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Weakening topologies on a countable abelian group of finite exponent

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Abstract. We prove that a countable locally minimal abelian group of finite exponent m is discrete. For prime m, this answers Question 7.35(b) from [2].

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Following [1], we say that a Hausdorff topological group (G, τ) is *locally minimal* if there exists a τ-neighbourhood U of the unit such that whenever σ is a Hausdorff group topology on G such that $\sigma \subseteq \tau$ and U is a σ-neighbourhood of the unit, then $\sigma = \tau$.

For $m = 2$, the following theorem was proved in [3] as an answer to Question 6.3 from [1], Question 7.35(a) from [2] is the same.

Theorem. *Let G be a countable abelian group of finite exponent m, τ be a non-discrete Hausdorff group topology on G, U be a τ-neighbourhood of 0. Then there exists a Hausdorff group topology σ on G such that $\sigma \subset \tau$ and U is a σ-neighbourhood of 0.*

Proof. Since τ is non-discrete and G is of finite exponent m, there exists a prime divisor p of m such that the subgroup $G_p = \{g \in G : pg = 0\}$ is non-discrete. Indeed, if k is a divisor of m then either $\{kg : g \in G\}$ is non-discrete or $\{g \in G : kg = 0\}$ is open.

We choose a sequence $(U_n)_{n<\omega}$ of neighbourhoods of 0 in (G, τ) such that $U_n = -U_n$ and

$$U_0 + U_0 \subseteq U,\ U_{n+1} + U_{n+1} \subseteq U_n,\ \bigcap_{n<\omega} U_n = \{0\}.$$

If $\{U_n : n < \omega\}$ is not a base of τ at 0 then we take $\{U_n : n < \omega\}$ as a base of σ at 0.

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We suppose that $\{U_n : n < \omega\}$ is a base of τ at 0 and denote by H the completion of G. For $x \in H$, $< x >$ denotes the subgroup of H generated by x, V_n denotes the closure of U_n in H. Clearly, $V_{n+1} + V_{n+1} \subseteq V_n$ and $\{V_n : n < \omega\}$ is a base of neighbourhoods of 0 in H.

Since G_p is non-discrete countable and metrizable, by the Baire category theorem, there is $h \in H \setminus G$ such that h is in the closure of $U_0 \cap G_p$. Since $ph = 0$, we can choose a sequence $(a_n)_{n<\omega}$ in $U_0 \cap G_p$ such that

$$(\star) \qquad a_n \in h + V_n, \; < a_n > \subseteq < h > + V_n.$$

We denote by A_n the subgroup of G_p generated by $\{a_n, a_{n+1}, \dots\}$, put $W_n = U_n + A_n$ and show that $\{W_n : n < \omega\}$ is a base at 0 for some group topology σ. We have $W_n = -W_n$ and

$$W_{n+1} + W_{n+1} = U_{n+1} + A_{n+1} + U_{n+1} + A_{n+1} \subseteq U_n + A_{n+1} \subseteq W_n.$$

To see that σ is Hausdorff, we use $(\star)$ to observe that

$$< a_n > + < a_{n+1} > + \dots + < a_{k+1} > \subseteq < a_n > + \dots$$
$$+ < a_k > + < h > + V_{k+1} \subseteq \dots \subseteq$$
$$< h > + V_n + V_{n+1} + \dots + V_{k+1} \subseteq < h > + V_{n-1},$$

so $A_n \subseteq < h > + V_{n-1}$. If $g \in G$ and $g \in W_n$ for each n then $g \in U_n + < h > + V_{n-1}$, so $g \in < h >$ and $g = 0$ because $< h > \cap G = \{0\}$. It follows that σ is Hausdorff because $\{W_n : n < \omega\}$ is a base of σ at 0.

Since $U_n \subseteq W_n$ and $a_n \in U_0$, we have $\sigma \subseteq \tau$ and U is a σ-neighbourhood of 0. The sequence $(a_n)_{n<\omega}$ converges 0 in σ but $(a_n)_{n<\omega}$ converges to h in H, so $\sigma \subset \tau$. $\square$

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