_{B_j} vF_j(a_j)\zeta_j^{k_1}dx\right)^{\frac{\alpha}{n\nu}}$$

$$+\left(\varepsilon_1+\frac{\gamma(\varepsilon_1)}{\delta_j v(B_j)}\sup_{t_0-\tau_j<t<t_0}\int_{B_j} vF_j(a_j)\zeta_j^{k_1}dx\right)^{h}\Bigg]$$

$$\times\frac{\rho_j^2}{w(B_j)}\iint_{L_j} w|\nabla(G_j\zeta_j^{\frac{k_1}{2}})|^2dxdt.$$

From this using (3.14), (3.17) and the evident inequality $l_j^{m^+-1}\le\gamma\delta_j^{m^--1}l_j^{m-m^-}$ we arrive at

$$I_9+I_{10}\le\left(\varepsilon_1^{\frac{\alpha}{n\nu}}\gamma(\varepsilon)+\varepsilon_1^h\gamma(\varepsilon)\right)I_{12}+\gamma(\varepsilon,\varepsilon_1)I_{12}^{\frac{\alpha}{n\nu}}+\gamma(\varepsilon,\varepsilon_1)I_{12}^{1+h},\quad(3.19)$$

$$I_{12}=\frac{1}{\tau_j v(B_j)}\iint_{L_j} v\frac{u-l_j}{\delta_j}\zeta_j^{k_1-2}dxdt$$

$$+\frac{l_j^{m^--1}}{\rho_j^2 v(B_j)}\iint_{L_j} wu^{m-m^-}\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1+\lambda}\zeta_j^{k_1-2}dxdt$$

$$+\frac{l_j^{m^+-1}}{r_j w(B_j)}\iint_{L_j} w\left(a_j\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1-\lambda}\zeta_j^{k_1-2}dxdt$$

$$+\frac{l_j^{m^+-1}}{r_j w(B_j)}\iint_{L_j} w\left(\frac{u^{m^-}-l_j^{m^-}}{\delta_j^{m^-}}\right)^{1-\lambda}\zeta_j^{k_1-2}dxdt+\frac{1}{\delta_j v(B_j)}\iint_{Q_j}|f|dxdt.$$

Collecting estimates (3.9), (3.16), (3.18), (3.19) we arrive at

$$\kappa\le\left(\varepsilon^{\frac{m^+}{m^-}}+\varepsilon^{1+\lambda}\right)\gamma\kappa+\left(\varepsilon_1^{\frac{\alpha}{n\nu}}+\varepsilon_1^h\right)\gamma(\varepsilon)(I_{11}+I_{12})$$

$$+\gamma(\varepsilon,\varepsilon_1)(I_{11}+I_{12})\Bigg[(I_{11}+I_{12})^{\frac{\alpha}{n\nu}}+(I_{11}+I_{12})^h(I_{11}+I_{12})^{m^+-m+\lambda}$$

$$+(I_{11}+I_{12})^{\frac{\beta_\nu(m^+)-\lambda n\nu}{n\nu}}\Bigg].\quad(3.20)$$

Let us estimate I_{11} and I_{12}. Since $\frac{u-l_{j-1}}{\delta_{j-1}}=\frac{u-l_j}{\delta_{j-1}}\ge1$ on L_j, by (3.17), choosing $k_1=2$ we obtain

$$
\begin{aligned}
&\frac{1}{\tau_j v(B_j)} \iint_{L_j} v \frac{u - l_j}{\delta_j} dxdt \\
&+ \frac{l_j^{m^- - 1}}{\rho_j^2 v(B_j)} \iint_{L_j} w u^{m-m^-} \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{1+\lambda} dxdt \\
&\le \frac{1}{r_j v(B_j)} \iint_{L_j} v u^{m^+ - 1} \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right)^{1+\lambda} \zeta_{j-1}^{k-2} dxdt \\
&+ \frac{1}{r_j w(B_j)} \iint_{L_j} w u^{m-m^-} \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{1+\lambda} \zeta_{j-1}^{k-2} dxdt \\
&\le \gamma A_{j-1}(l_j) \le \kappa. \qquad (3.21)
\end{aligned}
$$

Since $(1 - m^-)(m - m^-) = 0$ and $m^+ - 1 = m - m^-$, by (3.5) we obtain

$$
\begin{aligned}
&\frac{\delta_j^{m^+ - 1}}{r_j w(B_j)} \iint_{L_j} w \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{\frac{m - \lambda m^-}{m^-}} dxdt \\
&+ \frac{\delta_j^{m^+ - 1}}{r_j w(B_j)} \iint_{L_j} w \left(\frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{\frac{m - \lambda m^-}{m^-}} dxdt \\
&\le \frac{\gamma}{r_j w(B_j)} \iint_{L_j} w u^{m-m^-} \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{1-\lambda} dxdt \\
&\le \frac{\gamma}{r_j w(B_j)} \iint_{L_j} w u^{m-m^-} \left(a_{j-1} \frac{u^{m^-} - l_j^{m^-}}{\delta_{j-1}^{m^-}} \right)^{1-\lambda} \zeta_{j-1}^{k-2} dxdt \\
&\le \gamma A_{j-1}(l_j) \le \gamma\kappa. \qquad (3.22)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{l_j^{m^+ - 1}}{r_j w(B_j)} \iint_{L_j} w \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{1-\lambda} dxdt \\
&+ \frac{l_j^{m^+ - 1}}{r_j w(B_j)} \iint_{L_j} w \left(\frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right)^{1-\lambda} dxdt \\
&\le \frac{\gamma}{r_j w(B_j)} \iint_{L_j} w u^{m-m^-} \left(a_j \frac{u^{m^-} - l_j^{m^-}}{\delta_j^{m^-}} \right) dxdt \\
&\le \gamma A_{j-1}(l_j) \le \gamma\kappa. \qquad (3.23)
\end{aligned}
$$

Combining estimates (3.21)–(3.23) we obtain

$$I_{11} + I_{12} \le \gamma\kappa + \frac{1}{\delta_j} \iint_{Q_j} |f| dxdt.$$

Thus inequality (3.20) implies that

$$\kappa \le \left(\varepsilon^{\frac{m^+}{m^-}} + \varepsilon^{1+\lambda}\right)\gamma\kappa \ + \ \left(\varepsilon_1^{\frac{\alpha}{n\nu}} + \varepsilon_1^h\right)\gamma(\varepsilon)\kappa$$

$$+ \gamma(\varepsilon,\varepsilon_1)\kappa\left(\kappa^{\frac{\alpha}{n\nu}} + \kappa^h + \kappa^{m^+-m+\lambda} + \kappa^{\frac{\beta\nu(m^+)-\lambda n\nu}{n\nu}}\right)$$

$$+ \gamma(\varepsilon,\varepsilon_1)\frac{1}{\delta_j v(B_j)} \iint_{Q_j} |f| dxdt \Bigg[1 + \left(\frac{1}{\delta_j v(B_j)} \iint_{Q_j} |f| dxdt\right)^{\frac{\alpha}{n\nu}}$$

$$+ \left(\frac{1}{\delta_j v(B_j)} \iint_{Q_j} |f| dxdt\right)^{h} + \left(\frac{1}{\delta_j v(B_j)} \iint_{Q_j} |f| dxdt\right)^{m^+-m+\lambda}$$

$$+ \left(\frac{1}{\delta_j v(B_j)} \iint_{Q_j} |f| dxdt\right)^{\frac{\beta\nu(m^+)-\lambda n\nu}{n\nu}}\Bigg]. \tag{3.24}$$

First choose ε from the condition $\left(\varepsilon^{\frac{m^+}{m^-}} + \varepsilon^{1+\lambda}\right)\gamma = \frac{1}{8}$, then choose ε_1 from the condition $\left(\varepsilon_1^{\frac{\alpha}{n\nu}} + \varepsilon_1^h\right)\gamma(\varepsilon) = \frac{1}{8}$, and next fix κ by the condition

$$\gamma(\varepsilon,\varepsilon_1)\left(\kappa^{\frac{\alpha}{n\nu}} + \kappa^h + \kappa^{m^+-m+\lambda} + \kappa^{\frac{\beta\nu(m^+)-\lambda n\nu}{n\nu}}\right) = \frac{1}{8},$$

therefore inequality (3.24) implies that

$$\gamma(\varepsilon,\varepsilon_1,\kappa) \le \frac{1}{\delta_j v(B_j)} \iint_{Q_j} |f| dxdt,$$

which proves Lemma 3.1.

To complete the proof of Theorem 1.1 we sum inequality (3.6) with respect to j from 0 to $J-1$, as $\{l_j\}_{j\in N}$ is increasing sequense, and by the inequalities $\frac{r_0}{\psi_{x_0}(R) l_j^{m^+-m}} \le 1$, $\rho_j \le \tilde{\rho}_j$, $\tau_j \le \psi_{x_0}(\tilde{\rho}_j)\frac{\theta}{\psi_{x_0}(R)}$, $\psi_{x_0}^{-1}\left(\frac{r_j}{l_j^{m^+-m}}\right) \ \ge \ K_6^{-1}\left(\frac{r_0}{\psi_{x_0}(R) l_j^{m^+-m}}\right)^{\frac{1}{\alpha}}\tilde{\rho}_j$,

$v(B_j) \geq K_5^{-1}K_6^{-1}\left(\frac{r_0}{\psi_{x_0}(R)l_j^{m^+-m}}\right)^{\frac{n\nu}{\alpha}} v(B_{\tilde{\rho}_j}(x_0)),\ \tilde{\rho}_j = \psi_{x_0}^{-1}\left(\psi_{x_0}(R)\sigma^j\right),$ $j = 0, 1, 2, ...$ we obtain

$$l_J \leq \gamma\delta_{-1} + \gamma l_J^{(m^+-m)\frac{n\nu}{\alpha}} \left(\frac{\psi_{x_0}(R)}{r_0}\right)^{\frac{n\nu}{\alpha}}$$

$$\times \sum_{j=0}^{J} \frac{1}{v(B_{\tilde{\rho}_j}(x_0))} \iint_{Q_{\tilde{\rho}_j, \psi_{x_0}(\tilde{\rho}_j)\frac{\theta}{\psi_{x_0}(R)}}(x_0,t_0)} |f| dxdt,$$

where δ_{-1} has been defined in (3.2). Next we estimate the last term in the previous inequality. By (1.4), (3.1) we obtain

$$\sum_{j=0}^{J} \frac{1}{v(B_{\tilde{\rho}_j}(x_0))} \iint_{Q_{\tilde{\rho}_j, \psi_{x_0}(\tilde{\rho}_j)\frac{\theta}{\psi_{x_0}(R)}}(x_0,t_0)} |f| dxdt$$

$$\leq K_5K_6\sigma^{-\frac{n\nu}{\alpha}} \sum_{j=0}^{\infty} \int_{\tilde{\rho}_j}^{\tilde{\rho}_{j-1}} \frac{1}{v(B_\rho(x_0))} \iint_{Q_{\rho, \psi_{x_0}(\rho)\frac{\theta}{\psi_{x_0}(R)}}(x_0,t_0)} |f| dxdt$$

$$\leq \gamma \int_0^{\psi_{x_0}^{-1}\left(\psi_{x_0}(R)\sigma^{-1}\right)} \frac{1}{v(B_\rho(x_0))} \iint_{Q_{\rho, \psi_{x_0}(\rho)\frac{\theta}{\psi_{x_0}(R)}}(x_0,t_0)} |f| dxdt$$

$$\leq \gamma \int_0^{K_6\sigma^{-\frac{1}{\alpha}}R} \frac{1}{v(B_\rho(x_0))} \iint_{Q_{\rho, \psi_{x_0}(\rho)\frac{\theta}{\psi_{x_0}(R)}}(x_0,t_0)} |f| dxdt$$

$$= \gamma I_{v,w,f}\left(x_0, t_0, \gamma R, \frac{\theta}{\psi_{x_0}(R)}\right).$$

This implies that

$$l_J \leq \gamma\delta_{-1} + \gamma\left(\frac{\psi_{x_0}(R)}{r_0}\right)^{\frac{n\nu}{\beta_\nu(m^+)}} I_{v,w,f}^{\frac{\alpha}{\beta_\nu(m^+)}}\left(x_0, t_0, \gamma R, \frac{\theta}{\psi_{x_0}(R)}\right). \quad (3.25)$$

Hence the sequence $\{l_j\}_{j\in N}$ is convergent and $\delta_j \to 0$ $(j \to \infty)$ and we can pass to the limit $J \to \infty$ in (3.25), let $l_\infty = \lim\limits_{j\to\infty} l_j$, from (3.3) we conclude that

$$\frac{1}{v(B_{\tilde{\rho}_j}(x_0))} \iint_{Q_j} vu^{m^+-1}(u - l_\infty)^{1+\lambda} dxdt$$

$$\leq \gamma l_\infty^{-(m^+-m)\frac{n\nu}{\alpha}} \left(\frac{r_0}{\psi_R(x_0)}\right)^{\frac{n\nu}{\alpha}} \delta_j^{1+\lambda} \to 0 \quad (j \to \infty).$$

Choosing (x_0, t_0) as a Lebesgue point we obtain that $u(x_0, t_0) \le l_\infty$, and hence $u(x_0, t_0)$ is estimated from above by the right hand side of (3.25). This completes the proof of Theorem 1.1. □

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