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A nonlocal boundary value problem for a fourth order mixed type equation

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Abstract. The criterion of uniqueness of the solution of the problem with periodicity, nonlocal and boundary conditions is established by spectral analysis for the fourth-order mixed-type equation in a rectangular region. When constructing a solution in the form of the sum of a series, we use completeness in the space L_2 orthogonally conjugate to the system of eigenfunctions of the corresponding problem. When proving the convergence of a series, the problem of small denominators arises. Under conditions on the parameters of the data of the problem and given functions, the stability of the solution is proved.

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1. Introduction

The work is devoted to the study of a nonlocal boundary value problem for a fourth-order partial differential equation of mixed type.

Let $u(x,t)$ on the region $\Omega = \{(x,t)|\,(-1,\,1)\times(0,T),\,x\neq 0\}$ satisfies the equation

$$sgn\,x\,\frac{\partial^4 u}{\partial t^4} - Lu = 0. \tag{1.1}$$

where $Lu \equiv -\frac{\partial}{\partial x}\left(p(x)u_x(x,t)\right) + q(x)u(x,t)$ and $p(x)$, $p'(x)$, $q(x)$ continuous functions on the segment $[-1,\,1]$, $q(x)\geqslant 0$, $p(x) > p_0$, p_0 some positive constant.

Problem. Find function $u(x,t)$, satisfying equation (1.1) on the region Ω and the following conditions:

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nonlocal

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} + \left.\frac{\partial^i u}{\partial t^i}\right|_{t=T} = \varphi_i(x),\ i = 0, 1, 2, 3,\ x \in [-1,\ 1], \tag{1.2}$$

boundary

$$u(-1, t) = u(1, t) = 0,\ t \in [0,\ T] \tag{1.3}$$

and terms of bonding

$$\left.\frac{\partial^i u}{\partial x^i}\right|_{x=-0} = \left.\frac{\partial^i u}{\partial x^i}\right|_{x=+0},\ i = 0, 1,\ t \in [0,\ T]. \tag{1.4}$$

Nonlocal problems for mixed type differential equations with partial derivatives currently being studied very actively, and mainly equations of the first and second orders were considered. It should be noted the works of such authors as A. A. Dezin, G. Infante, T. Jankowski, V. A. Ilyin, M. A. Naimark, E. I. Moiseev, K. B. Sabitov, S. G. Krein and G. I. Laptev, O. A. Repin, I. E. Egorov, A. I. Kozhanov and others.

Fourth-order partial differential equations were studied in the works of the authors M. Smirnov [12], T. D. Djuraev and A. K. Sopuev [3], A. I. Kozhanov [9], V. B. Dmitriev [4], K. S. Fayazov and I. O. Khajiev [7], T. K. Yuldashev [14] and others.

Note that nonlocal problems can be ill-posed in the sense of J. Hadamard. In the work of P. N. Vabischevich [13] parabolic equation with nonlocal temporary variable conditions was investigated. For a stable approximate solution of such problems, an approach is used when a nonlocal condition is used instead of the initial condition. The regularizing properties of such a method are established in the usual class of bounded solutions. By S. P. Shishatsky in [11] was considered a boundary-value problem for a second-order differential equation in a Hilbert space H with a negative self-adjoint operator. K. S. Fayazov in [6] investigated boundary value problems for a second-order differential equation with self-adjoint operator coefficients in a Hilbert space.

The initial-boundary value problem for equation (1.1) was investigated in [7]. Using the methods of spectral decompositions and energy integrals, theorems on the uniqueness and conditional stability of a solution on a set of correctness are proved. An approximate solution is constructed by the regularization method and an error estimate of the norm of the difference between the exact and approximate solutions is obtained.

As a matter of fact, the problem (1.1)–(1.4) is incorrect as in the since of J. Hadamard, namely, there is no continuous dependence of the solution on the data of the problem. In addition, arises the problem of “small denominators" (see B. I. Ptashnik [10]), to be more exact, the problem has not a unique solution for all T. The Dirichlet problem for the wave equation, which is also incorrect studied in the works of D. G. Bourgin and R. Duffin [2] and Sh. A. Alimov [1].

This paper presents the conditions on the data of the problem in which the solution of the problem is unique and conditionally stable set of correctness. We give some facts from [5] necessary for the further presentation of our results.

We will seek a solution to problem (1.1)–(1.4) $u(x,t)$ in the form of a Fourier series in the eigenfunctions of the following spectral problem: Find the values of λ for which the problem

$$\begin{cases} sgn\,(x)\frac{d}{dx}\,(p(x)X'(x)) - sgn\,(x)q(x)X(x) + \lambda X(x) = 0, \\ X(-1) = X(1) = 0, \\ X(-0) = X(+0),\; X'(-0) = X'(+0). \end{cases} \tag{1.5}$$

has a non-trivial solution.

By $\{X_k^+(x)\}_{k=1}^{\infty}$, $\{X_k^-(x)\}_{k=1}^{\infty}$ we denote the eigenfunctions, through the corresponding positive $\{\lambda_k^+\}_{k=1}^{\infty}$ and negative $\{\lambda_k^-\}_{k=1}^{\infty}$ eigenvalues, and the numbers λ_k^+, $-\lambda_k^-$ form non-decreasing sequences.

According to [5], the eigenfunctions of problem (1.5) have the property

$$\left(sgn\,x\,X_k^{\pm}, X_j^{\pm}\right) = \pm\delta_{kj},\,,\;\left(sgn\,x\,X_k^{+}, X_j^{-}\right) = 0\;\forall k, j \in N,$$

where δ_{kj} is the Kronecker symbol.

Let $(u,v) = \int\limits_{-1}^{1} uvdx$ be the scalar product in $L_2(-1,1)$, and $\|u\| = \left(\int\limits_{-1}^{1} u^2(x,t)dx\right)^{1/2}$, and

$$\|u(x,t)\|_0^2 = \sum_{k=1}^{\infty}\left\{\left|\left(sgnx\;u(x,t),\,X_k^+\right)\right|^2 + \left|\left(sgnx\;u(x,t),\,X_k^-\right)\right|^2\right\}. \tag{1.6}$$

The eigenfunctions of problem (1.5) form a Riesz basis in H_0 and the norm in the space $L_2(-1,\,1)$ defined by equality (1.6) is equivalent to the original one [5].

2. Form of solution

By a generalized solution of the boundary value problem (1.1)–(1.4) we understand a function $u(x,t) \in W_2^{1,3}(\Omega)$ satisfying

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} + \left.\frac{\partial^j u}{\partial t^j}\right|_{t=T} = \varphi_j(x),\; j = 0,1,2,$$

$u(-1,t)=u(1,t)$ conditions and the following identity

$$\int_0^T\int_{-1}^1 (sgnx\, u_{ttt}V_t+u_xpV_x+uqV)\,dxdt$$
$$=\int_{-1}^1 sgnx\,\varphi_3(x)\,V(x,T)dx \quad (2.1)$$

for any function $V(x,t)\in W^{2,4}_{x,t}(\Omega)$, $\frac{\partial^j V(x,t)}{\partial t^j}\Big|_{t=0}+\frac{\partial^j V(x,t)}{\partial t^j}\Big|_{t=T}=0$, $V(-1,t)=V(1,t)=0$, $j=0,1,2,3$.

Let the solution of the problem (1.1)–(1.4) exists and has the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k^+(t)X_k^+ +\sum_{k=1}^{\infty}u_k^-(t)X_k^-$$

where $u_k^{\pm}(t)$ for each $k=1,2,3,...$ are solutions of the following problems:

$$\begin{cases}\{u_k^+(t)\}_{tttt}-\mu_k^4u_k^+(t)=0,\\ \frac{d^j}{dt^j}u_k^+(t)\Big|_{t=0}+\frac{d^j}{dt^j}u_k^+(t)\Big|_{t=T}=\varphi^+_{j_k},\ j=0,1,2,3.,\end{cases} \quad (2.2)$$

$$\begin{cases}\{u_k^-(t)\}_{tttt}+4\gamma_k^4u_k^-(t)=0,\\ \frac{d^j}{dt^j}u_k^-(t)\Big|_{t=0}+\frac{d^j}{dt^j}u_k^-(t)\Big|_{t=T}=\varphi^-_{j_k},\ j=0,1,2,3.,\end{cases} \quad (2.3)$$

where $\varphi^{\pm}_{j_k}=\pm\left(\operatorname{sgn}x\,X_k^{\pm}(x),\varphi_j(x)\right)$, $j=0,1,2,3.$, herewith $\gamma_k^4=\mu_k^4/4$, $\mu_k^4=\pm\lambda_k^{\pm}$.

We turn to the solution of the problem (2.2). Let $\frac{1}{\mu_k^2}\frac{d^2u_k^+}{dt^2}=\vartheta_k^+$, $w_k^+=u_k^+-\vartheta_k^+$, $v_k^+=u_k^++\vartheta_k^+$ then after some transformations we have

$$\begin{array}{l}\{v_k^+\}_{tt}-\mu_k^2v_k^+=0,\\ v_k^+(0)+v_k^+(T)=\varphi^+_{0_k}+\mu_k^{-2}\varphi^+_{2_k},\\ \{v_k^+(t)\}_t\big|_{t=0}+\{v_k^+(t)\}_t\big|_{t=T}=\varphi^+_{1_k}+\mu_k^{-2}\varphi^+_{3_k}.\end{array}$$

and

$$\begin{array}{l}\{w_k^+\}_{tt}+\mu_k^2w_k^+=0,\\ w_k^+(0)+w_k^+(T)=\varphi^+_{0_k}-\mu_k^{-2}\varphi^+_{2_k},\\ \{w_k^+(t)\}_t\big|_{t=0}+\{w_k^+(t)\}_t\big|_{t=T}=\varphi^+_{1_k}-\mu_k^{-2}\varphi^+_{3_k}.\end{array}$$

The solution of the latest problems can be represented as

$$v_k^+(t)=\frac{1}{2}\left(F(\mu_k,t)\left(\varphi^+_{0_k}+\mu_k^{-2}\varphi^+_{2_k}\right)+G(\mu_k,t)\left(\varphi^+_{1_k}+\mu_k^{-2}\varphi^+_{3_k}\right)\right),$$

$$w_k^+(t) = \frac{1}{2}\left(\overline{F}(\mu_k,t)\left(\varphi_{0_k}^+ - \mu_k^{-2}\varphi_{2_k}^+\right) + \overline{G}(\mu_k,t)\left(\varphi_{1_k}^+ - \mu_k^{-2}\varphi_{3_k}^+\right)\right),$$

where

$$F(\mu_k,t) = \frac{ch\mu_k t + ch\left(\mu_k(T-t)\right)}{1+ch\mu_k T}, \overline{F}(\mu_k,t) = \frac{\cos\mu_k t + \cos\left(\mu_k(T-t)\right)}{1+\cos\mu_k T},$$

$$G(\mu_k,t) = \frac{sh\mu_k t + sh\left(\mu_k(t-T)\right)}{1+ch\mu_k T}, \overline{G}(\mu_k,t) = \frac{sin\mu_k t + sin\left(\mu_k(t-T)\right)}{1+\cos\mu_k T}.$$

Then

$$\begin{aligned} u_k^+(t) &= \frac{1}{2}\left(v_k^+(t) + w_k^+(t)\right) \\ &= \frac{1}{4}\left(\left(F(\mu_k,t) + \overline{F}(\mu_k,t)\right)\varphi_{0_k}^+ + \mu_k^{-1}\left(G(\mu_k,t)+\overline{G}(\mu_k,t)\right)\varphi_{1_k}^+ \right.\\ &\left. + \mu_k^{-2}\left(F(\mu_k,t) - \overline{F}(\mu_k,t)\right)\varphi_{2_k}^+ + \mu_k^{-3}\left(G(\mu_k,t) - \overline{G}(\mu_k,t)\right)\varphi_{3_k}^+\right). \end{aligned} \quad (2.4)$$

Similarly, for problem (2.3) we have $\frac{1}{2i\gamma_k^2}\frac{d^2u_k^-}{dt^2} = \vartheta_k^-$, $w_k^- = u_k^- - \vartheta_k^-$, $v_k^- = u_k^- + \vartheta_k^-$

$$\begin{cases} \left\{v_k^-\right\}_{tt} - 2i\gamma_k^2 v_k^- = 0, \\ v_k^-(0) + v_k^-(T) = \varphi_{0_k}^- - 0,5i\gamma_k^{-2}\varphi_{2_k}^-, \\ \left\{v_k^-(t)\right\}_t\big|_{t=0} + \left\{v_k^-(t)\right\}_t\big|_{t=T} = \varphi_{1_k}^- - 0,5i\gamma_k^{-2}\varphi_{3_k}^-, \end{cases}$$

and

$$\begin{cases} \left\{w_k^-\right\}_{tt} + 2i\gamma_k^2 w_k^- = 0, \\ w_k^-(0) + w_k^-(T) = \varphi_{0_k}^- + 0,5i\gamma_k^{-2}\varphi_{2_k}^-, \\ \left\{w_k^-(t)\right\}_t\big|_{t=0} + \left\{w_k^-(t)\right\}_t\big|_{t=T} = \varphi_{1_k}^- + 0,5i\gamma_k^{-2}\varphi_{3_k}^-. \end{cases}$$

Then $u_k^-(t) = \frac{1}{2}\left(v_k^- + w_k^-\right)$, where

$$v_k^-(t) = \frac{1}{2}\left(f_{0_k}^- - 0,5i\gamma_k^{-2}f_{2_k}^-\right)F(z,t) + \frac{1}{2z}\left(f_{1_k}^- - 0,5i\gamma_k^{-2}f_{3_k}^-\right)G(z,t),$$

$$w_k^-(t) = \frac{1}{2}\left(f_{0_k}^- + 0,5i\gamma_k^{-2}f_{2_k}^-\right)F(\bar{z},t) + \frac{1}{2\bar{z}}\left(f_{1_k}^- + 0,5i\gamma_k^{-2}f_{3_k}^-\right)G(\bar{z},t),$$

here $z = \gamma_k + i\gamma_k$.

After simplification, we have

$$u_k^-(t) = \frac{P_1(\gamma_k,t)}{4\Delta_k^2}\varphi_{0_k}^- + \frac{P_2(\gamma_k,t)}{8\gamma_k\Delta_k^2}\varphi_{1_k}^- + \frac{P_3(\gamma_k,t)}{8\gamma_k^2\Delta_k^2}\varphi_{2_k}^- + \frac{P_4(\gamma_k,t)}{16\gamma_k^3\Delta_k^2}\varphi_{3_k}^- \quad (2.5)$$

where $\Delta_k = ch\gamma_k T + \cos\gamma_k T$,

$$\begin{aligned} P_1(\gamma_k,t) &= 2ch\gamma_k t\cos\gamma_k t + 2ch(\gamma_k(t-T))\cos(\gamma_k(t-T)) \\ &+ ch(\gamma_k(t+T))\cos(\gamma_k(t-T)) + ch(\gamma_k(t-T))\cos(\gamma_k(t+T)) \\ &+ ch\gamma_k t\cos(\gamma_k(t-2T)) + ch(\gamma_k(t-2T))\cos\gamma_k t, \end{aligned}$$

$$
\begin{aligned}
P_2(\gamma_k,t) = {}& 2sh\gamma_k t\cos\gamma_k t + 2sh(\gamma_k(t-T))\cos(\gamma_k(t-T)) \\
&+ sh(\gamma_k(t+T))\cos(\gamma_k(t-T)) + sh(\gamma_k(t-T))\cos(\gamma_k(t+T)) \\
&+ sh\gamma_k t\cos(\gamma_k(t-2T)) + sh(\gamma_k(t-2T))\cos\gamma_k t \\
&+ 2ch\gamma_k t\sin\gamma_k t + 2ch(\gamma_k(t-T))sin(\gamma_k(t-T)) \\
&+ ch(\gamma_k(t+T))sin(\gamma_k(t-T)) + ch(\gamma_k(t-T))sin(\gamma_k(t+T)) \\
&+ ch\gamma_k t\sin(\gamma_k(t-2T)) + ch(\gamma_k(t-2T))sin\gamma_k t,
\end{aligned}
$$

$$
\begin{aligned}
P_3(\gamma_k,t) = {}& 2sh\gamma_k t\sin\gamma_k t + 2sh(\gamma_k(t-T))sin(\gamma_k(t-T)) \\
&+ sh(\gamma_k(t+T))sin(\gamma_k(t-T)) + sh(\gamma_k(t-T))sin(\gamma_k(t+T)) \\
&+ sh\gamma_k t\sin(\gamma_k(t-2T)) + sh(\gamma_k(t-2T))sin\gamma_k t,
\end{aligned}
$$

$$
\begin{aligned}
P_4(\gamma_k,t) = {}& 2ch\gamma_k t\sin\gamma_k t + 2ch(\gamma_k(t-T))sin(\gamma_k(t-T)) \\
&+ ch(\gamma_k(t+T))sin(\gamma_k(t-T)) + ch(\gamma_k(t-T))sin(\gamma_k(t+T)) \\
&+ ch\gamma_k t\sin(\gamma_k(t-2T)) + ch(\gamma_k(t-2T))sin\gamma_k t \\
&- 2sh\gamma_k t\cos\gamma_k t - 2sh(\gamma_k(t-T))cos(\gamma_k(t-T)) \\
&- sh(\gamma_k(t+T))cos(\gamma_k(t-T)) - sh(\gamma_k(t-T))cos(\gamma_k(t+T)) \\
&- sh\gamma_k t\cos(\gamma_k(t-2T)) - sh(\gamma_k(t-2T))cos\gamma_k t.
\end{aligned}
$$

3. Theorems

Theorem 3.1. *For the uniqueness of a solution to problem* (1.1)–(1.4) *in the class* $W_2^{1,3}(\Omega)$, *it is necessary and sufficient that the equation*

$$\mu_k T = \pi + 2\pi n$$

had no solutions in integers k, n $(k, n \in N)$.

Proof. Necessity. If for some positive integer k, n the expression $1 + \cos\mu_k T$ vanishes, then the homogeneous problem (1.1)–(1.4), that is $\varphi_j(x) = 0,\ j = 0, 1, 2, 3.$, has nontrivial solutions of the form

$$u(x,t) = (sh\mu_k t + ch\mu_k t + \sin\mu_k t + \cos\mu_k t)\, X_k^+(x).$$

Then the solution to the inhomogeneous problem (1)–(4), if it exists, will not be unique.

Sufficiency. Let there exist two solutions $u_1(x,t)$, $u_2(x,t)$ of problem (1.1)–(1.4) from the space $(L_2(-1, 1); C[0,T])$. Then the function $u(x,t) = u_1(x,t) - u_2(x,t)$ is a solution to the homogeneous problem (1.1)–(1.4), where $\varphi_j(x) \equiv 0$, $j = 0, 1, 2, 3$. Hence we get that $u_k^+(t) \equiv 0$, $u_k^-(t) \equiv 0$. Therefore, $u(x,t) \equiv 0\ \forall (x,t) \in \Omega$. It is required to prove. □

Theorem 3.2. *Let* $\varphi_j(x) \in W_2^{4-j+\varepsilon}(-1;1)$, $j=0,1,2,3$ *and the conditions of Theorem 4 be satisfied. Then for* $T \neq \frac{(2n+1)\pi}{\mu_k}$ *(n, k natural numbers) there exists a unique solution to problem* (1)–(4), *which belongs to the space* $W_2^{1,3}(\Omega)$ *and continuously depends on the functions* $\varphi_j(x)$, $j=0,1,2,3$ *in the sense that the estimate*

$$\|u(x,t)\|_0^2 \leqslant C_0 \|\varphi_0(x)\|^2_{W_2^{4+\varepsilon}} + C_1 \|\varphi_1(x)\|^2_{W_2^{3+\varepsilon}} + C_2 \|\varphi_2(x)\|^2_{W_2^{2+\varepsilon}} + C_3 \|\varphi_3(x)\|^2_{W_2^{1+\varepsilon}}$$

is valid, where C_j *–constants,* $j=0,1,2,3$, $0<\varepsilon<1$.

Proof. The existence of a solution to problem (1.1)–(1.4) under the condition $T \neq \frac{(2n+1)\pi}{\mu_k}$ is associated with the problem of small denominators, since the expression $1+\cos\mu_k T$ in the denominators in formula (2.4), being nonzero, can become arbitrarily small for infinite set $k \in N$. Note that for an arbitrary integer $k>0$

$$\left|\cos\frac{\mu_k T}{2}\right| = \left|\sin\left(\frac{\mu_k T-\pi}{2} - n\pi\right)\right| = \left|\sin\left(\left(\frac{\mu_k T-\pi}{2\pi} - n\right)\pi\right)\right| > 2\left|\left(\frac{\mu_k T-\pi}{2\pi} - n\right)\right| = 2k\left|\left(\frac{\mu_k T-\pi}{2\pi k} - \frac{n}{k}\right)\right|,$$

where n is a non-negative integer satisfying the inequality

$$\left|\left(\frac{\mu_k T-\pi}{2\pi} - n\right)\right| < \frac{1}{2},$$

moreover, in deriving the upper inequality, we take into account that for all $x \in (0, \pi/2)$ the inequality $\sin x > 2x/\pi$ is satisfied. According to [10, Ch. 1] for almost all (in the sense of Lebesgue measure) numbers $T>0$, the inequality

$$\left|\frac{\mu_k T-\pi}{2\pi k} - \frac{n}{k}\right| < \frac{1}{k^{2+\varepsilon/2}},\ 0<\varepsilon<1,$$

with respect to $k>0$, $n>0$, it has at most finitely many integer solutions.

It is also known from number theory that for each $\frac{\mu_k T-\pi}{2\pi k}$ there are constants $\delta_1>0$ and $\varepsilon>0$ for which the inequality

$$\left|\frac{\mu_k T-\pi}{2\pi k} - \frac{n}{k}\right| > \frac{\delta_1}{k^{2+\varepsilon/2}},\ 0<\varepsilon<1$$

for all (except a finite number) pairs of integers n and k, $k\neq 0$. Hence we have

$$1+\cos\mu_k T = 2\cos^2\frac{\mu_k T}{2} > \frac{\delta^2}{k^{2+\varepsilon}}, \tag{3.1}$$

where $\delta^2 = 8\delta_1^2$. From equality (2.4), using the inequality $(a+b+c+d)^2 \leqslant 4(a^2+b^2+c^2+d^2)$ and the monotonically increasing function $ch\mu_k T$, as well as using (3.1), we obtain

$$\{u_k^+\}^2 \leqslant \left(2+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{0_k}^+\mu_k^{4+\varepsilon}\right)^2 + \left(\tfrac{2}{\mu_k}+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{1_k}^+\mu_k^{3+\varepsilon}\right)^2$$
$$+\left(\tfrac{2}{\mu_k^2}+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{2_k}^+\mu_k^{2+\varepsilon}\right)^2 + \left(\tfrac{2}{\mu_k^3}+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{3_k}^+\mu_k^{1+\varepsilon}\right)^2$$

We proceed to estimate the function $u_k^-(t)$. It follows from the representation (2.5) that the expression $\Delta_k = ch\gamma_k T + \cos\gamma_k T$ is in the denominator. This expression increases monotonically and $\Delta_k > 2$ for all γ_k, T. Therefore, in (2.5) there is no problem of small denominators.

We consider one of the terms in (2.5)

$$\frac{ch\left(\mu_k(t+T)\right)\cos\left(\mu_k(t-T)\right)}{\left(ch\left(\mu_k T\right)+\cos\left(\mu_k T\right)\right)^2} \leqslant \frac{ch\left(2\mu_k T\right)}{\left(ch\left(\mu_k T\right)+\cos\left(\mu_k T\right)\right)^2} < m,$$

where m –bounded constant. This estimate is true for any γ_k, T. The remaining terms are also bounded for all γ_k, T. Considering these facts, we estimate the function $u_k^-(t)$

$$\{u_k^-(t)\}^2 \leqslant 4m^2\{\varphi_{0_k}^-\}^2 + 32\frac{m^2}{\gamma_k}\{\varphi_{1_k}^-\}^2 + 8\frac{m^2}{\gamma_k^2}\{\varphi_{2_k}^-\}^2 + 16\frac{m^2}{\gamma_k^3}\{\varphi_{3_k}^-\}^2$$

Combining $u_k^+(t)$ and $u_k^-(t)$, we have

$$\{u_k^+(t)\}^2 + \{u_k^-(t)\}^2 \leqslant C_0\left(\mu_k^{4+\varepsilon}\right)^2\left(\{\varphi_{0_k}^+\}^2 + \{\varphi_{0_k}^-\}^2\right)$$
$$+ C_1\left(\mu_k^{3+\varepsilon}\right)^2\left(\{\varphi_{1_k}^+\}^2 + \{\varphi_{1_k}^-\}^2\right) + C_2\left(\mu_k^{2+\varepsilon}\right)^2\left(\{\varphi_{2_k}^+\}^2 + \{\varphi_{2_k}^-\}^2\right)$$
$$+ C_3\left(\mu_k^{1+\varepsilon}\right)^2\left(\{\varphi_{3_k}^+\}^2 + \{\varphi_{3_k}^-\}^2\right),$$

where $C_0 = \max\{2(1+\delta^{-2}), 4m^2\}$, $C_1 = \max\{2(\mu_1^{-1}+\delta^{-2}), 32m^2\gamma_1^{-1}\}$, $C_2 = \max\{2(\mu_1^{-2}+\delta^{-2}), 8m^2\gamma_1^{-2}\}$, $C_3 = \max\{2(\mu_1^{-3}+\delta^{-2}), 16m^2\gamma_1^{-3}\}$.

Adding the last inequalities in k, we obtain

$$\sum_{k=1}^{\infty}\{u_k^+(t)\}^2 + \{u_k^-(t)\}^2 \leqslant C_0\|\varphi_0(x)\|^2_{W_2^{4+\varepsilon}}$$
$$+ C_1\|\varphi_1(x)\|^2_{W_2^{3+\varepsilon}} + C_2\|\varphi_2(x)\|^2_{W_2^{2+\varepsilon}} + C_3\|\varphi_3(x)\|^2_{W_2^{1+\varepsilon}}$$

and this implies the proved inequality. □

4. Numerical calculations

For the numerical solution of problem (1.1)–(1.4), we take the initial data as follows

$$p(x)=1,\ q(x)=0,\ \varphi_0(x)=x^2-1,\ \varphi_j(x)=0,\ j=1,2,3.$$

Then $\pm\lambda_k^{\pm}$ are solutions of the equation $tg\sqrt{\pm\lambda_k^{\pm}}+th\sqrt{\pm\lambda_k^{\pm}}=0$. If we denote $\alpha=\sqrt{\pm\lambda_k^{\pm}}$, solutions of the equation $tg\,\alpha+th\,\alpha=0$ can easily be found using the Newton method. When $\varepsilon=10^{-15}$ error we calculate $\alpha_1\approx 2.36502037243135$, $\alpha_2\approx 5,49780391900084$, $\alpha_3\approx 8,63937982869974$ $\alpha_4\approx 11,7809724510202$, $\alpha_k\approx -\frac{\pi}{4}+\pi k$, $k>4$, $k\in N$. Then $\mu_k=\sqrt{\alpha_k}$, $\gamma_k=\sqrt{\alpha_k/2}$, $k=1,2...$.

Let the solution of the problem (1.1)–(1.4) exists and has the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k^+(t)X_k^+ +\sum_{k=1}^{\infty}u_k^-(t)X_k^- \tag{4.1}$$

where

$$u_k^+(t)=\frac{1}{4}\left(\frac{ch\mu_k t+ch\left(\mu_k(T-t)\right)}{1+ch\mu_k T}+\frac{\cos\mu_k t+\cos\left(\mu_k(T-t)\right)}{1+\cos\mu_k T}\right)\varphi_{0_k}^+,$$

$$u_k^-(t)=\frac{1}{4\Delta_k^2}\left(2ch\gamma_k t\cos\gamma_k t+2ch(\gamma_k(t-T))\cos(\gamma_k(t-T))+\right.$$
$$ch(\gamma_k(t+T))\cos(\gamma_k(t-T))+ch(\gamma_k(t-T))\cos(\gamma_k(t+T))$$
$$\left.+\ ch\gamma_k t\cos(\gamma_k(t-2T))+ch(\gamma_k(t-2T))\cos\gamma_k t\right)\varphi_{0_k}^-,$$

$$X_k^+(x)=\begin{cases}\frac{\sin\alpha_k(x-1)}{\cos\alpha_k},\ 0<x\leqslant 1,\\ \frac{sh\alpha_k(x+1)}{ch\alpha_k},\ -1\leqslant x<0,\end{cases}$$

$$X_k^-(x)=\begin{cases}\frac{sh\alpha_k(x-1)}{ch\alpha_k},\ 0<x\leqslant 1,\\ \frac{\sin\alpha_k(x+1)}{\cos\alpha_k},\ -1\leqslant x<0.\end{cases}$$

Let $\mu_k T=\pi+2\pi n$ equation has not solution when $k,n\in N$. Then the numerical table of the solution one can present in the form (case $T=2,5$, Table 4.1).

Let $\mu_k T=\pi+2\pi n$ equation has solution when $k,n\in N$. Then the numerical table of solution can be presented in the following form (Table 4.2) by approximate values closes to solution of the indicated equation because problem has not unique solution. As parameters we take

$n = 1$, $k = 5$ and we have $T = 2,43977280574315$. For calculation as approximate T we take $T = 2,4397728$.

Remark. The solution by the presented formula (4.1) is calculated with error $\varepsilon = 10^{-4}$.

	t=0,25	t=0,75	t=1,25	t=2,25	t=T
x=-1	0	0	0	0	0
x=-0,8	-0,059	-0,0123	0,0046	-0,059	-0,0891
x=-0,4	-0,1316	-0,0102	0,0352	-0,1316	-0,2079
x=0,2	-0,1053	0,1156	0,2047	-0,1053	-0,2377
x=0,6	-0,0579	0,1096	0,1756	-0,0579	-0,1585

Table 4.1. The numerical solution at $T = 2,5$

	t=0,25	t=0,75	t=1,25	t=2,25	t=T
x=-1	0	0	0	0	0
x=-0,8	-0,1946	-0,0511	0,1694	-0,1764	-0,0891
x=-0,4	-53,5711	-15,724	64,5786	-43,6283	-0,2079
x=0,2	17375,4	5111,31	-20982,9	14136,65	-0,2377
x=0,6	9134,79	2687,234	-11031,3	7432,073	-0,1585

Table 4.2. The numerical solution at $T = 2,4397728$

5. Conclusion

The study shows that the problem under consideration belongs to the class of ill-posed problems of mathematical physics. Based on the idea of the theory of ill-posed problems, the initial problem is investigated for conditional correctness. Since the problem belongs to the class of weakly incorrect problems, pairs of spaces for which the problem becomes correct are obtained. The proved theorems provide an opportunity for constructing an approximate solution algorithm and a numerical solution on a computer.

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