Український математичний вісник
Том 17 **(2020)**, № 2, 215 – 233
https://doi.org/10.37069/1810-3200-2020-17-2-5

Sobolev mappings and moduli inequalities on Carnot groups

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(Presented by V. Ya. Gutlyanskii)

Abstract. In the article we study mappings that satisfy moduli inequalities on Carnot groups. We prove that homeomorphisms satisfy the moduli inequalities (Q-homeomorphisms) with a locally integrable function Q are Sobolev mappings. On this base in the frameworks of the weak inverse mapping theorem we prove that on the Carnot groups $\mathbb{G}$ mappings inverse to Sobolev homeomorphisms of finite distortion of the class $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$ belong to the Sobolev class $W^1_{1,\mathrm{loc}}(\Omega';\Omega)$.

2010 MSC. 30C65, 22E30, 46E35.

Key words and phrases. Sobolev spaces, moduli inequalities, Carnot group.

1. Introduction

It is known that Sobolev mappings on Carnot groups $\mathbb{G}$ can not be characterized only in the terms of its coordinate functions. The basic approach to the Sobolev mappings theory on Carnot groups is based on the notion of absolutely continuity on almost all horizontal lines which allows to define a weak upper gradient of mappings. In the present article we prove that homeomorphisms satisfy moduli inequalities on Carnot groups are Sobolev mappings. On this base we prove the weak version of the inverse mapping theorem on Carnot groups. Namely we prove that mappings inverse to Sobolev homeomorphisms of finite distortion of the class $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$ are Sobolev mappings of the class $W^1_{1,\mathrm{loc}}(\Omega';\Omega)$. The problem of regularity of mappings inverse to Sobolev homeomorphisms represents a significant part of the weak inverse mapping theorem and was studied in [50] for a bi-measurable Sobolev homeomorphism $\varphi:\Omega\to\Omega'$, $\Omega,\Omega'\subset\mathbb{R}^n$ of the class $W^1_p(\Omega;\Omega')$, $p>n-1$. In [38] it was proved that the inverse of a homeomorphism $\varphi\in L^1_p(\Omega;\Omega')$, $p>n-1$, satisfies

Received 27.03.2020

$\varphi^{-1} \in BV_{\mathrm{loc}}(\Omega';\Omega)$. In the last decades the regularity of mappings inverse to Sobolev homeomorphisms was intensively studied in the frameworks of the non-linear elasticity theory [1], see, for example, [8, 14, 17, 18, 29].

The suggested approach on Carnot groups is based on the moduli inequalities, namely on the notion of Q-mappings introduced in [24] (see also [25–26]). Recall that a homeomorphism $\varphi : \Omega \to \Omega'$ of domains $\Omega, \Omega' \subset \mathbb{G}$ is called a Q-homeomorphism, with a non-negative measurable function Q, if

$$M(\varphi\Gamma) \leqslant \int\limits_{\Omega} Q(x) \cdot \rho^{\nu}(x)dx$$

for every family Γ of rectifiable paths in Ω and every admissible function ρ for Γ.

In the Euclidean space $\mathbb{R}^n$ it was proved [25] that a homeomorphism $\varphi \in W^1_{n,\mathrm{loc}}(\Omega)$ such that $\varphi^{-1} \in W^1_{n,\mathrm{loc}}$ is a Q-mapping with $Q = K_I(x,\varphi)$, where $K_I(x,\varphi)$ is the inner quasiconformal dilatation of φ. The systematic applications of the moduli theory to the geometric mapping theory can be found in [27].

The main result of the article concerns to the weak differentiability of mappings satisfy moduli inequalities on Carnot groups (Theorem 5.1). The proof is based on the capacity estimates and the Fubini type decomposition of measures associated with horizontal foliations defined by a left-invariant vector fields and moduli (capacity) inequalities on Carnot groups.

Using the property of the weak differentiability and connection between Sobolev mappings and moduli inequalities we prove the weak regularity of Sobolev homeomorphisms on Carnot groups: if $\varphi : \Omega \to \Omega'$ is a Sobolev homeomorphism of finite distortion of the class $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$, then the inverse mapping $\varphi^{-1} \in W^1_{1,\mathrm{loc}}(\Omega';\Omega)$.

The weak differentiability is a part of the analytic definition of quasiconformal mappings and mappings of bounded distortion (see, e.g., [31] and [23]). The ACL-property of Q-mappings defined on planar domains of the Euclidean space $\mathbb{R}^2$ was considered by Brakalova and Jenkins, who proved this property for solutions of Beltrami equations in the plane (see [3, Lemma 3]). Under the assumption that $Q \in L^1_{\mathrm{loc}}$, the ACL-property was proved in $\mathbb{R}^n$ for Q-homeomorphisms (see [32]), and for mappings with branching later (see e.g. [33, 34]).

Q-homeomorphisms are closely connected with mappings that generate bounded composition operators on Sobolev spaces (p,q-quasiconformal mappings) [12, 36, 47, 48] which were studied on Carnot groups in [39, 40, 47, 49]. In the recent decade the geometric theory of composition operators on Sobolev spaces was applied to spectral estimates of the Laplace operator in Euclidean non-convex domains (see, for example, [5, 6, 11, 13, 15, 16]) and so results of this article have applications to the Sobolev mappings theory, to the spectral theory of (sub)elliptic operators and to the non-linear elasticity problems associated with vector fields that satisfy Hörmander's hypoellipticity condition.

2. Sobolev mappings on Carnot groups

2.1 Carnot groups

Recall that a stratified homogeneous group [10], or, in another terminology, a Carnot group [30] is a connected simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra V is decomposed into the direct sum $V_1 \oplus \cdots \oplus V_m$ of vector spaces such that $\dim V_1 \geqslant 2$, $[V_1,\ V_i] = V_{i+1}$ for $1 \leqslant i \leqslant m-1$ and $[V_1,\ V_m] = \{0\}$. Let $X_{11}, \ldots, X_{1n_1}$ be left-invariant basis vector fields of V_1. Since they generate V, for each i, $1 < i \leqslant m$, one can choose a basis X_{ik} in V_i, $1 \leqslant k \leqslant n_i = \dim V_i$, consisting of commutators of order $i-1$ of fields $X_{1k} \in V_1$. We identify elements g of $\mathbb{G}$ with vectors $x \in \mathbb{R}^N$, $N = \sum_{i=1}^m n_i$, $x = (x_{ik})$, $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant n_i$ by means of exponential map $\exp(\sum x_{ik} X_{ik}) = g$. Dilations δ_t defined by the formula

$$\begin{aligned}\delta_t x &= (t^i x_{ik})_{1 \leqslant i \leqslant m,\, 1 \leqslant k \leqslant n_j} \\ &= (t x_{11}, ..., t x_{1n_1}, t^2 x_{21}, ..., t^2 x_{2n_2}, ..., t^m x_{m1}, ..., t^m x_{mn_m}),\end{aligned}$$

are automorphisms of $\mathbb{G}$ for each $t > 0$. Lebesgue measure dx on $\mathbb{R}^N$ is the bi-invariant Haar measure on $\mathbb{G}$ (which is generated by the Lebesgue measure by means of the exponential map), and $d(\delta_t x) = t^\nu\ dx$, where the number $\nu = \sum_{i=1}^m i n_i$ is called the homogeneous dimension of the group $\mathbb{G}$. The measure $|E|$ of a measurable subset E of $\mathbb{G}$ is defined by

$$|E| = \int\limits_E dx.$$

The system of basis vectors $X_1, X_2, \ldots, X_n$ of the space V_1 (here and throughout we set $n_1 = n$ and $X_{i1} = X_i$, where $i = 1, \ldots, n$) satisfies the Hörmander's hypoellipticity condition.

Euclidean space $\mathbb{R}^n$ with the standard structure is an example of an abelian group: the vector fields $\partial/\partial x_i$, $i = 1, \ldots, n$, have no nontrivial commutation relations and form the basis of the corresponding Lie algebra. One example of a non-abelian stratified group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$hh' = (x, y, z)(x', y', z') = (x + x', y + y', z + z' - 2xy' + 2yx'),$$

where $x,x',y,y' \in \mathbb{R}^n$, $z,z' \in \mathbb{R}$. Left translation $L_h(\cdot)$ is defined as $L_h(h') = hh'$. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial z},\ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial z},\ i = 1,...,n,\ Z = \frac{\partial}{\partial z},$$

constitute the basis of the Lie algebra V of the Heisenberg group $\mathbb{H}^n$. All non-trivial relations are only of the form $[X_i, Y_i] = -4Z$, $i = 1,...,n$, and all other commutators vanish.

The Lie algebra of the Heisenberg group $\mathbb{H}^n$ has dimension $2n+1$ and splits into the direct sum $V = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i, $i = 1,...n$, and the space V_2 is the one-dimensional center which is spanned by the vector field Z.

Recall that a homogeneous norm on the group $\mathbb{G}$ is a continuous function $|\cdot| : \mathbb{G} \to [0,\infty)$ that is C^∞-smooth on $\mathbb{G} \setminus \{0\}$ and has the following properties:

(a) $|x| = |x^{-1}|$ and $|\delta_t(x)| = t|x|$;

(b) $|x| = 0$ if and only if $x = 0$;

(c) there exists a constant $\tau_0 > 0$ such that $|x_1 x_2| \leqslant \tau_0(|x_1| + |x_2|)$ for all $x_1, x_2 \in \mathbb{G}$.

The homogeneous norm on the group $\mathbb{G}$ define a homogeneous (quasi)metric

$$\rho(x,y) = |y^{-1}x|.$$

Note that a continuous map $\gamma : [a,b] \to \mathbb{G}$ is called a continuous curve on $\mathbb{G}$. This continuous curve is rectifiable if

$$\sup \left\{ \sum_{k=1}^{m} |\left(\gamma(t_k)\right)^{-1} \gamma(t_{k+1})| \right\} < \infty,$$

where the supremum is taken over all partitions $a = t_1 < t_2 < ... < t_m = b$ of the segment $[a,b]$.

In [30] it was proved that any rectifiable curve is differentiable almost everywhere and $\dot{\gamma}(t) \in V_1$: there exists measurable functions $a_i(t)$, $t \in (a,b)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^{n} a_i(t) X_i(\gamma(t)) \text{ and } \left| (\gamma(t+\tau))^{-1} \gamma(t) exp(\dot{\gamma}(t)\tau) \right| = o(\tau) \text{ as } \tau \to 0$$

for almost all $t \in (a,b)$. The length $l(\gamma)$ of a rectifiable curve $\gamma : [a,b] \to \mathbb{G}$ can be calculated by the formula

$$l(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_0^{\frac{1}{2}}\ dt = \int_a^b \left(\sum_{i=1}^{n} |a_i(t)|^2 \right)^{\frac{1}{2}}\ dt$$

where $\langle \cdot, \cdot \rangle_0$ is the inner product on V_1. The result of [7] implies that one can connect two arbitrary points $x, y \in \mathbb{G}$ by a rectifiable curve. The Carnot–Carathéodory distance $d(x, y)$ is the infimum of the lengths over all rectifiable curves with endpoints x and y in $\mathbb{G}$. The Hausdorff dimension of the metric space $(\mathbb{G}, d)$ coincides with the homogeneous dimension ν of the group $\mathbb{G}$.

2.2 Sobolev spaces on Carnot groups

Let $\mathbb{G}$ be a Carnot group with one-parameter dilatation group δ_t, $t > 0$, and a homogeneous norm ρ, and let E be a measurable subset of $\mathbb{G}$. The Lebesgue space $L_p(E)$, $p \in [1, \infty]$, is the space of measurable pth-power integrable functions $f : E \to \mathbb{R}$ with the standard norm:

$$\|f \mid L_p(E)\| = \left(\int\limits_E |f(x)|^p \, dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty,$$

and $\|f \mid L_\infty(E)\| = \operatorname{ess\,sup}_E |f(x)|$ for $p = \infty$. We denote by $L_{p,\mathrm{loc}}(E)$ the space of functions $f : E \to \mathbb{R}$ such that $f \in L_p(F)$ for each compact subset F of E.

Let Ω be an open set in $\mathbb{G}$. The (horizontal) Sobolev space $W^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$, ($L^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$) consists of the functions $f : \Omega \to \mathbb{R}$ locally integrable in Ω, having a weak derivatives $X_i f$ along the horizontal vector fields X_i, $i = 1, \ldots, n$, and a finite (semi)norm

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla_H f \mid L_p(\Omega)\|$$

$$(\|f \mid L^1_p(\Omega)\| = \|\nabla_H f \mid L_p(\Omega)\|),$$

where $\nabla_H f = (X_1 f, \ldots, X_n f)$ is the horizontal subgradient of f. If $f \in W^1_p(U)$ for each bounded open set U such that $\overline{U} \subset \Omega$ then we say that f belongs to the class $W^1_{p,\mathrm{loc}}(\Omega)$.

Let $\varphi : \Omega \to \mathbb{G}$ be a mapping defined on open set $\Omega \subset \mathbb{G}$. A Lie group homomorphism $\psi : \mathbb{G} \to \mathbb{G}$ such that $\exp^{-1} \circ \psi \circ \exp(V_1) \subset V_1$ is called the P-differential of φ at the point a of the set Ω if the set

$$A_\varepsilon = \{z \in E : d(\psi(a^{-l}x)^{-1} \varphi(a)^{-1} \varphi(x)) < \varepsilon d(a^{-1}x)\}$$

is a neighborhood of a (relative to Ω) for every $\varepsilon > 0$. The notion of P-differentiability was introduced in [30] where it was proved that Lipschitz mappings defined on open subsets of Carnot groups are P-differentiable almost everywhere. The Stepanov type theorem on Carnot groups was obtained in [46] (see, also [42]) where it was proved that

Lipschitz mappings defined on measurable subsets of Carnot groups are (approximately) P-differentiable almost everywhere.

We say that a mapping $\varphi : \Omega \to \mathbb{G}$ is absolutely continuous on lines ($\varphi \in \mathrm{ACL}(\Omega; \mathbb{G})$) if for each domain U such that $\overline{U} \subset \Omega$ and each foliation Γ_i defined by a left-invariant vector field X_i, $i = 1, \dots, n$, φ is absolutely continuous on $\gamma \cap U$ with respect to one-dimensional Hausdorff measure for $d\gamma$-almost every curve $\gamma \in \Gamma_i$. Recall that the measure $d\gamma$ on the foliation Γ_i equals the inner product $i(X_i)dx$ of the vector field X_i and the bi-invariant volume dx (see, for example, [9, 46]).

Since $X_i\varphi(x) \in V_1$ for almost all $x \in \Omega$ [30], $i = 1, \dots, n$, the linear mapping $D_H\varphi(x)$ with matrix $(X_i\varphi_j(x))$, $i, j = 1, \dots, n$, takes the horizontal subspace V_1 to V_1 and is called the formal horizontal differential of the mapping φ at x. Let $|D_H\varphi(x)|$ be its norm:

$$|D_H\varphi(x)| = \sup_{\xi \in V_1,\, |\xi|=1} |D_H\varphi(x)(\xi)|.$$

We say that a mapping $\varphi : \Omega \to \mathbb{G}$ belongs to $\mathrm{ACL}_p(\Omega; \mathbb{G}))$ ($\mathrm{ACL}_{p,\mathrm{loc}}(\Omega; \mathbb{G})$)) if $\varphi \in \mathrm{ACL}(\Omega; \mathbb{G})$ and $|D_H\varphi| \in L_p(\Omega)$ ($|D_H\varphi| \in L_{p,\mathrm{loc}}(\Omega)$).

Smooth mappings with differentials respecting the horizontal structure are said to be contact. For this reason one could say that mappings in the class $\mathrm{ACL}(\Omega; \mathbb{G})$ are (weakly) contact. It was proved in [42,46] that a formal horizontal differential $D_H : V_1 \to V_1$ induces a homomorphism $D\varphi : V \to V$ of the Lie algebras which is called the formal differential. The determinant of the matrix $D\varphi(x)$ is called the (formal) Jacobian of the mapping φ, and it is denoted by $J(x, \varphi)$.

The definition of Sobolev mappings in terms of Lipschitz functions was introduced in [37, 42]:

Let Ω be a domain in a stratified group $\mathbb{G}$. The mapping $\varphi : \Omega \to \mathbb{G}$ belongs to $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{G})$ if for each function $f \in \mathrm{Lip}(\mathbb{G})$ the composition $f \circ \varphi$ belongs to $W^1_{p,\mathrm{loc}}(\Omega)$ and $|\nabla_H(f \circ \varphi)|(x) \leqslant \mathrm{Lip}\, f \cdot g(x)$, where $g \in L_{p,\mathrm{loc}}(\Omega)$ is independent of f. The function g is called the upper gradient of the mapping φ.

3. Foliations and Set Functions

3.1 The Fubini type decomposition

We consider families Γ_k of orbits of horizontal vector fields $X_{1k} \in V_1$, $1 \leqslant k \leqslant n_1$, generating smooth foliations of a domain $\Omega \subset \mathbb{G}$. Denote the flow corresponding to the vector field X_{1k} by the symbol f_t, then each fiber has the form $\gamma(t) = f_t(s)$, where s belongs to the surface S_k transversal to X_{1k} and a parameter $t \in \mathbb{R}$.

We suppose that the foliation Γ_k of Ω is furnished with a measure $d\gamma$ satisfying the inequality

$$c_1|B(x,r)|^{\frac{\nu-1}{\nu}} \leqslant \int\limits_{\gamma\in\Gamma,\gamma\cap B(x,r)\neq\emptyset} d\gamma \leqslant c_2|B(x,r)|^{\frac{\nu-1}{\nu}} \tag{3.1}$$

for sufficiently small balls $B(x,r) \subset \Omega$ where constants c_1 and c_2 independent on balls $B(x,r)$.

The measure $d\gamma$ can be obtained [46] as the interior multiplication $i(X_{1k})$ of the vector field X_{1k} with the bi-invariant volume form dx. Let J_{f_t} be a Jacobian of the flow f_t. Then

$$f_t^* i(X_{1k})dx = J_{f_t} i(X_{1k})dx \text{ or } f_t^*\left(J_{f_{-t}} i(X_{1k})dx\right) = i(X_{1k})dx.$$

The tangent vector to a one-parameter family of curves γ_t passing through points $s \exp tX_{1k}$ can be identified with the tangent vector X_{1k} at the point $s \in S$. The flow f_t takes the vector X_{1k} to $(f_t)_* X_{1k}$. Consequently, the form $J_{f_{-t}} i(X_{1k})\ dx$ determines the measure $d\gamma$ on the foliation Γ_k.

Note, that by the inequality (3.1) the measure $d\gamma$ is the locally doubling measure:

$$\int\limits_{\gamma\in\Gamma_k,\gamma\cap B(x,2r)\neq\emptyset} d\gamma \leqslant c_d \int\limits_{\gamma\in\Gamma_k,\gamma\cap B(x,r)\neq\emptyset} d\gamma \tag{3.2}$$

for sufficiently small balls $B = B(x,r) \subset \Omega$.

Because X_{1k} is a left-invariant vector field the flow f_t is the right translation on $\exp tX_{1k}$. Since dx is a bi-invariant form, we have $J_{f_t} = c_m$, where the constant c_m can be calculated exactly. Using the left invariance and homogeneity under dilatations, we obtain that

$$\int\limits_{\gamma\in\Gamma_k,\gamma\cap B(x,r)\neq\emptyset} d\gamma = c_m|B(x,r)|^{\frac{\nu-1}{\nu}}\|X_{1k}\| \tag{3.3}$$

where $\|X_{1k}\|$ is the length of the tangent vector X_{1k}.

3.2 Additive set functions

Recall that a mapping Φ defined on open subsets from $\Omega \subset \mathbb{G}$ and taking nonnegative values is called a *finitely quasiadditive* set function [49] if

1) for any point $x \in \Omega$, exists δ, $0 < \delta < \operatorname{dist}(x, \partial\Omega)$, such that $0 \leqslant \Phi(B(x,\delta)) < \infty$ (here and in what follows $B(x,\delta) = \{y \in \mathbb{G} : \rho(x,y) < \delta\}$);

2) for any finite collection $U_i \subset U \subset \Omega$, $i = 1, \ldots, k$, of mutually disjoint open sets the following inequality $\sum\limits_{i=1}^{k} \Phi(U_i) \leqslant \Phi(U)$ takes place.

Obviously, the inequality in the second condition of this definition can be extended to a countable collection of mutually disjoint open sets from Ω, so a finitely quasiadditive set function is also *countable quasiadditive.*

If instead of the second condition we suppose that for any finite collection $U_i \subset \Omega$, $i = 1, \ldots, k$, of mutually disjoint open sets the equality

$$\sum_{i=1}^{k} \Phi(U_i) = \Phi(U)$$

takes place, then such a function is said to be *finitely additive.* If the equality in this condition can be extended to a countable collection of mutually disjoint open sets from Ω, then such a function is said to be *countably additive.*

A mapping Φ defined on open subsets of Ω and taking nonnegative values is called a *monotone* set function [49] if $\Phi(U_1) \leqslant \Phi(U_2)$ under the condition that $U_1 \subset U_2 \subset \Omega$ are open sets.

Let us formulate a result from [49] in a form convenient for us.

Theorem 3.1. [49] *Let a finitely quasiadditive set function Φ be defined on open subsets of the domain $\Omega \subset \mathbb{G}$. Then for almost all points $x \in \Omega$ the finite derivative*

$$\Phi'(x) = \lim_{\delta \to 0, B_\delta \ni x} \frac{\Phi(B_\delta)}{|B_\delta|}$$

exists and for any open set $U \subset \Omega$, the inequality

$$\int\limits_{U} \Phi'(x)\ dx \leqslant \Phi(U)$$

holds.

We consider the cube $P = S_k \exp tX_{1k}$, where $|t| \leqslant M$ and S_k is the transversal hyperplane to X_{1k}:

$$S_k = \{(x_{ij}), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n_i : x_{1k} = 0 \text{ and } |x_{ij}| \leqslant M\}.$$

Given a point $s \in S_k$, denote by γ_s the element $s \exp tX_{1k}$ of the horizontal fibration which starts at the point s. Thus P is the union of all such intervals of integral lines. Consider the following tubular neighborhood of the fiber γ_s with radius r:

$$E(s,r) = \gamma_s B(e,r) \cap P = \left(\bigcup_{x \in \gamma_s} B(x,r) \right) \cap P.$$

The following lemma is valid (see [45]):

Lemma 3.1. *Let Φ be a quasiadditive set function on $\mathbb{G}$. Then*

$$\overline{\lim_{r \to 0}} \frac{\Phi(E(s,r))}{r^{\nu-1}} < \infty$$

for $d\gamma$-almost all $s \in S_k$.

4. Capacity and Modules

4.1 The basic definitions

A well-ordered triple $(F_0, F_1; \Omega)$ of nonempty sets, where Ω is an open set in $\mathbb{G}$, and F_0, F_1 are compact subsets of $\overline{\Omega}$, is called a condenser in the group $\mathbb{G}$.

The value

$$\operatorname{cap}_p(F_0, F_1; \Omega) = \inf \int\limits_{\Omega} |\nabla_H v|^p \, dx,$$

where the infimum is taken over all nonnegative functions $v \in C(\Omega) \cap L^1_p(\Omega)$, such that $v = 0$ in a neighborhood of the set F_0, and $v \geqslant 1$ in a neighborhood of the set F_1, is called the p-capacity of the condenser $(F_0, F_1; \Omega)$. If $G \subset \mathbb{G}$ is an open set, and E is a compact subset in G, then the condenser $(\partial G, E; \mathbb{G})$ will be denoted by (E, G). Properties of p-capacity in the geometry of vector fields satisfying Hörmander hypoellipticity condition, can be found in [43, 44].

The linear integral is denoted by

$$\int\limits_{\gamma} \rho \, ds = \sup \int\limits_{\gamma'} \rho \, ds = \sup \int\limits_{0}^{l(\gamma')} \rho(\gamma'(s)) \, ds$$

where the supremum is taken over all closed parts γ' of γ and $l(\gamma')$ is the length of γ'. Let Γ be a family of curves in $\mathbb{G}$. Denote by $adm(\Gamma)$ the set of Borel functions (admissible functions) $\rho : \mathbb{G} \to [0, \infty]$ such that the inequality

$$\int\limits_{\gamma} \rho \, ds \geqslant 1$$

holds for locally rectifiable curves $\gamma \in \Gamma$.

Let Γ be a family of curves in $\overline{\mathbb{G}}$, where $\overline{\mathbb{G}}$ is a one point compactification of a Carnot group $\mathbb{G}$. The quantity

$$M(\Gamma) = \inf \int\limits_{\mathbb{G}} \rho^{\nu}\, dx$$

is called the module of the family of curves Γ [20]. The infimum is taken over all admissible functions $\rho \in adm(\Gamma)$.

Let Ω be a bounded domain on $\mathbb{G}$ and F_0, F_1 be disjoint non-empty compact sets in the closure of Ω. Let $M(\Gamma(F_0, F_1; \Omega))$ stand for the module of a family of curves which connect F_0 and F_1 in Ω. Then [21]

$$M(\Gamma(F_0, F_1; \Omega)) = \operatorname{cap}_{\nu}(F_0, F_1; \Omega)\,. \tag{4.1}$$

4.2 The lower estimate of the p-capacity

The following lower estimate of the p-capacity was proved in [47, Lemma 5]. For readers convenience we reproduce here the detailed proof of this lemma.

Lemma 4.1. *Let $\nu - 1 < p < \infty$. Suppose that E is a compact connected set and $G \subset \{x \in \mathbb{G} : \rho(x, E) \leqslant c_0 \operatorname{diam} E\}$, where c_0 is a small number depending on the constant in the generalized triangle inequality. Then*

$$\operatorname{cap}_p^{\nu-1}(E, G) \geqslant c(\nu, p) \frac{(\operatorname{diam} E)^p}{|G|^{p-(\nu-1)}}, \tag{4.2}$$

where a constant $c(\nu, p)$ depends only on ν and p.

Proof. Since the inequality (4.2) is invariant under left translations and has the same degree of homogeneity under dilations, we can suppose, without loss of generality, that $0 \in E$ and $\operatorname{diam} \mathrm{E} = \rho(0, \sigma) = 1$ for some point $\sigma \in E$.

Consider a point $\sigma^{-1} \in S(0, 1)$. Then there exists a constant c_1 such that

$$\operatorname{diam} E = 1 \leqslant c_1(r_2 - r_1),$$

where $r_1 = |\sigma^{-1}| = 1$ and $r_2 = \rho(\sigma^{-1}, \sigma) = |\sigma^2|$.

Since c_0 was choosing such that $G \subset \{x \in \mathbb{G} : \rho(x, E) \leqslant c_0 \operatorname{diam} E\}$, then by the generalized triangle inequality

$$S(\sigma^{-1}, r) \cap (\mathbb{G} \setminus G) \neq \emptyset \;\text{ for all }\; r_1 \leqslant r \leqslant r_2.$$

Let $r_1 \leqslant r \leqslant r_2$. We choose some point $x_r \in E$ such that $\rho(\sigma^{-1}, x_r) = r$ and denote

$$P(r) = \left\{s \in S(\sigma^{-1}, r) : \rho(x_r, s) \leqslant \rho(x_r, \{(\mathbb{G} \setminus G) \cap S(\sigma^{-1}, r)\}\right\}.$$

Consider an arbitrary function $u \in \overset{\circ}{L}{}^1_p(G) \cap C(G)$ such that $u \geqslant 1$ on E. Then the function u takes the value 0 on the sphere $S(x, r)$, $r_1 < r < r_2$. Therefore, the following inequality is valid for almost all $r_1 < r < r_2$ [41, Theorem 1]

$$\int\limits_{S(x,r)\cap G} M_{\gamma r}(|\nabla_H u|)^p(\xi) d\sigma_r(\xi) \geqslant c_2 \omega_r(P(r))^{\frac{\nu-1-p}{\nu-1}},$$

where ω_r is the measure on $S(x, r)$ associated with the "spherical" coordinate system [41]. (Here $\gamma > 1$ is some constant and $M_\delta g$ denotes the maximal function defined for every locally summable function g as

$$M_\delta g(x) = \sup \left\{ |B(x,r)|^{-1} \int\limits_{B(x,r)} |g| dx : r \leqslant \delta \right\},$$

where $B(x, r) = \{y \in \mathbb{G} : \rho(x, y) < r\}$ is the ball of radius r centered at $x \in \mathbb{G}$.) Consequently,

$$\int\limits_G M_{\gamma r}(|\nabla_H u|)^p dx \geqslant c_2 \int\limits_{r_1}^{r_2} \omega_r(P(r))^{\frac{\nu-1-p}{\nu-1}} dr.$$

Now

$$\begin{aligned}
(\operatorname{diam} E)^p &\leqslant \left(c_1 \int\limits_{r_1}^{r_2} dr \right)^p \\
&\leqslant c_1^p \left(\int\limits_{r_1}^{r_2} \omega_r(P(r)) dr \right)^{p-(\nu-1)} \left(\int\limits_{r_1}^{r_2} \omega_r^{\frac{\nu-1-p}{\nu-1}}(P(r)) dr \right)^{\nu-1} \\
&\qquad \leqslant \frac{c_1^p}{c_2} |G|^{p-(\nu-1)} \left(\int\limits_G M_{\gamma r}(|\nabla_H u|)^p dx \right)^{\nu-1}.
\end{aligned}$$

By the maximal function theorem, we obtain

$$\left(\int\limits_G |\nabla_H u|^p dx \right)^{\nu-1} \geqslant c(\nu, p) \frac{(\operatorname{diam} E)^p}{|G|^{p-(\nu-1)}}$$

for arbitrary function $u \in \overset{\circ}{L}{}^1_p(G) \cap C(G)$ admissible for the condenser (E, G). Hence

$$\operatorname{cap}_p^{\nu-1}(E, G) \geqslant c(\nu, p) \frac{(\operatorname{diam} E)^p}{|G|^{p-(\nu-1)}}.$$

□

5. Sobolev spaces and Q-Homeomorphisms

In this section we consider connection between Sobolev mappings and Q-homeomorphisms on Carnot groups.

5.1 ACL-property of Q-homeomorphisms

We prove the ACL-property of Q-homeomorphisms with locally integrable function Q.

Theorem 5.1. *Let $\varphi : \Omega \to \Omega'$ be a Q-homeomorphism of domains $\Omega, \Omega' \subset \mathbb{G}$ with $Q \in L_{1,\mathrm{loc}}(\Omega)$. Then $\varphi \in W^1_{1,\mathrm{loc}}(\Omega; \Omega')$.*

Proof. Fix some field X_{1k}, $\leqslant k \leqslant n_1$, and let Γ_k be the fibration generated by this field. Take the cube $P = S_k \exp tX_{1k}$, where $|t| \leqslant M$ and S_k is the transversal hyperplane to X_{1k}:

$$S_k = \{(x_{ij}), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n_i : x_{1k} = 0 \text{ and } |x_{ij}| \leqslant M\}.$$

Given a point $s \in S_k$, denote by γ_s the element $s \exp tX_{1,k}$ of the fibration which starts at s. Thus, P is the union of all such intervals of integral lines. Consider the following tubular neighborhood of the fiber γ_s with radius r:

$$E(s, r) = \gamma_s B(e, r) \cap P = \left(\bigcup_{x \in \gamma_s} B(x, r) \right) \cap P.$$

Take a point $s \in S_k$ so that the assertion of Lemma 3.1 holds for γ_s. On γ_s take arbitrary pairwise disjoint closed segments $\gamma_{s1}, \ldots, \gamma_{sk}$ of lengths $\delta_1, \ldots, \delta_k$. Denoting by R_i the open set of points at a distance less than a given $r > 0$ from γ_{si}, $i = 1, ..., k$, and consider the condensers (γ_{si}, R_i), $i = 1, ..., k$. Suppose that $r > 0$ is chosen so small that the sets $R_1, \ldots, R_k$ are pairwise disjoint and the condenser $(\varphi(\gamma_{si}), \varphi(R_i))$ satisfies to the conditions of Lemma 4.1. Let Γ be a family of curves connected $\varphi(\gamma_{si})$ and $\partial\varphi(R_i)$ in Ω. Now, by (4.1)

$$M(\varphi(\Gamma)) = \operatorname{cap}_\nu(\varphi(\gamma_{si})), \varphi(R_i)). \tag{5.1}$$

Observe that the function

$$\rho(x) = \begin{cases} \frac{1}{r}, & x \in R_i, \\ 0, & x \in \mathbb{G} \setminus R_i \end{cases}$$

is admissible for Γ. Now by (5.1)

$$\operatorname{cap}_\nu(\varphi(\gamma_{si})), \varphi(R_i)) \leqslant \frac{1}{r^\nu} \int\limits_{R_i} Q(x)\, dx\,. \tag{5.2}$$

On the other hand, by Lemma 4.1

$$\operatorname{cap}_\nu(\varphi(\gamma_{si})), \varphi(R_i)) \geqslant c\left(\frac{(\operatorname{diam}\varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|}\right)^{1/(\nu-1)}. \tag{5.3}$$

Combining (5.2) and (5.3), we have the inequalities

$$\left(\frac{(\operatorname{diam}\varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|}\right)^{\frac{1}{\nu-1}} \leqslant \frac{c_\nu}{r^\nu} \int\limits_{R_i} Q(x)\ dx\ , \qquad i = 1, ..., k \tag{5.4}$$

where the constant c_ν depends only on ν.

By the discrete Hölder inequality, see e.g. (17.3) in [2], we obtain that

$$\sum_{i=1}^{k} \operatorname{diam}\varphi(\gamma_{si}) \leqslant \left(\sum_{i=1}^{k}\left(\frac{(\operatorname{diam}\varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|}\right)^{\frac{1}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \left(\sum_{i=1}^{k} |\varphi(R_i)|\right)^{\frac{1}{\nu}}, \tag{5.5}$$

i.e.,

$$\left(\sum_{i=1}^{k} \operatorname{diam}\varphi(\gamma_{si})\right)^\nu \leqslant \left(\sum_{i=1}^{k}\left(\frac{(\operatorname{diam}\varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|}\right)^{\frac{1}{\nu-1}}\right)^{\nu-1} |\varphi(E(s,r))|\,, \tag{5.6}$$

and in view of (5.4)

$$\left(\sum_{i=1}^{k} \operatorname{diam}\varphi(\gamma_{si})\right)^\nu \leqslant c_\nu\, \frac{|\varphi(E(s,r))|}{r^{\nu-1}} \left(\sum_{i=1}^{k} \frac{\int\limits_{R_i} Q(x)\, dx}{r^{\nu-1}}\right)^{\nu-1} \tag{5.7}$$

where a constant c_ν depends only on ν.

By [47, Lemma 4])

$$\lim_{r\to 0}\frac{|\varphi(E(s,r))|}{r^{\nu-1}}:=\omega(s)<\infty.$$

Denote

$$\omega_i(s)=\int\limits_{\delta_i} Q(s,t)\ dt,\ \ s\in S_k,$$

and note that because Q is locally integrable function then by Fubini theorem for any $\varepsilon>0$ there exists a number $\delta>0$ such that $\omega_i(s)<\varepsilon$ if $\delta_i<\delta$, $i=1,...,k$.

By Fubini type decomposition (3.3) we have

$$\lim_{r\to 0}\frac{\int\limits_{R_i} Q(x)\,dx}{r^{\nu-1}}=\frac{\omega_i(s)}{c_m\|X_{1k}\|}<\infty.$$

Passing in (5.7) while $r\to 0$ we get

$$\left(\sum_{i=1}^{k}\operatorname{diam}\varphi(\gamma_{si})\right)^{\nu}\leqslant\frac{c_\nu\omega(s)}{c_m\|X_{1k}\|}\left(\sum_{i=1}^{k}\omega_i(s)\right)^{\nu-1}. \tag{5.8}$$

Hence, φ is absolutely continuous on $\gamma\cap P$ with respect to one-dimensional Hausdorff measure for $d\gamma$-almost every curve $\gamma\in\Gamma_k$. Hence $\varphi\in W^1_{1,\mathrm{loc}}(\Omega;\Omega')$. □

5.2 Mappings of integrable distortion

Let a homeomorphism $\varphi:\Omega\to\Omega'$ belongs to the Sobolev space $W^1_{1,\mathrm{loc}}(\Omega;\Omega')$. Recall that a weakly differentiable mapping $\varphi:\Omega\to\Omega'$ is called a mapping of finite distortion if $|D_H\varphi(x)|=0$ for almost all $x\in Z=\{x\in\Omega: J(x,\varphi)=0\}$. We say that a homeomorphism $\varphi:\Omega\to\Omega'$ has the Luzin N-property, if an image of a set of measure zero has measure zero.

The outer dilatation of the mapping of finite distortion φ at x is defined by

$$K_O(x)=K_O(x,\varphi)=\begin{cases}\frac{|D_H\varphi(x)|^\nu}{J(x,\varphi)}, & \text{if } J(x,\varphi)\neq 0,\\ 0, & \text{if } D_H\varphi(x)=0.\end{cases}$$

Theorem 5.2. *Let $\varphi:\Omega\to\Omega'$ be a homeomorphism of finite distortion of the Sobolev class $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$. Then, for every family Γ of rectifiable paths in Ω and every $\rho\in adm(\Gamma)$*

$$M\left(\varphi^{-1}(\Gamma)\right)\leqslant\int\limits_{\Omega}K_O\left(\varphi^{-1}(y),\varphi\right)\rho^\nu(y)\ dy,$$

i. e., φ^{-1} is a Q-homeomorphism with $Q(y)=K_O\left(\varphi^{-1}(y),\varphi\right)\in L_{1,\mathrm{loc}}(\Omega')$.

Proof. Let F be a compact subdomain of Ω, $F'=\varphi(F)$. Denote

$$Z=\{x\in\Omega : J(x,\varphi)=0\}.$$

Because φ is a mapping of finite distortion then $|D_H\varphi|=0$ a. e. on Z and $K_O(x,\varphi)$ is well defined for almost all $x\in\Omega$. Since $\varphi\in W^1_{\nu,\mathrm{loc}}(\Omega)$ then φ possesses the Luzin N-property and the outer distortion $K_O(\varphi^{-1}(y),\varphi)$ be well defined for almost all $y\in\Omega'$. Then

$$\begin{aligned}\int\limits_{F'} K_O\left(\varphi^{-1}(y),\varphi\right)\,dy &= \int\limits_{F'\setminus\varphi(Z)} \frac{|D_H\varphi(\varphi^{-1}(y))|^\nu}{|J(\varphi^{-1}(y),\varphi)|}\,dy+\int\limits_{F'\cap\varphi(Z)} dy\\ &=\int\limits_{F\setminus Z}\frac{|D_H\varphi(x)|^\nu}{|J(x,\varphi)|}|J(x,\varphi)|\,dx+\int\limits_{F\cap Z}|J(x,\varphi)|\,dx\\ &=\int\limits_{F\setminus Z}|D_H\varphi(x)|^\nu\,dx+\int\limits_{F\cap Z}|J(x,\varphi)|\,dx<\infty.\end{aligned}$$

Since $\varphi:\Omega\to\Omega'$ belongs to $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$ then φ be a (weakly) contact mapping differentiable almost everywhere in Ω and absolutely continuous on almost all horizontal curves. By generalized Fuglede's theorem (see, [22, 35]), we have that if $\tilde\Gamma$ is the family of all paths $\gamma\in\varphi^{-1}(\Gamma)$ such that φ is absolutely continuous on all closed subpaths of γ, then $M(\varphi^{-1}(\Gamma))=M(\tilde\Gamma)$.

Hence, for given a function $\rho\in adm\Gamma$ we define

$$\begin{cases}\widetilde{\rho}(x)=\rho(\varphi(x))|D_H\varphi(x)| & \text{if } x\in\Omega,\\ 0 & \text{otherwise.}\end{cases} \tag{5.9}$$

Then, for almost all $\tilde\gamma\in\tilde\Gamma$

$$\int\limits_{\tilde\gamma}\tilde\rho\,ds\geqslant\int\limits_{\varphi\circ\widetilde{\gamma}}\rho\,ds\geqslant 1$$

and consequently $\tilde\rho\in adm\tilde\Gamma$.

Therefore, using the change of variable formula [46] we obtain:

$$
\begin{aligned}
M(\varphi^{-1}(\Gamma)) = M(\tilde{\Gamma}) &\leq \int\limits_{\Omega} \tilde{\rho}^{\nu}(x)\ dx = \int\limits_{\Omega} \rho^{\nu}(\varphi(x))|D_H\varphi(x)|^{\nu}\ dx \\
&= \int\limits_{\Omega\setminus Z} \rho^{\nu}(\varphi(x))|D_H\varphi(x)|^{\nu}\ dx = \int\limits_{\Omega\setminus Z} \rho^{\nu}(\varphi(x))\frac{|D_H\varphi(x)|^{\nu}}{|J(x,\varphi)|}|J(x,\varphi)|\ dx \\
&= \int\limits_{\Omega'\setminus\varphi(Z)} \rho^{\nu}(y)\frac{|D_H\varphi(\varphi^{-1}(y))|}{|J(\varphi^{-1}(y),\varphi)|}\ dy = \int\limits_{\Omega'} K_O\left(\varphi^{-1}(y),\varphi\right)\rho^{\nu}(y)\ dy.
\end{aligned}
\tag{5.10}
$$

Hence φ^{-1} is a Q-homeomorphism with $Q(y) = K_O\left(\varphi^{-1}(y),\varphi\right) \in L_{1,\mathrm{loc}}(\Omega')$. □

5.3 The weak inverse mapping theorem on Carnot groups

In this section we prove that mappings inverse to Sobolev homeomorphisms of finite distortion of the class $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$ are Sobolev mappings.

Theorem 5.3. *Let $\varphi:\Omega\to\Omega'$ be a Sobolev homeomorphism of finite distortion of the class $W^1_{\nu,\mathrm{loc}}(\Omega;\Omega')$. Then $\varphi^{-1}\in W^1_{1,\mathrm{loc}}(\Omega;\Omega')$.*

Proof. By Theorem 5.2 we obtain that the inverse mapping $\varphi^{-1}:\Omega'\to\Omega$ be a Q-homeomorphism with $Q\in L_{1,\mathrm{loc}}(\Omega')$. Hence using Theorem 5.1 we conclude that the inverse mapping $\varphi^{-1}\in W^1_{1,\mathrm{loc}}(\Omega';\Omega)$. □

Acknowledgments. The authors thank the anonymous reviewer for careful reading of the paper and really valuable comments.

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