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Mappings with finite length distortion and prime ends on Riemann surfaces

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Abstract. The present paper is a continuation of our research that was devoted to the theory of the boundary behavior of mappings in the Sobolev classes (mappings with generalized derivatives) on Riemann surfaces. Here we develop the theory of the boundary behavior of the mappings in the class FLD (mappings with finite length distortion) first introduced for the Euclidean spaces in the article of Martio–Ryazanov–Srebro–Yakubov at 2004 and then included in the known monograph of these authors at 2009 in the modern mapping theory. As it was shown in the recent papers of Kovtonyuk–Petkov–Ryazanov at 2017, such mappings, generally speaking, are not mappings in the Sobolev classes because their first partial derivatives can be not locally integrable. At the same time, this class is a natural generalization of the well-known significant classes of isometries and quasi–isometries.

We prove here a series of criteria in terms of dilatations for the continuous and homeomorphic extension to the boundary of the mappings with finite length distortion between domains on Riemann surfaces by prime ends of Caratheodory. The criterion for the continuous extension of the inverse mapping to the boundary is turned out to be the very simple condition on the integrability of the dilatations in the first power. The criteria for the continuous extension of the direct mappings to the boundary have a much more refined nature. One of such criteria is the existence of a majorant for the dilation in the class of functions with finite mean oscillation, i.e., having a finite mean deviation from its mean value over infinitisemal discs centered at boundary points. As consequences, it is obtained the corresponding criteria for a homeomorphic extension of mappings with finite length distortion to the closures of domains by prime ends of Caratheodory.

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1. Introduction

The present paper is a natural continuation of our previous papers [22–27], where the reader can find the corresponding historic comments and a discussion of many definitions and relevant results. The given papers were devoted to the theory of the boundary behavior of mappings with finite distortion by Iwaniec on Riemann surfaces first introduced for the plane case in the paper [6], and then extended to $\mathbb{R}^n$, $n \geqslant 2$, in the monograph [7].

At the present paper, it is developed the theory of the boundary behavior of the so–called mappings with finite length distortion first introduced in the paper [15] for $\mathbb{R}^n$, $n \geqslant 2$, see also Chapter 8 in the monograph [17]. As it was shown in the papers [8] and [9], such mappings, generally speaking, are not mappings with finite distortion by Iwaniec because of their first partial derivatives can be not locally integrable.

At the same time, this class is a generalization of the known class of mappings with bounded length distortion by Martio–Väisälä from the paper [18]. Moreover, this class contains as a subclass the so–called finitely bi–Lipschitz mappings introduced for $\mathbb{R}^n$, $n \geqslant 2$, in the paper [10], see also Section 10.6 in the monograph [17], that in turn is a natural generalization of the well–known classes of bi–Lipschitz mappings as well as isometries and quasi–isometries.

In the research of local and boundary behavior of mappings with finite length distortion in $\mathbb{R}^n$, the key fact was that they satisfy some modulus inequalities which was a motivation for the consideration more wide classes of mappings, the so–called $Q-$homeomorphisms, see e.g. the paper [16] and Chapters 4–6 in the monograph [17].

Hence it is natural that we start from establishing the corresponding modulus inequalities that are the main tool for us under the research of mappings with finite length distortion on Riemann surfaces. On this basis, we prove here a series of criteria in terms of dilatations for the continuous and homeomorphic extension to the boundary of the mappings with finite length distortion between domains on Riemann surfaces.

2. Definitions and preliminary remarks

Later on, we assume that all mappings under the consideration are continuous. Recall also that we refer the reader for the previous definitions to our papers [22–27] and here we restrict ourselves in the main by new conceptions.

Let us start from the main definitions of the paper [15] adopted to the case of domains D in the complex plane $\mathbb{C}$, see also Chapter 8 in the monograph [17]. It is said that a mapping $f : D \to \mathbb{C}$ is of **finite metric distortion**, written $f \in$ **FMD**, if f has (N)–property by Luzin with respect to the area in $\mathbb{C}$ and

$$0 \ < \ l(z,f) \ \leqslant L(z,f) \ < \ \infty \qquad \text{a.e.} \ , \tag{2.1}$$

where

$$l(z,f) \ := \ \liminf_{\zeta\to z, \zeta\in D} \frac{|f(\zeta)-f(z)|}{|\zeta - z|} \ , \quad L(z,f) \ := \ \limsup_{\zeta\to z, \zeta\in D} \frac{|f(\zeta)-f(z)|}{|\zeta - z|} \ . \tag{2.2}$$

Now, we say that a mapping $f : D \to \mathbb{C}$ has **(L)–property**, if, for a.e. path γ in D the path $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable and $f|_\gamma$ has (N)–property by Luzin with respect to the length measure. Recall that a path γ in D is a mapping $\gamma : \Delta \to D$, where Δ is an interval in $\mathbb{R}$. Moreover, it is said that a property holds for almost every **(a.e.)** path of a family, if the property fails only for its subfamily of paths of conformal modulus zero, see the definition of the conformal modulus on Riemann surfaces in our papers [22] and [23].

We say also that a homeomorphism f between domains D and D^* in $\mathbb{C}$ is of **finite length distortion**, written $f \in$ **FLD**, if $f \in$ FMD and, moreover, f and f^{-1} have (L)–property. **Finite bi–Lipschitz** homeomorphisms satisfying condition (2.1) everywhere but not only a.e. give examples of such homeomorphisms, see Theorem 5.7 in the paper [11] or Theorem 10.11 in the monograph [17]. A special case of the latter's are **bi–Lipschitz** homeomorphisms for which the quantities in (2.1) are uniformly in the domain D separated from zero as well as from infinity. Thus, homeomorphisms of finite length distortion are a far reaching generalization of isometries and quasiisometries.

Remark 1. By Theorem 6.10 in [15] or Theorem 8.6 in [17], a homeomorphism $f \in$ FLD between domains D and D^* in $\mathbb{C}$ satisfies the inequality

$$M(f\Gamma) \ \leqslant \ \int\limits_D Q(z) \cdot \rho^2(z) \ dm(z) \tag{2.3}$$

with $Q = K_f$ for any family Γ of paths γ in D and $\rho \in \operatorname{adm}\Gamma$, see [22] or [23] for definitions of the dilatation K_f, the conformal modulus M of families of paths and admissible functions $\rho : D \to [0, \infty]$.

Homeomorphisms f between domains D and D^* in the complex plane $\mathbb{C}$ satisfying conditions of the type (2.3) are called **Q-homeomorphisms**, see the paper [16], and also Chapters 4–6 in the monograph [17]. Correspondingly to Remark 1, such homeomorphisms form a more wide class of mappings than homeomorphisms with finite length distortion.

Let us pass to the corresponding definitions on Riemann surfaces. So, let f be a homeomorphism between domains D and D^* on Riemann surfaces $\mathbb{S}$ and $\mathbb{S}^*$. First of all, we say that f is a mapping with **finite length distortion**, written $f \in \mathbf{FLD}$, if f is so in charts of $\mathbb{S}$ and $\mathbb{S}^*$. In view of properties of conformal mappings, namely, (N)–properties of Luzin with respect to area as well as to length and invariance of local rectifiable paths, see e.g. Theorem 5.6 in the monograph [29], the definition is independent on the choice of charts. We also say that f is a **local Q–homeomorphism** for a measurable function $Q : \mathbb{S} \to (0, \infty)$, if the condition (2.3) holds for any family Γ of paths γ in D laying inside an arbitrary prescribed chart U of the Riemann surface $\mathbb{S}$.

Remark 2. As known, if a function $\rho : V \to [0, \infty]$ is admissible for a family $\mathcal{A}$ of paths α in an open set V of the complex plane $\mathbb{C}$, then the function $\rho^*(\zeta) = \rho(\varphi^{-1}(\zeta))/|\varphi'(\varphi^{-1}(\zeta))|$ is admissible for the family $\mathcal{B} := \varphi\mathcal{A}$ of paths $\beta := \varphi \circ \alpha$ under every conformal mapping $\varphi : V \to \mathbb{C}$, see again Theorem 5.6 in the monograph [29]. Thus, the right hand side in the inequality (2.3) is a conformal invariant because the Jacobian of $\varphi(z)$ is equal to $|\varphi'(z)|^2$.

Proposition 1. *Every homeomorphism f with finite length distortion between domains D and D^* on Riemann surfaces $\mathbb{S}$ and $\mathbb{S}^*$, correspondingly, is a local Q–homeomorphism with $Q = K_f$.*

Here and later on, we assume that K_f is extended by zero outside of D.

Proof. Let $g : U \to \mathbb{C}$ be a chart of the Riemann surface $\mathbb{S}$. Since the space $\mathbb{S}$ is separable, the open set $D \cap U$ consists of a countable collection of its components U_k every of which is homeomorphic to the plane domain $V_k := g(U_k)$. Thus, every domain $U_k^* := f(U_k)$ is also homeomorphic to the plane domains V_k and, consequently, by the general Koebe principle, see e.g. Section II.3 in [12], U_k^* is a chart of the Riemann surface $\mathbb{S}^*$.

Note also that the path family Γ is split into a countable collection of mutually disjoint path families Γ_k lying in the domains U_k. Hence also the path family $\Gamma^* := f\Gamma$ is split into a countable collection of mutually disjoint path families $\Gamma_k^* := f\Gamma_k$ lying in the domains U_k^*, i.e., in the corresponding charts of the Riemann surface $\mathbb{S}^*$. Thus, by Remark 1 of the paper [22] and by Remarks 1 and 2 of the present paper we obtain the desired conclusion.

3. The main lemma

Recall that the factor $\mathbb{D}/G$ of the unit disk $\mathbb{D}$ with a discrete group G of fractional mappings of $\mathbb{D}$ onto itself without fixed points is a Riemann surface with charts from the natural (locally homeomorphic) projection $\pi : \mathbb{D} \to \mathbb{D}/G$, see Theorem 6.2.1 in [1].

Lemma 1. *Let G be a discrete group of fractional maps of $\mathbb{D}$ onto itself with no fixed points, $f : D \to D^*$ be a homeomorphism of finite length distortion between domains D and D^* on Riemann surfaces $\mathbb{S} := \mathbb{D}/G$ and $\mathbb{S}^*$, $p_0 \in \overline{D}$.*

Then there is $\varepsilon(p_0)$ such that the natural projection $\pi : \mathbb{D} \to \mathbb{D}/G$ is injective on a hyperbolic disk $B_0 := \{z \in \mathbb{D} : h(z, z_0) < \varepsilon(p_0)\}$, where $z_0 \in \pi^{-1}(p_0)$, and

$$M(f(\Gamma)) \leqslant \int\limits_{D} K_f(p)\, \xi^2(p)\, dh(p) \tag{3.1}$$

for families Γ of paths in $D \cap \pi(B_0)$ and measurable functions $\xi : D \to [0, \infty]$, such that

$$\int\limits_{\gamma} \xi(p)\, ds_h(p) \geqslant 1 \qquad \forall\, \gamma \in \Gamma. \tag{3.2}$$

Remark 3. By the Klein–Poincare theorem on the uniformization, see e.g. II.3 in [12], and also 7.4 in [31], an arbitrary Riemann space $\mathbb{S}$ is conformally equivalent to the unit disk $\mathbb{D}$ factored by a discrete group G of fractional mappings of $\mathbb{D}$ onto itself without fixed points, excepting the simplest cases of $\mathbb{S}$ that are conformally equivalent to $\overline{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ or a torus.

In the latter case, $\mathbb{S}$ is conformally equivalent to $\mathbb{C}/G$ with respect to a group G of shifts in $\mathbb{C}$ with 2 generators $z \to z+\omega_1$ and $z \to z+\omega_2$, where ω_1 and $\omega_2 \in \mathbb{C} \setminus \{0\}$ and $\operatorname{Im} \omega_1/\omega_2 > 0$. In this case, a fundamental domain F is a parallelogram whose sides are parallel to ω_1 and ω_2 and gluing its opposite sides just gives a torus. Metrics and areas on surfaces $\mathbb{C}/G$ in the small coincide with Euclidean's because Euclidean's metric and area are invariant under the shifts. In the cases of $\overline{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$, we may also apply the spherical metric and area.

By the scheme of the proof below the relations (3.1) are also valid for all these special cases with the given metrics and areas instead of hyperbolic's. Later on, for the universality, we keep the same notations in these cases, too.

Proof. By Section 2 in either [22] or [24], here we may identify $\mathbb{D}/G$ with a fundamental set F in $\mathbb{D}$ for G with the metric d defined by (2.10) in [22] that contains a fundamental Poincare polygon P_{z_0} for G centered at $z_0 \in \pi^{-1}(p_0)$. Let us choose $\varepsilon(p_0) > 0$ such that $d(z_0, z) = h(z_0, z)$ for $d(z_0, z) \leqslant \varepsilon(p_0)$ and

$$\varepsilon(p_0) \ < \ \delta_0 \ := \ \min \left[\inf_{\zeta \in \partial P_{z_0}} d(z_0, \zeta), \ \sup_{z \in D} d(z_0, \, z) \right] .$$

Since $ds_h(z) = 2|dz|/(1-|z|^2)$, we see that, for every ξ satisfying (4.6),

$$\int\limits_{\gamma} \eta(z) \ |dz| \geqslant \ 1 \qquad \forall \, \gamma \in \Gamma \ , \qquad \text{where} \qquad \eta(z) \ := \ \frac{2\xi(z)}{1-|z|^2} \ ,$$

i.e., the function η is admissible for the family Γ of paths γ in $D \cap \pi(B_0)$. Moreover, since $dh(z) = 4dxdy/(1-|z|^2)^2$, $z = x+iy$, we obtain that

$$\int\limits_{D} K_f(z) \ \xi^2(z) \ dh(z) \ = \ \int\limits_{D} K_f(z) \ \eta^2(z) \ dm(z) \ , \tag{3.3}$$

where $dm(z) := dxdy$ corresponds to the Lebesgue area in the plane $\mathbb{C}$. Thus, the conclusion of Lemma 1 follows from Proposition 1.

4. On extending to the boundary of the inverse mappings

We refer the reader to the paper [26] for definitions, notations and comments in the theory of prime ends. Now, let us start from the following statement.

Lemma 2. *Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of components, and $f : D \to D^*$ be a homeomorphism of finite length distortion with $K_f \in L^1_{\rm loc}$. Then, for all prime ends $P_1 \neq P_2$ of the domain D,*

$$C(P_1, f) \cap C(P_2, f) = \varnothing \ . \tag{4.1}$$

Here we use the notation of the **cluster set** of the mapping f at $P \in E_D$,

$$C(P, f) := \left\{ \ P' \in E_{D'} : P' = \lim_{k \to \infty} f(p_k), \ p_k \to P, \ p_k \in D \ \right\}$$

Proof. First of all, by the Uryson theorem, see e.g. Theorem 22.II.1 in [13], $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$ are metrizable spaces. Hence their compactness is equivalent to their sequential compactness, see e.g. Remark 41.I.3 in [14], and

$\overline{D}$, $\overline{D}'$, ∂D and $\partial D'$ are compact subsets of $\mathbb{S}$ and $\mathbb{S}'$, correspondingly, see e.g. Proposition I.9.3 in [2]. Thus, by Lemma 2, Remarks 1 and 2 in [26], we may assume that P_1 and P_2 are associated with the same nondegenerate component ∂ of ∂D, $K_f \in L^1(D)$, D' is a ring $R = \{z \in \mathbb{C} : 0 \leq r < |z| < 1\}$ and

$$A_k \ := \ C(P_k, f) \ , \qquad k \ = \ 1, \ 2$$

are sets of points in the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$, ∂D consists of 2 components: ∂ and a closed Jordan curve γ, f is extended to a homeomorphism of $D \cup \gamma$ onto $D' \cup C_1$, $C(C_r, f^{-1}) = \partial$, see Proposition 2.5 in [19] or Proposition 13.5 in [17]. Note that the sets A_k are continua, i.e. closed arcs of the circle C_r, because

$$A_k \ = \ \bigcap_{m=1}^{\infty} \overline{f\left(d_m^{(k)}\right)} \ , \qquad k \ = \ 1, \ 2 \ ,$$

where $d_m^{(k)}$ are domains corresponding to chains of cross–cuts $\{\sigma_m^{(k)}\}$ in the prime ends P_k, $k = 1, 2$, see e.g. I(9.12) in [30] and also I.9.3 in [2]. In addition, by Remark 1 in [26] we may assume also that $\sigma_m^{(k)}$ are open arcs of the circles $C_m^{(k)} := \{p \in \mathbb{S} : h(p, p_k) = r_m^{(k)}\}$ on $\mathbb{S}$ with $p_k \in \partial D$ and $r_m^{(k)} \to 0$ as $m \to \infty$, $k = 1, 2$.

Set $p_0 = p_1$. By the definition of the topology of the prime ends in the space $\overline{D}_P$, we have that $d_m^{(1)} \cap d_m^{(2)} = \varnothing$ for all large enough m because $P_1 \neq P_2$. For such m, set $R_1 = r_{m+1}^{(1)} < R_2 = r_m^{(1)} < \varepsilon(p_0)$, where $\varepsilon(p_0)$ is from Lemma 1, and

$$U_k = d_m^{(k)} \ , \quad \Sigma_k = \sigma_m^{(k)} \ , \quad C_k = \{p \in \mathbb{S} : h(p, p_0) = R_k\}, \quad k = 1, 2 \ .$$

Let K_1 and K_2 be arbitrary continua in U_1 and U_2, correspondingly. Applying Proposition 2 and Lemma 1 in [26] with $T = D$, $E_1 = d_{m+1}^{(1)}$ and $E_2 = D \setminus d_m^{(1)}$, and taking into account the inclusion $\Delta(K_1, K_2, D) \subset \Delta(E_1, E_2, D)$, we obtain that

$$\Delta(K_1, K_2, D) \ > \ \Delta(C_1, C_2, A) \ , \quad A := \{p \in \mathbb{S} : R_1 < h(p, p_0) < R_2\} \ , \tag{4.2}$$

which means that any path $\alpha : [a, b] \to \mathbb{S}$ joining K_1 and K_2 in D, $\alpha(a) \in K_1$, $\alpha(b) \in K_2$ and $\alpha(t) \in D$, $t \in (a, b)$, has a subpath joining C_1 and C_2 in A. Thus, since f is a homeomorphism, we have also that

$$\Delta(fK_1, fK_2, fD) \ > \ \Delta(fC_1, fC_2, fA) \tag{4.3}$$

and by the minorization principle, see e.g. [3], p. 178, we obtain that

$$M(\Delta(fK_1, fK_2, fD)) \ \leq \ M(\Delta(fC_1, fC_2, fA)) \ . \tag{4.4}$$

Consequently, by Proposition 2.4 in [19], see also Proposition 13.4 in [26], and Lemma 1 we conclude that

$$M(\Delta(fK_1, fK_2, fD)) \leqslant \int\limits_A K_f(p) \cdot \xi^2(h(p,p_0))\, dh(p) \tag{4.5}$$

for all measurable functions $\xi : (R_1, R_2) \to [0,\infty]$ such that

$$\int\limits_{R_1}^{R_2} \xi(R)\, dR \geqslant 1\ . \tag{4.6}$$

In particular, for $\xi(R) \equiv 1/\delta$, $\delta = R_2 - R_1 > 0$, we get from here that

$$M(\Delta(fK_1, fK_2, fD)) \leqslant M_0 := \frac{1}{\delta}\int\limits_D K_f(p)\, dh(p) < \infty\ . \tag{4.7}$$

Since f is a homeomorphism, (4.7) means that

$$M(\Delta(\mathcal{K}_1, \mathcal{K}_2, D')) \leqslant M_0 < \infty \tag{4.8}$$

for all continua $\mathcal{K}_1$ and $\mathcal{K}_2$ in the domains $V_1 = fU_1$ and $V_2 = fU_2$, correspondingly.

Let us assume that $A_1 \cap A_2 \neq \varnothing$. Then by the construction there is a point $p_* \in \partial R \cap \partial V_1 \cap \partial V_2$. However, the latter contradicts (4.8) because the ring $D' = R$ is a QED (quasiextremal distance) domains, see e.g. Theorem 3.2 in [17], see also Theorem 10.12 in [29]. □

By contrast with the direct mappings, see the next section, we have the following simple criterion for the inverse mappings.

Theorem 1. *Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$, correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of components, and let $f : D \to D'$ be a homeomorphism of finite length distortion with $K_f \in L^1_{\rm loc}$. Then the inverse mapping $g = f^{-1} : D' \to D$ can be extended to a continuous mapping $\tilde{g}$ of $\overline{D'}_P$ onto $\overline{D}_P$.*

Proof. Recall that by Remark 2 in [26] the spaces $\overline{D}_P$ and $\overline{D'}_P$ are compact and metrizable with metrics ρ and ρ'. Let a sequence $p_n \in D'$ converges as $n \to \infty$ to a prime end $P' \in E_{D'}$. Then any subsequence of $p_n^* := g(p_n)$ has a convergent subsequence by compactness of $\overline{D}_P$. By Lemma 2 any such convergent subsequence should have the same limit. Thus, the sequence p_n^* is convergent in $\overline{D}_P$, see e.g. Theorem 2 of Section 2.20.II in [13]. Similarly, by Lemma 2 the sequence $\tilde{p}_n^* := g(\tilde{p}_n)$ has the same limit for any other sequence $\tilde{p}_n \in D'$ as $n \to \infty$. Hence g generates the natural mapping $\tilde{g} : \overline{D'}_P \to \overline{D}_P$.

Note that p_n^* cannot converge to an inner point of D because $I(P') \subseteq \partial D'$ by Proposition 1 in [26] and, consequently, the cluster set of p_n^* belongs to ∂D, see e.g. Proposition 2.5 in [19] or Proposition 13.5 in [17]. Thus, $E_{D'}$ is mapped into E_D under this extension $\tilde{g}$ of g. In fact, $\tilde{g}$ maps $E_{D'}$ onto E_D because $p_n = f(p_n^*)$ has a convergent subsequence for every sequence $p_n^* \in D$ that is convergent to a prime end P of the domain D because $\overline{D'}_P$ is compact.

The map $\tilde{g}$ is continuous. Indeed, let a sequence $P_n' \in \overline{D'}_P$ be convergent to $P' \in \overline{D'}_P$. Then by the first item there is a sequence $p_n \in D'$ with $\rho'(P_n', p_n) < 2^{-n}$ and $\rho(p_n^*, P_n^*) < 2^{-n}$ where $p_n^* := g(p_n)$ and $P_n^* := \tilde{g}(P_n)$. Then $p_n \to P'$ and, again by the first item, $p_n^* \to P^*$ as well as $P_n^* \to P^*$ as $n \to \infty$, where $P^* = \tilde{g}(P')$. □

Corollary 1. *Under the hypothesis of Lemma 2, if ∂ is a nondegenerate component of ∂D, then $C(\partial, f)$ is a nondegenerate component of $\partial D'$.*

5. On extending to the boundary of the direct mappings

As it was already established in the plane, no degree of integrability of Q leads to the extension to the boundary of direct mappings of $Q-$homeomorphisms, see Proposition 6.3 in [17]. The corresponding criterion for that given below is much more refined.

Lemma 3. *Let $\mathbb{S}$, $\mathbb{S}'$ be Riemann surfaces, D, D' be domains in $\overline{\mathbb{S}}$, $\overline{\mathbb{S}'}$, correspondingly, $\partial D \subset \mathbb{S}$, $\partial D' \subset \mathbb{S}'$ have finite collections of components, and let $f : D \to D'$ be a homeomorphism of finite length distortion.*

Suppose that, for all $p_0 \in \partial D$,

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} K_f(p)\cdot\psi^2_{p_0,\varepsilon,\varepsilon_0}(h(p,p_0))\, dh(p) = o\left(I^2_{p_0,\varepsilon_0}(\varepsilon)\right) \qquad as \quad \varepsilon \to 0 \tag{5.1}$$

for some $\varepsilon_0 > 0$ depending on p_0, where $\psi_{p_0,\varepsilon,\varepsilon_0}(t)$, $\varepsilon \in (0, \varepsilon_0)$, is a family of nonnegative measurable functions such that, for all small enough $\varepsilon \in (0, \varepsilon_0)$,

$$0 \ < \ I_{p_0,\varepsilon_0}(\varepsilon) \ := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{p_0,\varepsilon,\varepsilon_0}(t)\ dt \ < \ \infty \ . \tag{5.2}$$

Then f can be extended to a continuous mapping $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$.

Note that conditions (5.1)–(5.2) imply that $I_{p_0,\varepsilon_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and that ε_0 can be chosen arbitrarily small with keeping (5.1)–(5.2).

Proof. By Lemma 2, Remarks 1 and 2 in [26], arguing as in the beginning of the proof of Lemma 2 of the present paper, we may assume with no loss of generality that $\overline{D}$ is a compact set in $\mathbb{S}$, ∂D consists of 2 components: a closed Jordan curve γ and one more nondegenerate component ∂, D' is a ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$, $\overline{D'}_P = \overline{R}$,

$$C(\partial, f) = C_r := \{z \in \mathbb{C} : |z| = r\}, \quad C(\gamma, f) = C_1 := \{z \in \mathbb{C} : |z| = 1\}$$

and that f is extended to a homeomorphism of $D \cup \gamma$ onto $D' \cup C_1$.

Let us first prove that the set $L := C(P, f)$ consists of a single point of C_r for a prime end P of the domain D associated with ∂. Note that $L \neq \varnothing$ by compactness of the set $\overline{R}$ and, moreover, $L \subseteq C_r$ by Proposition 1 in [26].

Let us assume that there is at least two points ζ_0 and $\zeta_* \in L$. Set $U = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < \rho_0\}$ where $0 < \rho_0 < |\zeta_* - \zeta_0|$.

Let σ_k, $k = 1, 2, \ldots$, be a chain in the prime end P from Remark 1 in [26] lying on the circles $S_k := \{p \in \mathbb{S} : h(p, p_0) = r_k\}$ where $p_0 \in \partial$ and $r_k \to 0$ as $k \to \infty$. Let d_k be the domains associated with σ_k. Then there exist points ζ_k and ζ_k^* in the domains $d_k' = f(d_k) \subset R$ such that $|\zeta_0 - \zeta_k| < \rho_0$ and $|\zeta_0 - \zeta_k^*| > \rho_0$ and, moreover, $\zeta_k \to \zeta_0$ and $\zeta_k^* \to \zeta_*$ as $k \to \infty$. Let γ_k be paths joining ζ_k and ζ_k^* in d_k'. Note that by the construction $\partial U \cap \gamma_k \neq \varnothing$, $k = 1, 2, \ldots$.

By the condition of strong accessibility of the point ζ_0 in the ring R, there is a continuum $E \subset R$ and a number $\delta > 0$ such that

$$M(\Delta(E, \gamma_k; R)) \ \geqslant \ \delta \tag{5.3}$$

for all large enough k. Note that $C = f^{-1}(E)$ is a compact subset of D because f is a homeomorphism and hence $d_0 := h(p_0, C)) > 0$. Let $\varepsilon_0 \in (0, d_0)$. Without loss of generality, we may assume that $r_k < \varepsilon_0$ and that (5.3) holds for all $k = 1, 2, \ldots$.

Let Γ_m be the family of paths joining the circle $S_0 := \{p \in \mathbb{S} : h(p, p_0) = \varepsilon_0\}$ and σ_m, $m = 1, 2, \ldots$, in the intersection of $D \setminus d_m$ and the ring $R_m := \{p \in \mathbb{S} : r_m < h(p, p_0) < \varepsilon_0\}$. Applying Proposition 2 and Lemma 2 in [26] with $T = D$, $E_1 = d_m$ and $E_2 = B_0 := \{p \in \mathbb{S} : h(p, p_0) > \varepsilon_0\}$, and taking into account the inclusion $\Delta(C, C_k, D) \subset \Delta(E_1, E_2, D) = \Delta(B_0, d_m, D)$ where $C_k = f^{-1}(\gamma_k)$, we have that $\Delta(C, C_k, D) > \Gamma_m$ for all $k \geqslant m$ because by the construction $C_k \subset d_k \subset d_m$. Thus, since f is a homeomorphism, we have also that $\Delta(E, \gamma_k, D) > f\Gamma_m$ for all $k \geqslant m$, and by the principle of minorization, see e.g. [3], p. 178, we obtain that $M(f(\Gamma_m)) \geqslant \delta$ for all $m = 1, 2, \ldots$.

On the other hand, every function $\xi(t)=\xi_m(t):=\psi_{p_0,r_m,\varepsilon_0}(t)/I_{p_0,\varepsilon_0}(r_m)$, $m=1,2,\ldots$, satisfies the condition (4.6) with $R_2=\varepsilon_0$ and $R_1=r_m$ and we obtain by Lemma 1, see also Proposition 2.4 in [19] or Proposition 13.4 in [17], that

$$M(f\Gamma_m) \leqslant \int\limits_{R_m} K_f(p)\cdot\xi_m^2(h(p,p_0))\,dh(p)\ ,$$

i.e., $M(f\Gamma_m)\to 0$ as $m\to\infty$ in view of (5.1).

The obtained contradiction disproves the assumption that the cluster set $C(P,f)$ consists of more than one point.

Thus, we have the extension $\tilde{f}$ of f to $\overline{D}_P$ such that $\tilde{f}(E_D)\subseteq E_{D'}$. In fact, $\tilde{f}(E_D)=E_{D'}$. Indeed, if $\zeta_0\in D'$, then there is a sequence ζ_n in D' that is convergent to ζ_0. We may assume with no loss of generality that $f^{-1}(\zeta_n)\to P_0\in\overline{D}_P$ because $\overline{D}_P$ is compact, see Remark 2 in [26]. Hence $\zeta_0\in E_D$ because $\zeta_0\notin D$, see e.g. Proposition 2.5 in [19] or Proposition 13.5 in [17].

Finally, let us show that the extended mapping $\tilde{f}:\overline{D}_P\to\overline{D'}_P$ is continuous. Indeed, let $P_n\to P_0$ in $\overline{D}_P$. The statement is obvious for $P_0\in D$. If $P_0\in E_D$, then by the last item we are able to choose $P_n^*\in D$ such that $\rho(P_n,P_n^*)<2^{-n}$ and $\rho'(\tilde{f}(P_n),\tilde{f}(P_n^*))<2^{-n}$ where ρ and ρ' are some metrics on $\overline{D}_P$ and $\overline{D'}_P$, correspondingly, see Remark 2 in [26]. Note that by the first part of the proof $f(P_n^*)\to f(P_0)$ because $P_n^*\to P_0$. Consequently, $\tilde{f}(P_n)\to\tilde{f}(P_0)$. □

Lemma 3 makes possible to derive a series of criteria on the continuous extension to the boundary of mappings with finite length distortion, for instance:

Theorem 2. *Let $\mathbb{S}$, $\mathbb{S}^*$ be Riemann surfaces, D, D^* be domains on $\overline{\mathbb{S}}$, $\overline{\mathbb{S}^*}$, correspondingly, $\partial D\subset\mathbb{S}$, $\partial D^*\subset\mathbb{S}^*$ have finite collections of components, and let $f:D\to D'$ be a homeomorphism of finite length distortion. Suppose that*

$$\int\limits_0^{\delta(p_0)}\frac{dr}{||K_f||\,(p_0,r)} = \infty \qquad \forall\ p_0\in\partial D \tag{5.4}$$

for some $\delta(p_0)>0$, where

$$||K_f||\,(p_0,r) := \int\limits_{h(p,p_0)=r} K_f(p)\ ds_h(p)\ .$$

Then f can be extended to a continuous mapping $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$.

Proof. Indeed, setting $\psi_{p_0}(t) = 1/||K_f||\,(p_0,t)$ for all $t \in (0,\varepsilon_0)$, $\varepsilon_0 := \varepsilon(p_0)$, where $\varepsilon(p_0)$ is from Lemma 1, and $\psi_{p_0}(t) = 1$ for all $t \in (\varepsilon_0,\infty)$, we obtain from condition (5.4) that

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} K_f(p)\cdot \psi^2_{p_0}(h(p,p_0))\, d\,h(p) = I_{p_0,\varepsilon_0}(\varepsilon) = o(I^2_{p_0,\varepsilon_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

where, in view of the condition $K_f(p) \in [1,\infty)$ a.e. in D, for small enough ε,

$$0 \;<\; I_{p_0,\varepsilon_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{p_0}(t)\, dt \;<\; \infty\ .$$

Thus, the conclusion of Theorem 2 follow from Lemma 3. □

Corollary 2. *In particular, the conclusion of Theorem 2 holds if*

$$K_f(p) \quad = O\left(\log \frac{1}{h(p,p_0)}\right) \qquad as \quad p \to p_0 \qquad \forall\ p_0 \in \partial D \tag{5.5}$$

or, more generally,

$$k_{p_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad as \quad \varepsilon \to 0 \qquad \forall\ p_0 \in \partial D \tag{5.6}$$

where $k_{p_0}(\varepsilon)$ is the mean value of the function K_f over the circle $h(p,p_0) = \varepsilon$.

By Theorem 3.1 in [20] we have the following consequence of Theorem 2, too.

Theorem 3. *Let $\mathbb{S}$, $\mathbb{S}^*$ be Riemann surfaces, D, D^* be domains on $\overline{\mathbb{S}}$, $\overline{\mathbb{S}^*}$, correspondingly, $\partial D \subset \mathbb{S}$, $\partial D^* \subset \mathbb{S}^*$ have finite collections of components, and let $f : D \to D'$ be a homeomorphism of finite length distortion. Suppose that*

$$\int\limits_{U} \Phi(K_f(p))\ dh(p) \;<\; \infty \tag{5.7}$$

in a neighborhood U of ∂D where $\Phi : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ is a nondecreasing convex function with the condition

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty\ , \qquad\qquad \delta > \Phi(0)\ . \tag{5.8}$$

Then f can be extended to a continuous mapping $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$.

Remark 4. Note that by Theorem 5.1 and Remark 5.1 in [11] condition (5.8) is not only sufficient but also necessary for the continuous extension to the boundary of all mappings f of finite length distortion with integral restrictions of the form (5.7). Note also that by Theorem 2.1 in [20] and Theorem 2.5 in [21] condition (5.8) is equivalent to each of the following conditions where $H(t) = \log \Phi(t)$:

$$\int\limits_{\Delta}^{\infty} H'(t)\, \frac{dt}{t} = \infty \ , \tag{5.9}$$

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty \ , \tag{5.10}$$

$$\int\limits_{\Delta}^{\infty} H(t)\, \frac{dt}{t^2} = \infty \tag{5.11}$$

for some $\Delta > 0$, and also to each of the equality:

$$\int\limits_{0}^{\delta} H\left(\frac{1}{t}\right)\, dt = \infty \tag{5.12}$$

for some $\delta > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{5.13}$$

for some $\Delta_* > H(+0)$.

Here the integral in (5.10) is understood as the Lebesgue—Stiltjes integral, and the integrals in (5.9), (5.11)–(5.13) as the usual Lebesgue integrals.

It is necessary to give more explanations. In the right hand sides of conditions (5.9)–(5.13), we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then $H(t) = -\infty$ for $t \in [0, t_*]$, and we complete the definition in (5.9) setting $H'(t) = 0$ for $t \in [0, t_*]$. Note that conditions (5.10) and (5.11) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (5.10) and (5.11) either are equal $-\infty$ or not determined. Hence we may assume that in (5.9–(5.12) $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup_{\Phi(t)=0} t$ and $t_0 = 0$ if $\Phi(0) > 0$.

Among the conditions counted above, the most interesting one is condition (5.11) that can be written in the form:

$$\int\limits_{\Delta}^{\infty} \log \Phi(t)\ \frac{dt}{t^2}\ =\ +\infty \qquad\qquad \text{for some}\ \ \Delta > 0\ . \tag{5.14}$$

Corollary 3. *In particular, the conclusion of Theorem 3 holds if, for $\alpha > 0$,*

$$\int\limits_{U} e^{\alpha K_f(p)}\ dh(p)\ <\ \infty\ . \tag{5.15}$$

The next statement follows by Remarks 3–4 and Lemma 2 with $\psi(t) = 1/t$.

Theorem 4. *Let $\mathbb{S}$, $\mathbb{S}^*$ be Riemann surfaces, D, D^* be domains on $\overline{\mathbb{S}}$, $\overline{\mathbb{S}^*}$, correspondingly, $\partial D \subset \mathbb{S}$, $\partial D^* \subset \mathbb{S}^*$ have finite collections of components, and let $f : D \to D'$ be a homeomorphism of finite length distortion. Suppose that*

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} K_f(p)\ \frac{dh(p)}{h(p,p_0)^2}\ =\ o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \quad as \quad \varepsilon\to 0 \quad \forall\ p_0\in\partial D\ . \tag{5.16}$$

Then f can be extended to a continuous mapping $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$.

Remark 5. Choosing in Lemma 3 the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$, we obtain that condition (5.16) can be replaced by the condition

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} \frac{K_f(p)\ dh(p)}{\left(h(p,p_0)\ \log\frac{1}{h(p,p_0)}\right)^2}\ =\ o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \quad \text{as} \quad \varepsilon\to 0\ . \tag{5.17}$$

Similarly, condition (5.6) by Theorem 2 can be replaced by the weaker condition

$$k_{z_0}(\varepsilon)\ =\ O\left(\log\frac{1}{\varepsilon}\log\,\log\frac{1}{\varepsilon}\right) \quad \text{as} \quad \varepsilon\to 0\ . \tag{5.18}$$

Of course, we could give here a series of the corresponding conditions of the logarithmic type applying suitable functions $\psi(t)$.

Following paper [19], cf. [5], see also Section 13.4 in [17], Section 2.3 in [4], we say that a function $\varphi : \mathbb{S}\to\mathbb{R}$ has **finite mean oscillation** at a point $p_0\in\mathbb{S}$, written $\varphi\in \mathrm{FMO}(p_0)$, if

$$\limsup_{\varepsilon\to 0}\ \fint_{B(p_0,\,\varepsilon)} |\ \varphi(p) - \widetilde{\varphi}_{\varepsilon}|\ dh(p)\ <\ \infty \tag{5.19}$$

where $\widetilde{\varphi}_\varepsilon$ is the mean value of φ over the disk $B(p_0, \varepsilon) = \{p \in \mathbb{S} : h(p, p_0) < \varepsilon\}$.

By Remarks 3–4 and Lemma 3 with the choice $\psi_{p_0,\varepsilon}(t) \equiv 1/t \log \frac{1}{t}$, in view of Lemma 4.1 and Remark 4.1 in [19], see also Lemma 13.2 and Remark 13.3 in [17], we come to the following result.

Theorem 5. *Let $\mathbb{S}$, $\mathbb{S}^*$ be Riemann surfaces, D, D^* be domains on $\overline{\mathbb{S}}$, $\overline{\mathbb{S}^*}$, correspondingly, $\partial D \subset \mathbb{S}$, $\partial D^* \subset \mathbb{S}^*$ have finite collections of components, and let $f : D \to D'$ be a homeomorphism of finite length distortion. Suppose that*

$$K_f(p) \leqslant Q(p) \in \text{FMO}(p_0) \quad \forall\, p_0 \in \partial D, \text{ for some } Q : \mathbb{S} \to \mathbb{R}^+. \tag{5.20}$$

Then f can be extended to a continuous mapping $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$.

By Corollary 4.1 in [19], see also Corollary 13.3 in [17], we have the next:

Corollary 4. *In particular, the conclusion of Theorem 5 holds if*

$$\limsup_{\varepsilon \to 0} \fint_{B(p_0,\varepsilon)} K_f(p)\ dh(p) < \infty \qquad \forall\, p_0 \in \partial D\,. \tag{5.21}$$

Remark 6. Combining the above results with Theorem 1, we get that, under the hypotheses of Theorems 3 and 5 and Corollaries 3 and 4, f can be extended to a homeomorphism $\tilde{f}$ of $\overline{D}_P$ onto $\overline{D'}_P$. To obtain this conclusion in the cases of Lemma 3, Theorem 2, Corollary 2, Remark 4 and Theorem 4, we should add the hypothesis that K_f is integrable at least in a neighborhood of ∂D. We do not formulate the corresponding theorems on a homeomorphic extension to the boundary of mappings with finite length distortion in the explicit form because of the restrictions on the volume of the paper.

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