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Approximation of functions by linear summation methods in the Orlicz type spaces

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Abstract. Approximative properties of linear summation methods of Fourier series are considered in the Orlicz type spaces $\mathcal{S}_M$. In particular, in terms of approximations by such methods, constructive characteristics are obtained for classes of functions whose moduli of smoothness do not exceed a certain majorant.

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1. Introduction

Linear methods (or processes) of summation of Fourier series are an important object of research in approximation theory. In particular, this is due to the fact that most of these methods naturally generate the corresponding aggregate of approximation. These topics are well studied in classical functional spaces such as Lebesgue and Hilbert spaces, the spaces of continues functions, etc. However, there are relatively fewer papers devoted to similar topics in the Banach spaces of Orlicz type. It particularly concerns the direct and inverse theorems of approximation by linear summation methods.

In the paper, approximative properties of linear summation methods of Fourier series are studied in the Orlicz type spaces $\mathcal{S}_M$. The spaces $\mathcal{S}_M$ are defined in the following way. An Orlicz function $M(t)$ is a non-decreasing convex function defined for $t \geq 0$ such that $M(0) = 0$ and

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$M(t) \to \infty$ as $t \to \infty$. Let $\mathcal{S}_M$ be the space of all 2π-periodic Lebesgue summable functions f ($f \in L_1$) such that the following quantity (which is also called the Luxemburg norm of f) is finite:

$$\|f\|_M := \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_M(\mathbb{Z})} = \inf\Big\{a>0:\ \sum_{k\in\mathbb{Z}} M(|\widehat{f}(k)|/a) \le 1\Big\}, \quad (1.1)$$

where $\widehat{f}(k) := [f]^{\widehat{\ }}(k) = (2\pi)^{-1}\int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t$, $k \in \mathbb{Z}$, are the Fourier coefficients of f. Functions $f \in L_1$ and $g \in L_1$ are equivalent in the space $\mathcal{S}_M$, when $\|f-g\|_M = 0$.

The spaces $\mathcal{S}_M$ defined in this way are Banach spaces. They were considered in [6]. In particular, direct and inverse approximation theorems in terms of the best approximations of functions and moduli of fractional smoothness are proved for the spaces $\mathcal{S}_M$ in [6].

In case $M(t) = t^p$, $p \ge 1$, the spaces $\mathcal{S}_M$ coincide with the well-known spaces $\mathcal{S}^p$ [18] of functions $f \in L_1$ with the finite norm

$$\|f\|_{\mathcal{S}^p} = \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_p(\mathbb{Z})} = \Big(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^p\Big)^{1/p}.$$

In $\mathcal{S}^p$, approximative properties of linear summation methods of Fourier series were studied in [16, 17]. The purpose of this paper is to continue this study of approximative properties of linear summation methods in the spaces $\mathcal{S}_M$. In this case, our attention is drawn to the connection of the approximative properties of these methods with the differential properties of the functions, namely, direct and inverse theorems of approximation by the methods of Zygmund, Abel–Poisson, Taylor–Abel–Poisson are proved, and in terms of approximations by such methods, constructive characteristics are given for classes of functions of $\mathcal{S}_M$ such that the moduli of smoothness of their generalized derivatives do not exceed a certain majorant.

2. Preliminaries

For any function $f \in L_1$ with the Fourier series of the form

$$S[f](x) := \sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx},$$

consider the following linear transformations S_n, $Z_n^{(s)}$, $P_{\varrho,s}$ and $A_{\varrho,r}$:

$$S_n(f)(x) := \sum_{k=-n}^{n} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}, \quad n = 0, 1, \ldots,$$

$$Z_n^{(s)}(f)(x) := \sum_{k=-n}^{n} \left(1 - \left(\frac{|k|}{n+1}\right)^s\right) \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx}, \quad s > 0,$$

$$P_{\varrho,s}(f)(x) := \sum_{k\in\mathbb{Z}} \varrho^{|k|^s} \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx}, \quad s > 0, \ \varrho \in [0,1),$$

and

$$A_{\varrho,r}(f)(x) := \sum_{k\in\mathbb{Z}} \lambda_{|k|,r}(\varrho) \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}, \tag{2.1}$$

where for $k = 0, 1, \ldots, r-1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1$ and

$$\lambda_{k,r}(\varrho) := \sum_{j=0}^{r-1} \binom{k}{j} (1-\varrho)^j \varrho^{k-j}, \quad k = r, r+1, \ldots, \quad \varrho \in [0,1]. \tag{2.2}$$

The expressions $S_n(f)$, $Z_n^{(s)}(f)$ and $P_{\varrho,s}(f)$ are called the partial sum of the Fourier series, the Zygmund sum and the generalised Abel–Poisson sum of the function f, respectively. The expression $A_{\varrho,r}(f)$ is called the Taylor–Abel–Poisson sum of the function f. If $s = 1$, then the sum $Z_n^{(s)}(f)$ coincides with the Fejér sum of the function f, i.e.,

$$Z_n^{(1)}(f)(x) = \sigma_n(f)(x) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f)(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx}.$$

Note that the transformation $A_{\varrho,r}$ can be considered as a linear operator on L_1 into itself. Indeed, for $k = 0, 1, \ldots, r-1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1$ and

$$\sum_{j=0}^{r-1} \binom{k}{j} (1-\varrho)^j \varrho^{k-j} \le r q^k k^{r-1}, \text{ where } q = \max\{1-\varrho, \varrho\},$$

and hence, for any $f \in L_1$ and $0 < \varrho < 1$, the series on the right-hand side of (2.1) is majorized by the convergent series $2r\|f\|_{L_1} \sum_{k=r}^{\infty} q^k k^{r-1}$.

Denote by $P(f)\,(\varrho, x)$, $0 \le \varrho < 1$, the Poisson integral (the Poisson operator) of f, i.e.,

$$P(f)\,(\varrho, x) := \frac{1}{2\pi} \int_0^{2\pi} f(t) P(\varrho, x-t) dt, \tag{2.3}$$

where $P(\varrho, t) = \frac{1-\varrho^2}{|1-\varrho \mathrm{e}^{\mathrm{i}t}|^2}$ is the Poisson kernel.

According to the decomposition of the Poisson kernel in powers of ϱ, for any function $f \in L_1$, its Poisson integral $P(f)(\varrho, x)$, with $\varrho \in [0,1)$ and $x \in [0; 2\pi]$ can be written in the form

$$P(f)\,(\varrho, x) = \sum_{k\in\mathbb{Z}} \varrho^{|k|} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}. \tag{2.4}$$

The sum of the right-hand side of this equality coincides with the sum of the Abel–Poisson of the series $\sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}$, or, what is the same, with the sum of $P_{\varrho,1}(f)(x)$. For $x = 0$, we denote by $F(\varrho)$ the sum of this series and consider it as a function of the variable ϱ. It is clear that the function F is analytic on $[0, 1)$. Therefore, in the neighborhood of $\varrho \in [0, 1)$ for the functions F, the following Taylor's formula is satisfied:

$$F(t) = \sum_{k=0}^{\infty} \frac{F^{(k)}(\varrho)}{k!}(t-\varrho)^k.$$

By direct computation we see that the partial sum of this series of order $r-1$ for $t = 1$ coincides with the sum $A_{\varrho,r}(f)(0)$. In particular, for $r = 1$, we obtain $F(\varrho) = A_{\varrho,1}(f)(0) = P_{\varrho,1}(f)(0)$.

Consequently, on the one hand, the sum of $A_{\varrho,r}(f)(0)$ can be interpreted as the Taylor sum of order $r-1$ of the function F, and on the other hand, for $r = 1$, it can be interpreted as the Abel–Poisson sum.

The operators $A_{\varrho,r}$ were first studied in [15], where in the terms of these operators, the author gives the structural characteristic of Hardy–Lipschitz classes $H^r_p \operatorname{Lip}\alpha$ of one variable functions, holomorphic in the unit disc in the complex plane. Approximative properties of these operators were also considered in [13, 16]. In general case, the operators $P_{\varrho,s}$ were perhaps first considered as the aggregates of approximation of functions of one variable in [3, 4]. In special cases when $r = s = 1$, the operators $A_{\varrho,1}$ and $P_{\varrho,1}$ coincide with each other and generate the Abel–Poisson summation method of Fourier series. The problem of approximation of 2π-periodic functions by Abel–Poisson sums has a long history, full of many results. Here we mention only the books [1, 5, 20], which contain fundamental results in this subject.

3. Derivatives and moduli of smoothness

Let $\psi = \{\psi(k)\}_{k\in\mathbb{Z}}$ be a numerical sequence whose members are not all zero and

$$\mathcal{Z}(\psi) := \{k \in \mathbb{Z} : \psi(k) = 0\}.$$

In what follows, assume that the number of elements of the set $\mathcal{Z}(\psi)$ is finite.

If for the function $f \in L_1$, there exists the function $g \in L_1$ with the Fourier series of the form

$$S[g](x) = \sum_{k\in\mathbb{Z}\setminus\mathcal{Z}(\psi)} \widehat{f}(k)e^{\mathrm{i}kx}/\psi(k), \tag{3.1}$$

then we say that for the function f, there exists ψ-derivative g, for which we use the notation $g = f^{\psi}$.

This definition of ψ-derivative is adapted to the needs of the research described in this paper and it is not fundamentally different from the established concept of ψ-derivative of A.I. Stepanets [19, Ch. XI].

In the paper, we consider ψ-derivatives defined by the sequences of the following two forms: 1) $\psi(k) = |k|^{-s}$, $k \in \mathbb{Z}$, $s > 0$, and 2) $\psi(k) = 0$ for $|k| \le r-1$ and $\psi(k) = (|k|-r)!/(|k|!)$ for $|k| \ge r$, where $r \in \mathbb{N}$. In the first case, for ψ-derivative of f, we use the notation $f^{(s)}$ and in the second case, we use the notation $f^{[r]}$. If $r = 0$, then we set $f^{(0)} = f^{[0]} = f$. Also note that $f^{(1)} = f^{[1]}$.

In the terms of Poisson integrals, we give the following interpretation of the derivative $f^{[r]}$. Assume that $\varrho \in [0,1)$, then

$$P(f^{[r]})(\varrho, x) = \varrho^r \frac{\partial^r}{\partial \varrho^r} P(f)(\varrho, x) \tag{3.2}$$

and by virtue of the well-known theorem on radial limit values of the Poisson integral (see, eg, [14]), for almost all $x \in [0; 2\pi]$

$$f^{[r]}(x) = \lim_{\varrho \to 1-} \frac{\partial^r}{\partial \varrho^r} P(f)(\varrho, x).$$

The modulus of smoothness of $f \in \mathcal{S}_M$ of the index $\alpha > 0$ is defined by

$$\omega_\alpha(f,\delta)_M := \sup_{|h|\le\delta} \|\Delta_h^\alpha f\|_M = \sup_{|h|\le\delta} \Big\| \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x-jh) \Big\|_M,$$

where $\delta > 0$, $\binom{\alpha}{0} := 1$, $\binom{\alpha}{j} = \alpha(\alpha-1)\cdot \ldots \cdot (\alpha - j + 1)/j!$, $j \in \mathbb{N}$.

Let ω be a function defined on the interval $[0,1]$. For $\alpha > 0$, we set

$$\mathcal{S}_M H_\omega^\alpha := \left\{ f \in \mathcal{S}_M : \quad \omega_\alpha(f,\delta)_M = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+ \right\}.$$

Further, we consider the functions $\omega(t)$, $0 \le t \le 1$, satisfying the following conditions 1)-4): 1) $\omega(t)$ is continuous on $[0,1]$; 2) $\omega(t)$ is monotonically increasing; 3) $\omega(t) \neq 0$ for $t \in (0,1]$; 4) $\omega(t) \to 0$ as $t \to 0$; and the well-known Zygmund–Bari–Stechkin conditions $(\mathcal{B})$ and $(\mathcal{B}_s)$, $s \in \mathbb{N}$ (see, e.g., [2]):

$$(\mathcal{B}): \sum_{v=n+1}^{\infty} v^{-1}\, \omega(v^{-1}) = \mathcal{O}[\omega(n^{-1})], \quad n \to \infty;$$

$$(\mathcal{B}_s): \sum_{v=1}^{n} v^{s-1}\, \omega(v^{-1}) = \mathcal{O}[n^s \omega(n^{-1})], \quad n \to \infty.$$

Remark 3.1. From condition $(\mathcal{B}_s)$ it follows that $\liminf\limits_{\delta\to 0+}(\delta^{-s}\omega(\delta)) > 0$ or that for any $r \geq s$, the quantity $(1-\varrho)^{r-s}\omega(1-\varrho) \gg (1-\varrho)^r$ as $\varrho \to 1-$.

4. The main results

Proposition 4.1. *Assume that $f \in L_1$, $s > 0$ and ω is the function satisfying conditions 1)–4) and $(\mathcal{B})$. The following statements are equivalent:*

1) $\|S_n(f^{(s)})\|_M = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty;$

2) $\left\|f - Z_n^{(s)}(f)\right\|_M = \mathcal{O}(\omega(n^{-1})), \quad n \to \infty;$

3) $f \in \mathcal{S}_M H_\omega^s$.

Let us note that in the case when $s \in \mathbb{N}$ and the function ω satisfies conditions 1)–4), $(\mathcal{B})$ and $(\mathcal{B}_s)$, the relation 1) of Proposition 4.1 is equivalent to the corresponding relation for the derivative $f^{[s]}$:

$$\|S_n(f^{[s]})\|_M = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty. \tag{4.1}$$

Indeed, by the definition for $|k| < s$ we have $0 = |\widehat{f^{[s]}}(k)| \leq |\widehat{f^{(s)}}(k)|$ and for $|k| \geq s$,

$$|\widehat{f^{[s]}}(k)| = |k|(|k|-1)\cdot\ldots\cdot(|k|-s+1)\widehat{f}(k) \leq |k|^s|\widehat{f}(k)| = |\widehat{f^{(s)}}(k)|.$$

Therefore, if the statement 1) of Proposition 4.1 holds, then

$$\|S_n(f^{[s]})\|_M \leq \|S_n(f^{(s)})\|_M = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty.$$

On the other hand, for $|k| \geq s$, we have

$$|\widehat{f^{[s]}}(k)| = |k|^s\cdot\Big(1-\frac{1}{|k|}\Big)\cdot\ldots\cdot\Big(1-\frac{s-1}{|k|}\Big)|\widehat{f}(k)| \geq \frac{|k|^s}{s^s}|\widehat{f}(k)| = s^{-s}|\widehat{f^{(s)}}(k)|.$$

Therefore, taking into account Remark 3.1, we see that relation (4.1) yields the statement 1):

$$\|S_n(f^{(s)})\|_M \leq \|S_{s-1}(f^{(s)})\|_M + \Big\|\sum_{s\leq|k|\leq n}|k|^s\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M$$

$$\leq \|S_{s-1}(f^{(s)})\|_M + s^s\|S_n(f^{[s]})\|_M = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty.$$

Hence, the following assertion is valid:

Proposition 4.2. *Assume that $f \in L_1$, $s \in \mathbb{N}$ and ω is the function, satisfying conditions 1)–4),* $(\mathcal{B})$ *and* $(\mathcal{B}_s)$. *The following statements are equivalent:*

1) $\|S_n(f^{\{s\}})\|_M = \mathcal{O}(n^s\omega(n^{-1}))$, $\quad n \to \infty$, *where $f^{\{s\}}$ is one of the derivatives $f^{[s]}$ or $f^{(s)}$;*

2) $\left\|f - Z_n^{(s)}(f)\right\|_M = \mathcal{O}(\omega(n^{-1}))$, $\quad n \to \infty$;

3) $f \in \mathcal{S}_M H^s_\omega$.

In the case when $s = 1$, we have $f^{(1)} = f^{[1]}$ and $Z_n^{(1)}(f) = \sigma_n(f)$.

Corollary 4.1. *Assume that $f \in L_1$ and ω is a function satisfying conditions 1)–4) and* $(\mathcal{B})$. *The following statements are equivalent:*

1) $\|S_n(f^{[1]})\|_M = \mathcal{O}(n\omega(n^{-1}))$, $\quad n \to \infty$;

2) $\|f - \sigma_n(f)\|_M = \mathcal{O}(\omega(n^{-1}))$, $\quad n \to \infty$;

3) $f \in \mathcal{S}_M H^1_\omega$.

The proof of these and others assertions will be given in Section 6. Let us give some comments. First, let us note that in the proposed assertions, the equivalence 2) $\Leftrightarrow$ 3) is the statement of the type direct and inverse theorem for Zygmund and Fejér method [5].

In the papers [9–12], Móricz investigated properties of 2π-periodic functions represented by Fourier series, which convergent absolutely. In particular, in [9] and [12], the author found the conditions under which such functions satisfy the Lipshitz and Zygmund condition respectively.

In the cases where $M(t) = t$ and $\omega(t) = t^\beta$, the implication 1) $\Rightarrow$ 3) of Corollary 4.1 ($\beta \in (0, 1)$) coincides with the statements (i) of Theorem 1 [9] and the implication 1) $\Rightarrow$ 3) of Proposition 4.1 ($\beta \in (0, 2)$) coincides with the statements (i) of Theorem 1 [10].

In the following theorem, we give the direct and inverse theorem of the approximation of functions by the linear operator $A_{\varrho,r}$ in the space $\mathcal{S}_M$ and constructive characteristics for classes of functions of $\mathcal{S}_M$ such that the moduli of smoothness of their generalized derivatives do not exceed majorants ω.

Theorem 4.1. *Assume that $f \in L_1$, $s, r \in \mathbb{N}$, $s \le r$ and ω is a function satisfying conditions 1)–4),* $(\mathcal{B})$ *and* $(\mathcal{B}_s)$. *The following statements are equivalent:*

1) $\|f - A_{\varrho,r}(f)\|_M = \mathcal{O}((1-\varrho)^{r-s}\omega(1-\varrho))$, $\quad \varrho \to 1-$;

2) $\left\|P(f^{[r]})(\varrho, \cdot)\right\|_M = \mathcal{O}((1-\varrho)^{-s}\omega(1-\varrho))$, $\quad \varrho \to 1-$;

3) $\|S_n(f^{[r]})\|_M = \mathcal{O}(n^s\omega(n^{-1}))$, $\quad n \to \infty$;

4) $f^{[r-s]} \in \mathcal{S}_M H^s_\omega$.

Let us note that the implication 2) $\Rightarrow$ 3) is the statement of the Hardy–Littlewood type theorems [8].

Remark 4.1. In Remark 3.1 it is noted that from the condition $(\mathcal{B}_s)$ it follows that $(1-\varrho)^{r-s}\omega(1-\varrho) \gg (1-\varrho)^r$ as $\varrho \to 1-$. Therefore, if the condition $(\mathcal{B}_s)$ is satisfied, then the quantity on the right-hand side of the relation in statement 1) decreases to zero as $\varrho \to 1-$ not faster, than the function $(1-\varrho)^r$. Also note that the relation $\|f - A_{\varrho,r}(f)\|_M = o\left((1-\varrho)^r\right)$, $\varrho \to 1-$, holds only in the trivial case when $f(x) = \sum_{|k|\le r-1} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}$, and in such case, the theorems are easily true. This fact is related to the so-called saturation property of the approximation method, generated by the operator $A_{\varrho,r}$. In particular, in [15], it was shown that the operator $A_{\varrho,r}$ generates the linear approximation method of holomorphic functions, which is saturated in the Hardy space H_p with the saturation order $(1-\varrho)^r$ and the saturation class $H_p^{r-1}\,\mathrm{Lip}\,1$.

Consider approximative properties of the sums $P_{\varrho,s}(f)$ in the space $\mathcal{S}_M$.

Let us prove that for any function $f \in \mathcal{S}_M$ such that the derivative $f^{(s)} \in \mathcal{S}_M$, the following relation holds as $\varrho \to 1-$:

$$\|f - P_{\varrho,s}(f)\|_M \sim \|f^{(s-1)} - P_{\varrho,1}(f^{(s-1)})\|_M \sim (1-\varrho)\|f^{(s)}\|_M. \tag{4.2}$$

For this, let us show that

$$\|f - P_{\varrho,s}(f)\|_M \sim (1-\varrho)\|f^{(s)}\|_M, \quad \varrho \to 1-. \tag{4.3}$$

The second relation in (4.2) is proved similarly.

For any $n \in \mathbb{N}$, we have $1-\varrho^n = (1-\varrho)(1+\varrho+\ldots+\varrho^{n-1})$. Then setting $b_1 := (1-\varrho)\|f^{(s)}\|_M$, we get for all $\varrho \in (0,1)$,

$$\sum_{k\in\mathbb{Z}} M\Big((1-\varrho^{|k|^s})|\widehat{f}(k)|/b_1\Big) \le \sum_{k\in\mathbb{Z}} M\Big((1-\varrho)|k|^s|\widehat{f}(k)|/b_1\Big) \le 1.$$

Therefore, $\|f - P_{\varrho,s}(f)\|_M \le (1-\varrho)\|f^{(s)}\|_M$.

On the other hand side, since $f^{(s)} \in \mathcal{S}_M$, then for any $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that for all $n \ge N$

$$\|S_n(f^{(s)})\|_M \ge \|f^{(s)}\|_M - \varepsilon/4$$

and by the definition of the norm

$$\sum_{|k|\le N} M\bigg(\frac{|k|^s|\widehat{f}(k)|}{\|f^{(s)}\|_M - \varepsilon/2}\bigg) \ge \sum_{|k|\le N} M\bigg(\frac{|k|^s|\widehat{f}(k)|}{\|S_n(f^{(s)})\|_M - \varepsilon/4}\bigg) > 1.$$

Choosing ϱ_0 such that for all $\varrho \in (\varrho_0, 1)$ and $|k| \le N$, the following inequality holds:

$$(\|f^{(s)}\|_M - \varepsilon/2)(1+\varrho+\ldots+\varrho^{|k|^s-1}) > |k|^s(\|f^{(s)}\|_M - \varepsilon)$$

we see that for such ϱ and $b_2 := (1-\varrho)(\|f^{(s)}\|_M - \varepsilon)$

$$\sum_{k\in\mathbb{Z}} M\Big((1-\varrho^{|k|^s})|\widehat{f}(k)|/b_2\Big) \ge \sum_{|k|\le N} M\Big((1-\varrho)(1+\varrho+\ldots+\varrho^{|k|^s-1})|\widehat{f}(k)|/b_2\Big)$$

$$= \sum_{|k|\le N} M\bigg(\frac{(1+\ldots+\varrho^{|k|^s-1})|\widehat{f}(k)|}{\|f^{(s)}\|_M - \varepsilon}\bigg) > \sum_{|k|\le N} M\bigg(\frac{|k|^s|\widehat{f}(k)|}{\|f^{(s)}\|_M - \varepsilon/2}\bigg) > 1.$$

Thus, for all $\varrho \in (\varrho_0, 1)$, we have $\|f - P_{\varrho,s}(f)\|_M \ge (1-\varrho)(\|f^{(s)}\|_M - \varepsilon)$ and hence relation (4.3) holds.

It is clear that

$$P_{\varrho,1}(f)(x) = A_{\varrho,1}(f)(x).$$

Therefore, applying Theorem 2.1 to the function $f = g^{(s-1)}$ with $r = 1$ and taking into account relation (4.2), we obtain the following result.

Theorem 4.2. *Assume that $f \in L_1$, $s \in \mathbb{N}$, and ω is the function, satisfying conditions 1)–4), $(\mathcal{B})$ and $(\mathcal{B}_s)$. The following statements are equivalent:*

1) $\|f - P_{\varrho,s}(f)\|_M = \mathcal{O}(\omega(1-\varrho))$, $\quad \varrho \to 1-$;

2) $\big\|P(f^{(s)})(\varrho,\cdot)\big\|_M = \mathcal{O}(\frac{\omega(1-\varrho)}{1-\varrho})$, $\quad \varrho \to 1-$;

3) $f^{(s-1)} \in \mathcal{S}_M H^1_\omega$.

Let us note that in the case where $M(t) = t^p$, $p \ge 1$, that is in the spaces $\mathcal{S}^p$, Proposition 4.1, Theorem 2.1 (for $s = 1$) and Theorem 4.2 were proved in [16].

5. The equivalence between moduli of smoothness and K-functionals

It is known that approximative properties of functions are well expressed by their K-functionals. In [16] the authors showed the dependence of the order of approximation of a given function by the Taylor–Abel–Poisson means and the behavior of its modulus of smoothness in the spaces $\mathcal{S}^p$. In [13] the dependence was found for the order of approximation of a given function by the Taylor–Abel–Poisson means and the behavior of K-functionals of the function generated by its radial derivatives in the spaces L_p. It is natural to study the relations the modulus of smoothness and such K-functionals of functions in the spaces $\mathcal{S}_M$.

In the space $\mathcal{S}_M$, the Petree K-functional of a function f (see, e.g. [7, Ch. 6]), which generated by its radial derivative of order $n \in \mathbb{N}$, is the following quantity:

$$K_n(\delta, f)_M = \inf \Big\{ \|f - g\|_M + \delta^n \|g^{[n]}\|_M : \ g^{[n]} \in \mathcal{S}_M \Big\}, \quad \delta > 0. \quad (5.1)$$

Theorem 5.1. *For any $n \in \mathbb{N}$, there exist constants $C_1(n)$, $C_2(n) > 0$, such that for each $f \in \mathcal{S}_M$ and all $\delta > 0$*

$$C_1(n)\omega_n(f,\delta)_M \le K_n(\delta, f)_M$$

$$+\delta^n \Big\| \sum_{0<|k|\le n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M \le C_2(n)\omega_n(f,\delta)_M. \quad (5.2)$$

Remark 5.1. Let $f \in \mathcal{S}_M$. For any $\alpha > 0$, $h \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have

$$[\Delta_h^\alpha f]^\frown(k) = \Big[\sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\cdot - jh) \Big]^\frown(k)$$

$$= \widehat{f}(k) \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \mathrm{e}^{-\mathrm{i}kjh} = (1 - \mathrm{e}^{-\mathrm{i}kh})^\alpha \widehat{f}(k). \quad (5.3)$$

For a fixed $r = 0, 1, \ldots$ we denote by f_r the function from $\mathcal{S}_M$ such that $\widehat{f_r}(k) = 0$ when $|k| \le r$, and $\widehat{f_r}(k) = \widehat{f}(k)$ when $|k| > r$. Then according to (5.3), we have $\|\Delta_h^\alpha f\|_M = \|\Delta_h^\alpha f_0\|_M$ and therefore,

$$\omega_\alpha(f,\delta)_M = \omega_\alpha(f_0,\delta)_M. \quad (5.4)$$

On the other hand, by virtue of (5.1) and the definition of the radial derivative, it is clear that infimum on the right-hand side of (5.1) is attained at the set $G_{n,f}$ of all functions $g \in \mathcal{S}_M$ such that $g^{[n]} \in \mathcal{S}_M$ and $\widehat{g}(k) = \widehat{f}(k)$ for $|k| \le n-1$. Hence,

$$K_n(\delta, f)_M = K_n(\delta, f_{n-1})_M. \quad (5.5)$$

Thus, in (5.2), we use the term $\delta^n \Big\| \sum_{0<|k|\le n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M$ which takes into account the peculiarities of relations (5.4) and (5.5).

6. Proof of the results

Proof of Proposition 4.1. Implication 1) $\Rightarrow$ 2). For any $n \in \mathbb{N}$, we have

$$\Big\| f - Z_n^{(s)}(f) \Big\|_M \le (n+1)^{-s} \Big\| \sum_{|k|\le n} |k|^s \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M + \Big\| \sum_{|k|>n} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M. \quad (6.1)$$

Therefore, if relation 1) holds, then

$$(n+1)^{-s}\Big\|\sum_{|k|\le n}|k|^s\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M=(n+1)^{-s}\Big\|\sum_{|k|\le n}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M$$

$$=(n+1)^{-s}\|S_n(f^{(s)})\|_M=\mathcal{O}(\omega(n^{-1})),\quad n\to\infty. \tag{6.2}$$

To estimate the second term in (6.1), fix an integer $N>n$ and apply the Abel transformation,

$$\Big\|\sum_{n<|k|\le N}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M=\Big\|\sum_{n<|k|\le N}|k|^{-s}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M$$

$$=\Big\|\sum_{j=n+1}^{N-1}\Big(\frac{1}{j^s}-\frac{1}{(j+1)^s}\Big)\sum_{|k|\le j}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}$$

$$+N^{-s}\sum_{|k|\le N}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}-(n+1)^{-s}\sum_{|k|\le n}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M.$$

Then

$$\Big\|\sum_{n<|k|\le N}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M\le s\sum_{j=n+1}^{N-1}j^{-s-1}\|S_j(f^{(s)})\|_M$$

$$+N^{-s}\|S_N(f^{(s)})\|_M+(n+1)^{-s}\|S_n(f^{(s)})\|_M.$$

If relation 1) holds, then there exist a number $C_1>0$ such that for all integers $N>n$,

$$\Big\|\sum_{n<|k|\le N}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M\le C_1\Big(\sum_{j=n+1}^{N-1}\omega(j^{-1})/j+\omega(N^{-1})+\omega(n^{-1})\Big)$$

$$\le C_1\Big(\sum_{j=n+1}^{\infty}\omega(j^{-1})/j+2\omega(n^{-1})\Big).$$

In view of the condition $(\mathcal{B})$, this yields that

$$\Big\|\sum_{|k|>n}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_M=\mathcal{O}(\omega(n^{-1})),\quad n\to\infty. \tag{6.3}$$

Combining relations (6.1)–(6.3), we get the relation 2). Furthermore, since $\omega(\delta)\to 0$ as $\delta\to 0+$, then from 2), it follows that $f\in\mathcal{S}_M$.

2) $\Rightarrow$ 3). Let us set $n := [1/\delta]-1$. By virtue of (5.3), for any $|h|\le\delta$ and $|k|\le n$, we have

$$|[\Delta_h^s f]^\frown(k)| = |1-\mathrm{e}^{-\mathrm{i}kh}|^s|\widehat{f}(k)| = \Big|2\sin\frac{hk}{2}\Big|^s|\widehat{f}(k)|$$

$$\le \delta^s|k|^s|\widehat{f}(k)| \le (n+1)^{-s}|k|^s|\widehat{f}(k)|$$

and $|[\Delta_h^s f]^\frown(k)|\le|\widehat{f}(k)|$ when $|k|>n$. Let $a_1 := \|f-Z_n^{(s)}(f)\|_{M}$. Then

$$\sum_{k\in\mathbb{Z}} M(|[\Delta_h^s f]^\frown(k)|/a_1) \le \sum_{|k|\le n} M((n+1)^{-s}|k|^s|\widehat{f}(k)|/a_1)$$

$$+\sum_{|k|>n} M(|\widehat{f}(k)|/a_1)\le 1.$$

Therefore, for any $|h|\le\delta$,

$$\|\Delta_h^s f\|_{M} \le \|f-Z_n^{(s)}(f)\|_{M} = \mathcal{O}(\omega(n^{-1})) = \mathcal{O}(\omega(\delta)), \quad \delta\to 0+,$$

and hence $f\in\mathcal{S}_M H_\omega^s$.

3) $\Rightarrow$ 1). Setting $h_n := \pi/n$, $n\in\mathbb{N}$, and $a_2 := (n/2)^s\|\Delta_{h_n}^s f\|_{M}$, by virtue of the inequality $th_n\le\pi\sin(th_n/2)$, which is valid for all $t\in[0,n]$, we see that

$$\sum_{|k|\le n} M\Big(|\widehat{f^{(s)}}(k)|/a_2\Big) = \sum_{|k|\le n} M\Big(h_n^s|k|^s|\widehat{f}(k)|/(a_2h_n^s)\Big)$$

$$\le \sum_{|k|\le n} M\Big(\pi^s\Big|\sin\frac{kh_n}{2}\Big|^s|\widehat{f}(k)|/(a_2h_n^s)\Big)$$

$$\le \sum_{k\in\mathbb{Z}} M\Big(\Big|2\sin\frac{kh_n}{2}\Big|^s\frac{|\widehat{f}(k)|}{\|\Delta_{h_n}^s f\|_{M}}\Big)\le 1.$$

Thus,

$$\|S_n(f^{(s)})\|_{M} \le (n/2)^s\|\Delta_{h_n}^s f\|_{M}$$

$$\le (n/2)^s\omega_s(f,\pi/n)_{M} = \mathcal{O}(n^s\omega(n^{-1})), \quad n\to\infty.$$

It should be noted that in the case where $M(t)=t$, $\omega(t)=t^\beta$, $\beta>0$, the equivalence of the relations 1) and (6.3) was also proved in [9, Lemma 1].

Proof of Theorem 2.1. It is shown above that the Theorem 4.2 follows from Theorem 2.1. Therefore, it remains to prove the truth of Theorem 2.1.

1) $\Rightarrow$ 2). Since

$$\sum_{j=0}^{\nu}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}=\big((1-\varrho)+\varrho\big)^{\nu}=1,\ \nu=0,1,\ldots, \tag{6.4}$$

then for $a_3 := \|f-A_{\varrho,r}(f)\|_M$, we have

$$1\ge\sum_{|k|\ge r}M\Big(|1-\lambda_{|k|,r}(\varrho)||\widehat{f}(k)|/a_3\Big)$$

$$=\sum_{|k|\ge r}M\bigg(\bigg|1-\sum_{j=0}^{r-1}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}\bigg||\widehat{f}(k)|/a_3\bigg)$$

$$=\sum_{|k|\ge r}M\bigg(\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}|\widehat{f}(k)|/a_3\bigg)$$

$$\ge\sum_{|k|\ge r}M\bigg(\binom{|k|}{r}(1-\varrho)^r\varrho^{|k|-r}|\widehat{f}(k)|/a_3\bigg). \tag{6.5}$$

On the other hand, by virtue of (3.2),

$$\|P(f^{[r]})(\varrho,\cdot)\|_M=\Big\|\varrho^r\frac{\partial^r}{\partial\varrho^r}P(f)(\varrho,\cdot)\Big\|_M$$

$$=\inf\bigg\{a>0:\sum_{|k|\ge r}M\bigg(r!\binom{|k|}{r}\varrho^{|k|}|\widehat{f}(k)|/a\bigg)\le 1\bigg\}.$$

Combining these relations and equality (3.2), we see that for $\varrho\to 1-$,

$$\|P(f^{[r]})(\varrho,\cdot)\|_M\le r!\varrho^r(1-\varrho)^{-r}\|f-A_{\varrho,r}(f)\|_M=\mathcal{O}((1-\varrho)^{-s}\omega(1-\varrho)).$$

2) $\Rightarrow$ 3). For $a_4 := \|P(f^{[r]})(\varrho,\cdot)\|_M$ and for any numbers $n>r$ and $\varrho\in[0,1)$, we have

$$1\ge\sum_{|k|\ge r}M\bigg(\binom{|k|}{r}\frac{r!\varrho^{|k|}|\widehat{f}(k)|}{a_4}\bigg)$$

$$\ge\sum_{r\le|k|\le n}M\bigg(\varrho^n\binom{|k|}{r}\frac{r!|\widehat{f}(k)|}{a_4}\bigg)=\sum_{r\le|k|\le n}M\bigg(\frac{\varrho^n|\widehat{f^{[r]}}(k)|}{a_4}\bigg).$$

This yields $\|S_n(f^{[r]})\|_M\le\varrho^{-n}\|P(f^{[r]})(\varrho,\cdot)\|_M$ and putting $\varrho=1-1/n$ and taking into account statement 2), we see that

$$\|S_n(f^{[r]})\|_M\le(1-1/n)^{-n}\mathcal{O}(n^s\omega(n^{-1}))=\mathcal{O}(n^s\omega(n^{-1})),\ \text{as } n\to\infty.$$

3) $\Rightarrow$ 4). Let us set $g := f^{[r-s]}$. By the definition, for $|k| \geq r$, we have

$$|\widehat{f}^{[r]}(k)| = \frac{|k|!|\widehat{f}(k)|}{(|k|-r)!}$$

$$= |g^{[s]}(k)| \frac{(|k|-r+1)(|k|-r+2)\cdot \ldots \cdot (|k|-r+s)}{|k|(|k|-1)\cdot \ldots \cdot (|k|-s+1)}$$

$$\geq |g^{[s]}(k)| \Big(1 - \frac{r-1}{|k|}\Big)^s \geq r^{-s}|g^{[s]}(k)|.$$

Therefore, taking into account Remark 3.1, we get

$$\|S_n(g^{[s]})\|_M \leq \|S_{r-1}(g^{[s]})\|_M + \Big\| \sum_{r\leq |k| \leq n} g^{[s]}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M$$

$$\leq \|S_{r-1}(g^{[s]})\|_M + r^s \|S_n(f^{[r]})\|_M = \mathcal{O}(n^s \omega(n^{-1})), \quad n \to \infty.$$

Then by virtue of Proposition 4.2, we see that $\|g - Z_n^{(s)}(g)\|_M = \mathcal{O}(\omega(n^{-1}))$, $n \to \infty$, hence, $g = f^{[r-s]} \in \mathcal{S}_M$, $f \in \mathcal{S}_M$ and $f^{[r-s]} \in \mathcal{S}_M H_\omega^s$.

4) $\Rightarrow$ 3). If $g := f^{[r-s]}$, then according to Proposition 4.2, we get

$$\|S_n(g^{[s]})\|_M = \mathcal{O}(n^s \omega(n^{-1})), \quad n \to \infty.$$

For $|k| < r$ we have $\widehat{f}^{[r]}(k) = 0$ and for $|k| \geq r$,

$$\widehat{f}^{[r]}(k)| = \frac{|k|!}{(|k|-r)!}|\widehat{f}(k)| \leq \frac{|k|!}{(|k|-s)!}\frac{|k|!}{(|k|-r+s)!}|\widehat{f}(k)| = |g^{[s]}(k)|.$$

Thus

$$\|S_n(f^{[r]})\|_M \leq \|S_n(g^{[s]})\|_M = \mathcal{O}(n^s \omega(n^{-1})), \quad n \to \infty.$$

3) $\Rightarrow$ 1). From identity (6.4), it follows that for any $\varrho \in [0,1]$,

$$\sum_{j=r}^{\nu} \binom{\nu}{j} (1-\varrho)^j \varrho^{\nu-j} \leq 1, \quad \nu \geq r.$$

This implies the relation

$$\sum_{|k|\geq r} M\Big(|1 - \lambda_{|k|,r}(\varrho)| \frac{|\widehat{f}(k)|}{a_5}\Big)$$

$$= \sum_{|k|\geq r} M\Big(\sum_{j=r}^{|k|} \binom{|k|}{j} (1-\varrho)^j \varrho^{|k|-j} \frac{|\widehat{f}(k)|}{a_5}\Big) \leq \sum_{|k|\geq r} M\Big(\frac{|\widehat{f}(k)|}{a_5}\Big) \leq 1,$$

where $a_5 := \|f\|_M$, and therefore, we have $\|f - A_{\varrho,r}(f)\|_M \le \|f\|_M < \infty$. From this relation, we conclude that for any $\varepsilon > 0$ there exists the number n_0 such that for all $n > n_0$ and all $\varrho \in [0,1)$,

$$\|f - A_{\varrho,r}(f)\|_M \le \Bigg\| \sum_{r\le|k|\le n} \sum_{j=r}^{|k|} \binom{|k|}{j} (1-\varrho)^j \varrho^{|k|-j} \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx} \Bigg\|_M + \varepsilon. \quad (6.6)$$

Let us use the following inequality

$$\sum_{j=r}^{\nu} \binom{\nu}{j} (1-\varrho)^j \varrho^{\nu-j} \le \binom{\nu}{r} (1-\varrho)^r \quad (6.7)$$

which is valid for all $\nu \ge r$ and $\varrho \in [0,1]$ (see, for example [16]). Putting $a_6 := (1-\varrho)^r \|S_n(f^{[r]})\|_M / r!$, we get

$$\sum_{r\le|k|\le n} M\Bigg(\sum_{j=r}^{|k|} \binom{|k|}{j} (1-\varrho)^j \varrho^{|k|-j} \frac{|\widehat{f}(k)|}{a_6} \Bigg)$$

$$\le \sum_{r\le|k|\le n} M\Bigg((1-\varrho)^r \binom{|k|}{r} \frac{|\widehat{f}(k)|}{a_6} \Bigg) \le 1.$$

Thus,

$$\Bigg\| \sum_{r\le|k|\le n} \sum_{j=r}^{|k|} \binom{|k|}{j} (1-\varrho)^j \varrho^{|k|-j} \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx} \Bigg\|_M \le \frac{(1-\varrho)^r}{r!} \|S_n(f^{[r]})\|_M. \quad (6.8)$$

Combining relations (6.6) and (6.8) and putting $n := n_\varrho = [(1-\varrho)^{-1}]$, where $[\cdot]$ means the integer part of the number, we get

$$\|f - A_{\varrho,r}(f)\|_M \le \frac{(1-\varrho)^r}{r!} \|S_n(f^{[r]})\|_M + \varepsilon$$

$$= (1-\varrho)^r \mathcal{O}(n_\varrho^s \omega(n_\varrho^{-1})) + \varepsilon = \mathcal{O}((1-\varrho)^{r-s} \omega(1-\varrho)) + \varepsilon,$$

as $\varrho \to 1-$. By virtue of arbitrary ε, from this relation it follows that the implication 3) $\Rightarrow$ 1) is true.

Proof of Theorem 5.1. Before proving Theorem 5.1, let us formulate some known auxiliary statements.

Lemma 6.1. [6] *Assume that* $f, g \in \mathcal{S}_M$, $\alpha, \delta > 0$, $h \in \mathbb{R}$. *Then*
(i) $\|\Delta_h^\alpha f\|_M \le K(\alpha)\|f\|_M$, *where* $K(\alpha) := \sum_{j=0}^\infty |\binom{\alpha}{j}| \le 2^{\{\alpha\}}$, $\{\alpha\} = \inf\{k \in \mathbb{N} : k \ge \alpha\}$.
(ii) $\omega_\alpha(f+g,\delta)_M \le \omega_\alpha(f,\delta)_M + \omega_\alpha(g,\delta)_M$.
(iii) $\omega_\alpha(f,\delta)_M \le 2^{\{\alpha\}}\|f\|_M$.

Lemma 6.2. [6] *Assume that $\alpha > 0$, $n \in \mathbb{N}$ and $0 \le h \le 2\pi/n$. Then for any polynomial $\tau_n(x) = \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kh}$*

$$\Big(\frac{\sin(nh/2)}{n/2}\Big)^{\alpha} \|\tau_n^{(\alpha)}\|_M \le \|\Delta_h^{\alpha}\tau_n\|_M \le h^{\alpha}\|\tau_n^{(\alpha)}\|_M. \tag{6.9}$$

Lemma 6.3. [6] *If $f \in \mathcal{S}_M$, then for any numbers $\alpha > 0$ and $m \in \mathbb{N}$ the following inequality holds:*

$$\|f - S_m(f)\|_M = E_{m+1}(f)_M \le C(\alpha)\, \omega_\alpha(f, m^{-1})_M. \tag{6.10}$$

where $C = C(\alpha)$ is a constant that does not depend on f and n.

Consider an arbitrary function g from the set $G_{n,f}$ defined in Remark 5.1. By virtue (5.3), if $|h| < \delta$, then $[\Delta_h^n g]^\frown(0) = 0$, for all $0 < |k| \le n-1$,

$$\Big|[\Delta_h^n g]^\frown(k)\Big| = \Big|2\sin\frac{kh}{2}\Big|^n |\widehat{g}(k)| \le \delta^n |k|^n |\widehat{g}(k)|$$

$$\le \delta^n (n-1)^n |\widehat{g}(k)| \le \delta^n (n-1)^n |\widehat{f}(k)|,$$

and for $|k| \ge n$,

$$\Big|[\Delta_h^n g]^\frown(k)\Big| \le |k|^n \delta^n |\widehat{g}(k)| \le \delta^n n^n |k| \ldots (|k| - n + 1)|\widehat{g}(k)|$$

$$= \delta^n n^n |\widehat{g}^{[n]}(k)|.$$

Therefore, for any $|h| < \delta$, we have

$$\|\Delta_h^n g\|_M \le \delta^n (n-1)^n \Big\| \sum_{0<|k|\le n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M + \delta^n n^n \|g^{[n]}\|_M$$

and hence,

$$\omega_n(g, \delta) \le \delta^n (n-1)^n \Big\| \sum_{0<|k|\le n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M + \delta^n n^n \|g^{[n]}\|_M. \tag{6.11}$$

By virtue of Lemma 6.1 (ii) and (iii) and relation (6.11), for any $g \in G_{n,f}$, we have

$$\omega_n(f, \delta)_M \le \omega_n(f - g, \delta)_M + \omega_n(g, \delta)_M$$

$$\le 2^n \|f - g\|_M + \delta^n (n-1)^n \Big\| \sum_{0<|k|\le n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M.$$

Taking the infimum of the right hand side of the last relation over all $h \in G_{n,f}$, we get the left-hand side of (5.2) with the constant $C_1 = \min\{2^{-n}, n^{-n}\}$.

Now we shall prove the right-hand side of (5.2). Let $S_m := S_m(f_0)$, $m \geq n$, be the Fourier sum of f_0 defined in Remark 5.1. Then for $n \leq |k| \leq m$ the Fourier coefficients of the derivative $S_m^{[n]}$

$$|[S_m^{[n]}]\widehat{\ }(k)| = |k|(|k|-1)\dots(|k|-n+1)|\widehat{f}(k)| \leq |k|^n|\widehat{f}(k)| = |[S_m^{(n)}]\widehat{\ }(k)|$$

and $[S_m^{[n]}]\widehat{\ }(k) = 0$ for $|k| \in \mathbb{N} \setminus [n, m]$. Therefore, $\|S_n^{[n]}\|_M \leq \|S_m^{(n)}\|_M$.

Now let $\delta \in (0, 2\pi)$ and $m \in \mathbb{N}$ such that $\pi/m < \delta < 2\pi/m$. Using Lemma 6.2 with $h = \pi/m$ and property (i) of Lemma 6.1, we obtain

$$\begin{aligned} \|S_n^{[n]}\|_M &\leq \|S_m^{(n)}\|_M \leq (m/2)^n \|\Delta^n_{\pi/m} S_m\|_M \\ &\leq (m/2)^n \|\Delta^n_{\pi/m} f\|_M \leq (\pi/\delta)^n \omega_n(f,\delta)_M \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} \Big\| \sum_{0<|k|\leq n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M &\leq \Big\| \sum_{0<|k|\leq m} |k|^n \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M \\ &\leq (m/2)^n \|\Delta^n_{\pi/m} f\|_M \leq (\pi/\delta)^n \omega_n(f,\delta)_M. \end{aligned} \tag{6.13}$$

By virtue of Lemma 6.3, we have

$$\|f_0 - S_m\|_M = E_{m+1}(f_0)_M \leq C(n)\omega_n(f_0,\delta)_M = C(n)\omega_n(f,\delta)_M. \tag{6.14}$$

Setting $C_2(n) := C(n) + 2\pi^n$ and combining (6.12)–(6.14) we obtain the right-hand side of (5.2):

$$K_n(\delta,f)_M + \delta^n \Big\| \sum_{0<|k|\leq n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M = K_n(\delta,f_0)_M + \delta^n \Big\| \sum_{0<|k|\leq n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M$$

$$\leq \|f_0 - S_m\|_M + \delta^n \|S_m^{[n]}\|_M + \delta^n \Big\| \sum_{0<|k|\leq n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \Big\|_M \leq C_2(\alpha)\omega_n(f,\delta)_M.$$

□

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