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Hadamard compositions of Gelfond–Leont'ev derivatives of analytic functions

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Abstract. For analytic functions f and g, the growth of the Hadamard composition of their Gelfond–Leont'ev derivatives is investigated in terms of generalized orders. A relation between the behaviors of the maximal terms of the Hadamard composition of Gelfond–Leont'ev derivatives and those of the Gelfond–Leont'ev derivative of a Hadamard composition is established.

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1. Introduction

For the power series

$$f(z)=\sum_{k=0}^{\infty} f_k z^k \tag{1.1}$$

and $g(z)=\sum\limits_{k=0}^{\infty} g_k z^k$ with convergence radii $R[f]$ and $R[g]$, the series $(f*g)(z)=\sum\limits_{k=0}^{\infty} f_k g_k z^k$ is called the Hadamard composition. It is well known [1–2] that $R[f*g]\ge R[f]R[g]$. The properties of this composition found the applications [2–3] in the theory of the analytic continuation of functions represented by power series. It is worth to note that the singular points of a Hadamard composition were investigated in work [4].

For $0\le r<R[f]$, let $M(r,f)=\max\{|f(z)|:\ |z|=r\}$, $\mu(r,f)=\max\{|f_k|r^k:\ k\ge 0\}$ be the maximal term, and let $\nu(r,f)=\max\{k:\ |f_k|r^k=\mu(r,f)\}$ be the central index of the power expansion of f. Studying [5–6] a connection between the growth of the maximal terms of a derivative of the Hadamard composition of two entire functions f and g

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and the Hadamard composition of their derivatives M. Sen [6] proved in particular that if the function $(f*g)$ has the order ϱ and the lower order λ, then, for every $\varepsilon>0$ and all $r\ge r_0(\varepsilon)$,

$$r^{(n+2)\lambda-1-\varepsilon}\le\frac{\mu(r,f^{(n+1)}*g^{(n+1)})}{\mu(r,(f*g)^{(n)})}\le r^{(n+2)\varrho-1+\varepsilon}.$$

For the power series (1.1) with the convergence radius $R[f]\in[0,+\infty]$ and the power series $l(z)=\sum\limits_{k=0}^{\infty}l_kz^k$ with the convergence radius $R[l]\in[0,+\infty]$ and coefficients $l_k>0$, $k\ge0$, the power series

$$D_l^{(n)}f(z)=\sum_{k=0}^{\infty}\frac{l_k}{l_{k+n}}f_{k+n}z^k \tag{1.2}$$

is called [7] the Gelfond–Leont'ev derivative of the n-th order of f with respect to l. If $l(z)=e^z$, then $D_l^{(n)}f(z)=f^{(n)}(z)$ is the usual derivative of the n-th order. Naturally, the radius of convergence of the Gelfond–Leont'ev derivative of series (1.1) coincides not always with the radius of convergence of the latter. However, using the Cauchy–Hadamard formula, it is not difficult to verify the validity of such statement. The following lemmas were proved in [8].

Lemma 1. *In order that, for an arbitrary series* (1.1), *the equalities* $R[f]=+\infty$ *and* $R[D_l^{(n)}f]=+\infty$ *be equivalent, it is necessary and sufficient that*

$$0<q=\varliminf_{k\to\infty}\sqrt[k]{l_k/l_{k+1}}\le\varlimsup_{k\to\infty}\sqrt[k]{l_k/l_{k+1}}=Q<+\infty. \tag{1.3}$$

Lemma 2. *In order that, for an arbitrary series* (1.1), *the equalities* $R[f]=1$ *and* $R[D_l^{(n)}f]=1$ *be equivalent, it is necessary and sufficient that*

$$\lim_{k\to\infty}\sqrt[k]{l_k/l_{k+1}}=1. \tag{1.4}$$

For the functions of finite order, the following analogs of the result by M. Sen were proved in [8].

Proposition 1. *If* $R[f]=R[g]=+\infty$ *and* (1.3) *with* $q>1$ *holds, then*

$$\varlimsup_{r\to+\infty}\frac{1}{\ln r}\ln\frac{\mu(r,D_l^{(n+1)}f*D_l^{(n+1)}g)}{\mu(r,D_l^{(n)}(f*g))}=(n+2)\varrho[f*g]-1$$

and

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{\mu(r, D_l^{(n+1)} f * D_l^{(n+1)} g)}{\mu(r, D_l^{(n)}(f*g))} = (n+2)\lambda[f*g] - 1,$$

where $\varrho[f]$ is the order, and $\lambda[f]$ is the lower order of the entire function f.

Proposition 2. *If $R[f] = R[g] = 1$, and if (1.4) holds, then*

$$(n+2)\varrho^{(1)}[f*g] \le \overline{\lim}_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln^+ \frac{\mu(r, D_l^{(n+1)} f * D_l^{(n+1)} g)}{\mu(r, D_l^{(n)}(f*g))}$$

$$\le (n+2)(\varrho^{(1)}[f*g]+1)$$

and

$$(n+2)\lambda^{(1)}[f*g] \le \underline{\lim}_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln^+ \frac{\mu(r, D_l^{(n+1)} f * D_l^{(n+1)} g)}{\mu(r, D_l^{(n)}(f*g))}$$

$$\le (n+2)(\lambda^{(1)}[f*g]+1),$$

where $\varrho^{(1)}[f]$ is the order and $\lambda^{(1)}[f]$ is the lower order of the analytic function f in a unit disk.

In [8], the behavior of $\mu(r, D_l^{(n)} f * D_l^{(n)} g)$ was studied as well. The following statements were proved.

Proposition 3. *Let $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $m > n$, and $R[f] = R[g] = +\infty$. If (1.3) holds with $q > 1$, then*

$$\overline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D_l^{(m)} f * D_l^{(m)} g)}{\mu(r, D_l^{(n)} f * D_l^{(n)} g)} = \varrho[f*g]$$

and

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D_l^{(m)} f * D_l^{(m)} g)}{\mu(r, D_l^{(n)} f * D_l^{(n)} g)} = \lambda[f*g].$$

If

$$0 < \underline{\lim}_{k\to\infty} \frac{l_k}{(k+1)l_{k+1}} \le \overline{\lim}_{k\to\infty} \frac{l_k}{(k+1)l_{k+1}} < +\infty, \tag{1.5}$$

then

$$\overline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(m)} f * D_l^{(m)} g)}{\mu(r, D_l^{(n)} f * D_l^{(n)} g)} = \varrho[f*g]$$

and

$$\varliminf_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(m)} f * D_l^{(m)} g)}{\mu(r, D_l^{(n)} f * D_l^{(n)} g)} = \lambda[f * g].$$

Proposition 4. *Let $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $m > n$, and $R[f] = R[g] = R[f * g] = 1$. If (1.5) holds, then*

$$2(m-n)\varrho^{(1)}[f * g] \le \varlimsup_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln^+ \frac{\mu(r, D_l^{(m)} f * D_l^{(m)} g)}{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}$$

$$\le 2(m-n)(\varrho^{(1)}[f * g] + 1)$$

and

$$2(m-n)\lambda^{(1)}[f * g] \le \varliminf_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln^+ \frac{\mu(r, D_l^{(m)} f * D_l^{(m)} g)}{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}$$

$$\le 2(m-n)(\lambda^{(1)}[f * g] + 1).$$

Here, in terms of generalized orders, we investigate the behavior of $\mu(r, D_l^{(n)} f * D_\lambda^{(m)} g)$, where $m \ne n$, $\lambda(z) = \sum_{k=0}^{\infty} \lambda_k z^k$, and $\lambda_k > 0$ for all $k \ge 0$.

2. Entire transcendental functions

Suppose that the functions f and g are transcendental, and the sequences (l_k) and (λ_k) satisfy condition (1.3). Since $R[f] = R[g] = +\infty$, $R[D_l^{(n)} f] = R[D_l^{(m)} g] = +\infty$ by Lemma 1, and, thus, $R[D_l^{(n)} f * D_l^{(m)} g] = +\infty$.

As in [9], let L be a class of continuous functions α nonnegative on $(-\infty, +\infty)$ and such that $\alpha(x) = \alpha(x_0) \ge 0$ for $x \le x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \le x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for every fixed $c \in (0, +\infty)$, i.e., α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

For $\alpha \in L$, $\beta \in L$, and the entire transcendental function (1.1), the quantities

$$\varrho_{\alpha,\beta}[f] := \varrho_{\alpha,\beta}[\ln M, f] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln M(r, f))}{\beta(\ln r)}$$

and

$$\lambda_{\alpha,\beta}[f] := \lambda_{\alpha,\beta}[\ln M, f] = \varliminf_{r\to+\infty} \frac{\alpha(\ln M(r,f))}{\beta(\ln r)}$$

are called the generalized order and the lower generalized order, respectively. If we substitute $\ln \mu(r,f)$ or $\nu(r,f)$ instead of $\ln M(r,f)$, then we obtain the definitions of the quantities $\varrho_{\alpha,\beta}[\ln \mu, f]$, $\lambda_{\alpha,\beta}[\ln \mu, f]$ and $\varrho_{\alpha,\beta}[\nu, f]$, $\lambda_{\alpha,\beta}[\nu, f]$, respectively.

Lemma 3. *Let $\alpha \in L_{si}$, $\beta \in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Then*

$$\varrho_{\alpha,\beta}[f] = \varlimsup_{k\to\infty} \frac{\alpha(k)}{\beta\left(\dfrac{1}{k}\ln\dfrac{1}{|f_k|}\right)}. \tag{2.6}$$

If, moreover, $|f_k/f_{k+1}| \nearrow +\infty$ as $k_0 \le k \to \infty$, then

$$\lambda_{\alpha,\beta}[f] = \varliminf_{k\to\infty} \frac{\alpha(k)}{\beta\left(\dfrac{1}{k}\ln\dfrac{1}{|f_k|}\right)}. \tag{2.7}$$

Formula (2.6) was proved in [9], and formula (2.7) follows from the corresponding formula for entire Dirichlet series proved in [10].

Lemma 4. *If $\alpha \in L_{si}$ and $\beta \in L^0$, then $\varrho_{\alpha,\beta}[f] = \varrho_{\alpha,\beta}[\ln \mu, f]$ and $\lambda_{\alpha,\beta}[f] = \lambda_{\alpha,\beta}[\ln \mu, f]$. If, moreover, $\alpha(e^x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \to +\infty$, then $\varrho_{\alpha,\beta}[\ln \mu, f] = \varrho_{\alpha,\beta}[\nu, f]$ and $\lambda_{\alpha,\beta}[\ln \mu, f] = \lambda_{\alpha,\beta}[\nu, f]$.*

Proof. In view of the conditions $\alpha \in L_{si}$ and $\beta \in L^0$, the equalities $\varrho_{\alpha,\beta}[f] = \varrho_{\alpha,\beta}[\ln \mu, f]$ and $\lambda_{\alpha,\beta}[f] = \lambda_{\alpha,\beta}[\ln \mu, f]$ follow from the estimates

$$\mu(r,f) \le M(r,f) \le \sum_{k=0}^{\infty} |f_k| r^k = \sum_{k=0}^{\infty} |f_k|(2r)^k 2^{-k} \le 2\mu(2r,f).$$

It is well known [11, p. 13] that

$$\ln \mu(r,f) - \ln \mu(r_0,f) = \int_{r_0}^{r} \frac{\nu(t,f)}{t} dt \, (0 \le r_0 \le r).$$

From whence for $r_0 = 1$, we get

$$\nu(r/2,f)\ln 2 \le \int_{r/2}^{r} \frac{\nu(t,f)}{t} dt \le \ln \mu(r,f) - \ln \mu(1.f) \le \nu(r,f)\ln r. \tag{2.8}$$

Therefore, in view of the conditions $\alpha(e^x) \in L_{si}$, $\beta \in L^0$ and $\alpha(x) = o(\beta(x))$ as $x \to +\infty$, we have

$$(1+o(1))\frac{\alpha(\nu(r,f))}{\beta(\ln r)} \le (1+o(1))\frac{\alpha(\ln \mu(r,f))}{\beta(\ln r)}$$

$$\le \frac{\alpha(\exp\{\ln \nu(r,f) + \ln \ln r\})}{\beta(\ln r)} \le \frac{\alpha(\exp\{2\max\{\ln \nu(r,f),\, \ln \ln r\}\})}{\beta(\ln r)}$$

$$= (1+o(1))\frac{\alpha(\exp\{\max\{\ln \nu(r,f),\, \ln \ln r\}\})}{\beta(\ln r)}$$

$$= (1+o(1))\frac{\max\{\alpha(\nu(r,f)),\, \alpha(\ln r)\}}{\beta(\ln r)} \le (1+o(1))\frac{\alpha(\nu(r,f)) +\, \alpha(\ln r)}{\beta(\ln r)}$$

$$= (1+o(1))\frac{\alpha(\nu(r,f))}{\beta(\ln r)} + o(1), \quad r \to +\infty.$$

Thus, $\varrho_{\alpha,\beta}[\ln \mu, f] = \varrho_{\alpha,\beta}[\nu, f]$ and $\lambda_{\alpha,\beta}[\ln \mu, f] = \lambda_{\alpha,\beta}[\nu, f]$. The proof of Lemma 4 is completed. □

Using Lemmas 3 and 4, we prove the following statements.

Proposition 5. *If $\alpha \in L_{si}$ and $\beta \in L^0$, then $\varrho_{\alpha,\beta}[f * g] \le \min\{\varrho_{\alpha,\beta}[f], \varrho_{\alpha,\beta}[g]\}$ and $\lambda_{\alpha,\beta}[f * g] \le \min\{\lambda_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}$.*

Proof. Since $|g_k| \le 1$ for $k \ge k_0$ and $r^{k_0} = o(\mu(r, f)$ as $r \to +\infty$, we have

$$\mu(r, f * g) = \max\{\max\{|f_k g_k| r^k : 0 \le k \le k_0\}, \max\{|f_k g_k| r^k : k \ge k_0\}\}$$

$$\le \max\{O(r^{k_0}), \max\{|f_k| r^k : k \ge n_0\}\} \le (1+o(1))\mu(r, f), \quad r \to +\infty.$$

From whence, we get $\varrho_{\alpha,\beta}[\ln \mu, f * g] \le \varrho_{\alpha,\beta}[\ln \mu, f]$, $\lambda_{\alpha,\beta}[\ln \mu, f * g] \le \lambda_{\alpha,\beta}[\ln \mu, f]$, and, by Lemma 4, $\varrho_{\alpha,\beta}[f * g] \le \varrho_{\alpha,\beta}[f]$ and $\lambda_{\alpha,\beta}[f * g] \le \lambda_{\alpha,\beta}[f]$. Similarly, $\varrho_{\alpha,\beta}[f * g] \le \varrho_{\alpha,\beta}[g]$ and $\lambda_{\alpha,\beta}[f * g] \le \lambda_{\alpha,\beta}[g]$. □

Proposition 6. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, and $\dfrac{d\beta^{-1}(c\alpha(x))}{d \ln x} = O(1)$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Suppose that $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow +\infty$ as $k_0 \le k \to \infty$. Then*

$$\varrho_{\alpha,\beta}[f * g] \ge \max\{\min\{\varrho_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}, \min\{\varrho_{\alpha,\beta}[g], \lambda_{\alpha,\beta}[f]\}\}, \tag{2.9}$$

and if, moreover, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$ as $k_0 \le k \to \infty$, then

$$\lambda_{\alpha,\beta}[f * g] \ge \min\{\lambda_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}. \tag{2.10}$$

Proof. We can consider that $\varrho_{\alpha,\beta}[f] > 0$ and $\lambda_{\alpha,\beta}[g] > 0$. Then $\dfrac{1}{k_j}\ln\dfrac{1}{|f_{k_j}|} \le \beta^{-1}\left(\dfrac{\alpha(k_j)}{\varrho}\right)$ for every $\varrho \in (0, \varrho_{\alpha,\beta}[f])$ and some sequence $(k_j) \uparrow \infty$, and $\dfrac{1}{k}\ln\dfrac{1}{|g_k|} \le \beta^{-1}\left(\dfrac{\alpha(k)}{\lambda}\right)$ for every $\lambda \in (0, \lambda_{\alpha,\beta}[g])$ and all $k \ge k_0(\lambda)$. Therefore, by Lemma 3 in view of the condition $\beta \in L_{si}$, we have

$$\varrho_{\alpha,\beta}[f * g] = \varlimsup_{k\to\infty} \frac{\alpha(k)}{\beta\left(\dfrac{1}{k}\ln\dfrac{1}{|f_k g_k|}\right)} \ge \varlimsup_{j\to\infty} \frac{\alpha(k_j)}{\beta\left(\dfrac{1}{k_j}\ln\dfrac{1}{|f_{k_j}|} + \dfrac{1}{k_j}\ln\dfrac{1}{|g_{k_j}|}\right)}$$

$$\ge \varlimsup_{j\to\infty} \frac{\alpha(k_j)}{\beta(\beta^{-1}(\alpha(k_j)/\varrho) + \beta^{-1}(\alpha(k_j)/\lambda))}$$

$$\ge \varlimsup_{j\to\infty} \frac{\alpha(k_j)}{\beta(2\beta^{-1}(\alpha(k_j)/\min\{\varrho,\lambda\}))} = \min\{\varrho,\lambda\}. \tag{2.11}$$

In view of the arbitrariness of ϱ and λ, inequality (2.11) implies the inequality $\varrho_{\alpha,\beta}[f * g] \ge \min\{\varrho_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}$ follows. Similarly, $\varrho_{\alpha,\beta}[f * g] \ge \min\{\varrho_{\alpha,\beta}[g], \lambda_{\alpha,\beta}[f]\}$. Thus, estimate (2.9) is proved.

Now, we suppose that $\lambda_{\alpha,\beta}[f] > 0$. Then $\dfrac{1}{k}\ln\dfrac{1}{|g_k|} \le \beta^{-1}\left(\dfrac{\alpha(k)}{\lambda^*}\right)$ for every $\lambda \in (0, \lambda_{\alpha,\beta}[g])$ and all $k \ge k_0(\lambda)$. Therefore, by Lemma 3 as above, we obtain

$$\lambda_{\alpha,\beta}[f * g] \ge \varliminf_{k\to\infty} \frac{\alpha(k)}{\beta(\beta^{-1}(\alpha(k)/\lambda^*) + \beta^{-1}(\alpha(k)/\lambda))}.$$

From whence, $\lambda_{\alpha,\beta}[f * g] \ge \min\{\lambda^*, \lambda\}$. In view of the arbitrariness of λ^* and λ, we get (2.10). The proof of Proposition 6 is completed. □

A next statement establishes a connection between the growth of an entire function and its Gelfond–Leont'ev derivative.

Proposition 7. *Let $\alpha \in L^0$ and $\beta \in L^0$. If condition* (1.4) *holds, and if f is an entire function, then $\lambda_{\alpha,\beta}[D_l^{(n)} f] = \lambda_{\alpha,\beta}[f]$ and $\varrho_{\alpha,\beta}[D_l^{(n)} f] = \varrho_{\alpha,\beta}[f]$.*

Proof. It is sufficient to consider the case where $n = 1$. Condition (1.4) yields the existence of numbers $0 < q_1 \le Q_1 < +\infty$ such that $q_1^k \le l_k/l_{k+1} \le Q_1^k$ for all $k \ge 0$. Therefore,

$$r\mu(r, D_l^1 f) = \max\left\{\frac{l_k}{l_{k+1}}|f_{k+1}|r^{k+1} : k \geq 0\right\}$$

$$\leq \frac{1}{Q_1}\max\{|f_{k+1}|(Q_1 r)^{k+1} : k \geq 0\} \leq \frac{\mu(Q_1 r, f)}{Q_1}$$

and, by analogy,

$$r\mu(r, D_l^1 f) \geq \frac{\mu(q_1 r, f)}{q_1}$$

for all r sufficiently large. Since $\ln r = o(\ln \mu(r, f))$ as $r \to +\infty$ for every entire transcendental function, we get the asymptotic inequalities

$$(1+o(1))\ln \mu(q_1 r, f) \leq \ln \mu(r, D_l^1 f) \leq (1+o(1))\ln \mu(Q_1 r, f),\ r \to +\infty.$$

From whence, we have $\lambda_{\alpha,\beta}[\ln \mu, D_l^{(n)} f] = \lambda_{\alpha,\beta}[\ln \mu, f]$ and $\varrho_{\alpha,\beta}[\ln \mu, D_l^{(n)} f] = \varrho_{\alpha,\beta}[\ln \mu, f]$. Therefore, in view of Lemma 4, Proposition 7 is proved. □

Let us go to the main result.

Theorem 1. *Let $\alpha(e^x) \in L_{si}$, $\beta \in L_{si}$, and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Suppose that $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow +\infty$ as $k_0 \leq k \to \infty$. Then:*

1) *if the sequences (l_k) and (λ_k) satisfy (1.3) with $q > 1$, then*

$$\max\{\min\{\varrho_{\alpha,\beta}[f],\, \lambda_{\alpha,\beta}[g]\},\, \min\{\lambda_{\alpha,\beta}[f], \varrho_{\alpha,\beta}[g]\}\}$$

$$\leq \varlimsup_{r\to+\infty} \frac{1}{\beta(\ln r)}\alpha\left(\ln \frac{r^j \mu(r, D_l^{(n+j)} f * D_\lambda^{(m+j)} g)}{\mu(r, D_l^{(n)} f * D_\lambda^{(m)} g)}\right)$$

$$\leq \min\{\varrho_{\alpha,\beta}[f],\, \varrho_{\alpha,\beta}[g]\} \tag{2.12}$$

and if, moreover, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$ and $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2 \nearrow 1$ as $k_0 \leq k \to \infty$, then

$$\varliminf_{r\to+\infty} \frac{1}{\beta(\ln r)}\alpha\left(\ln \frac{r^j \mu(r, D_l^{(n+j)} f * D_\lambda^{(m+j)} g)}{\mu(r, D_l^{(n)} f * D_\lambda^{(m)} g)}\right)$$

$$= \min\{\lambda_{\alpha,\beta}[f],\, \lambda_{\alpha,\beta}[g]\}; \tag{2.13}$$

2) *if*

$$l_k/l_{k+1} \asymp k, \quad \lambda_k/\lambda_{k+1} \asymp k \tag{2.14}$$

as $k \to \infty$, then

$$\max\{\min\{\varrho_{\alpha,\beta}[f],\, \lambda_{\alpha,\beta}[g]\},\, \min\{\lambda_{\alpha,\beta}[f], \varrho_{\alpha,\beta}[g]\}\} \leq$$

$$\leq \varlimsup_{r\to+\infty} \frac{1}{\beta(\ln r)}\alpha\left(\frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}\right)$$

$$\leq \min\{\varrho_{\alpha,\beta}[f],\ \varrho_{\alpha,\beta}[g]\} \tag{2.15}$$

and if, moreover, $l_kl_{k+2}/l_{k+1}^2 \nearrow 1$ *and* $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2 \nearrow 1$ *as* $k_0 \leq k \to \infty$*, then*

$$\varliminf_{r\to+\infty} \frac{1}{\beta(\ln r)}\alpha\left(\frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}\right) = \min\{\lambda_{\alpha,\beta}[f],\ \lambda_{\alpha,\beta}[g]\}. \tag{2.16}$$

Proof. It is clear that

$$(D_l^{(n)}f * D_\lambda^{(m)}g)(z) = \sum_{k=0}^{\infty} \frac{l_k}{l_{k+n}}\frac{\lambda_k}{\lambda_{k+m}}g_{k+m}f_{k+n}z^k$$

and

$$(D_l^{(n+j)}f * D_\lambda^{(m+j)}g)(z) = \sum_{k=0}^{\infty} \frac{l_k}{l_{k+n+j}}\frac{\lambda_k}{\lambda_{k+m+j}}g_{k+m+j}f_{k+n+j}z^k.$$

At first, we prove that

$$\frac{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}\frac{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}} \leq \frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}$$

$$\leq \frac{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}\frac{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}. \tag{2.17}$$

Indeed,

$$\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)$$

$$= \frac{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}f)}}{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+n+j}}\frac{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+m+j}}$$

$$\times|f_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+m+j}||f_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+n+j}|r^{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}$$

$$= \frac{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}\frac{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}r^{-j}$$

$$\times\frac{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j+n}}\frac{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j+m}}$$

$$\times|f_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j+n}||f_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j+m}|r^{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}$$

$$\leq \frac{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{l_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}\frac{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}}{\lambda_{\nu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)+j}}r^{-j}\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g). \tag{2.18}$$

On the other hand,

$$\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)$$

$$= \frac{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)+n}}\frac{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)+m}}$$

$$\times|f_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)+m}||f_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)+n}|r^{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}$$

$$= \frac{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}\frac{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)--jj}}$$

$$\times\frac{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j+n+j}}\frac{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j+m+j}}$$

$$\times|f_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j+m+j}||f_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j+n+j}|r^{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}r^j$$

$$\leq \frac{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}{l_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}\frac{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j}}{\lambda_{\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}}r^j\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g). \tag{2.19}$$

Inequalities (2.18) and (2.19) yield (2.17).

Condition (1.3) with $q>1$ implies that there exist numbers $1<q_1\leq q_2<+\infty$ such that $q_1^{kn}\leq l_k/l_{k+n}\leq q_2^{kn}$ and $q_1^{km}\leq \lambda_k/\lambda_{k+m}\leq q_2^{km}$ for all $k\geq k_0$. Therefore, from (2.17), we get

$$q_1^{2(\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j)j}\leq \frac{r^j\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}$$

$$\leq q_2^{2(\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g)-j)j}.$$

From whence, we get

$$\ln\frac{r^j\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}\asymp \nu(r,D_l^{(n)}f*D_\lambda^{(m)}g),\quad r\to+\infty. \tag{2.20}$$

Condition (2.14) yields the existence of numbers $0 < h_1 \le h_2 < +\infty$ such that $h_1 k^j \le l_k/l_{k+j} \le h_2 k^j$ and $h_1 k^j \le \lambda_k/\lambda_{k+j} \le h_2 k^j$. Therefore, from (2.17), we get

$$h_1^2(\nu(r, D_l^{(n)} f * D_\lambda^{(m)} g) - j)^{2j} \le \frac{r^j \mu(r, D_l^{(n+j)} f * D_\lambda^{(m+j)} g)}{\mu(r, D_l^{(n)} f * D_\lambda^{(m)} g)}$$

$$\le h_1^2 \nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)^{2j},$$

i.e.,

$$\frac{r^j \mu(r, D_l^{(n+j)} f * D_\lambda^{(m+j)} g)}{\mu(r, D_l^{(n)} f * D_\lambda^{(m)} g)} \asymp \nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)^{2j}. \quad r \to +\infty. \quad (2.21)$$

We note that if $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for every $c \in (0, +\infty)$, then $\alpha(x) = o(\beta(x)\ x \to +\infty$. Since $\alpha(e^x) \in L_{si}$, we have

$$\alpha(\nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)^{2j}) = \alpha(\exp\{2j \ln\, \nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)\})$$

$$= (1 + o(1))\alpha(\exp\{\ln\, \nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)\})$$

$$= (1 + o(1))\alpha(\nu(r, D_l^{(n)} f * D_\lambda^{(m)} g))$$

as $r \to +\infty$ and, by Lemma 4,

$$\varlimsup_{r\to+\infty} \frac{\alpha(\nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)^{2j})}{\beta(\ln\, r)} = \varrho_{\alpha,\beta}[\nu, D_l^{(n)} f * D_\lambda^{(m)} g)]$$

$$= \varrho_{\alpha,\beta}[D_l^{(n)} f * D_\lambda^{(m)}]$$

and

$$\varliminf_{r\to+\infty} \frac{\alpha(\nu(r, D_l^{(n)} f * D_\lambda^{(m)} g)^{2j})}{\beta(\ln\, r)} = \lambda_{\alpha,\beta}[\nu, D_l^{(n)} f * D_\lambda^{(m)} g)]$$

$$= \lambda_{\alpha,\beta}[D_l^{(n)} f * D_\lambda^{(m)}].$$

Therefore, from (2.20), we get

$$\varlimsup_{r\to+\infty} \frac{1}{\beta(\ln\, r)} \alpha\left(\ln\, \frac{r^j \mu(r, D_l^{(n+j)} f * D_\lambda^{(m+j)} g)}{\mu(r, D_l^{(n)} f * D_\lambda^{(m)} g)}\right) = \varrho_{\alpha,\beta}[D_l^{(n)} f * D_\lambda^{(m)}], \quad (2.22)$$

$$\underline{\lim}_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{r^j\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}\right)=\lambda_{\alpha,\beta}[D_l^{(n)}f*D_\lambda^{(m)}],\tag{2.23}$$

and (2.21) yields

$$\overline{\lim}_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\frac{r^j\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}\right)=\varrho_{\alpha,\beta}[D_l^{(n)}f*D_\lambda^{(m)}],\tag{2.24}$$

$$\underline{\lim}_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\frac{r^j\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}\right)=\lambda_{\alpha,\beta}[D_l^{(n)}f*D_\lambda^{(m)}].\tag{2.25}$$

We note that the conditions $|f_k/f_{k+1}|\nearrow+\infty$, $|g_k/g_{k+1}|\nearrow+\infty$, $l_kl_{k+2}/l_{k+1}^2\nearrow 1$, and $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2\nearrow 1$ as $k_0\le k\to\infty$ were not used in the proof of equalities (2.22)–(2.25).

Let now $|f_k/f_{k+1}|\nearrow+\infty$ and $|g_k/g_{k+1}|\nearrow+\infty$ as $k_0\le k\to\infty$. Then, in view of Propositions 5, 6, and 7, we have

$$\varrho_{\alpha,\beta}[D_l^{(n)}f*D_\lambda^{(m)}g]\le\min\{\varrho_{\alpha,\beta}[D_l^{(n)}f],\ \varrho_{\alpha,\beta}[D_\lambda^{(m)}g]\}$$
$$=\min\{\varrho_{\alpha,\beta}[f],\ \varrho_{\alpha,\beta}[g]\}$$

and

$$\varrho_{\alpha,\beta}[D_l^{(n)}f*D_\lambda^{(m)}g]$$
$$\ge\max\{\min\{\varrho_{\alpha,\beta}[D_l^{(n)}f],\ \lambda_{\alpha,\beta}[D_\lambda^{(m)}g]\},\min\{\lambda_{\alpha,\beta}[D_l^{(n)}f],\ \varrho_{\alpha,\beta}[D_\lambda^{(m)}g]\}\}$$
$$=\max\{\min\{\varrho_{\alpha,\beta}[f],\ \lambda_{\alpha,\beta}[g]\},\ \min\{\lambda_{\alpha,\beta}[f],\varrho_{\alpha,\beta}[g]\}\}.$$

Therefore, relations (2.22) and (2.24) yield (2.12) and (2.15).

If, moreover, $l_kl_{k+2}/l_{k+1}^2\nearrow 1$ and $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2\nearrow 1$ as $k_0\le k\to\infty$, then

$$\frac{l_kl_{k+n+1}}{l_{k+1}l_{k+n}}=\prod_{j=0}^{n-1}\frac{l_{k+j}l_{k+j+2}}{l_{k+j+1}^2}\nearrow 1,$$

$$\frac{\lambda_k\lambda_{k+n+1}}{\lambda_{k+1}\lambda_{k+n}}=\prod_{j=0}^{n-1}\frac{\lambda_{k+j}\lambda_{k+j+2}}{\lambda_{k+j+1}^2}\nearrow 1, k_0\le k\to\infty,$$

as $k_0\le k\to\infty$ and, thus,,

$$\left(\frac{l_k|f_{k+n}|}{l_{k+n}}\right)/\left(\frac{l_{k+1}|f_{k+n+1}|}{l_{k+n+1}}\right)\nearrow+\infty,$$

$$\left(\frac{\lambda_k|g_{k+n}|}{\lambda_{k+n}}\right)/\left(\frac{\lambda_{k+1}|g_{k+n+1}|}{\lambda_{k+n+1}}\right)\nearrow+\infty$$

as $k_0 \le k \to \infty$. Therefore, by Propositions 5, 6, and 7, we get

$$\lambda_{\alpha,\beta}[D_l^{(n)} f * D_\lambda^{(m)} g] = \min\{\lambda_{\alpha,\beta}[D_l^{(n)} f],$$

$$\lambda_{\alpha,\beta}[D_\lambda^{(m)} g]\} = \min\{\lambda_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}.$$

Moreover, from (2.23) and (2.25), we obtain (2.13) and (2.16). The proof of Theorem 1 is completed. □

Remark 1. Choosing $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ from the definitions of $\varrho_{\alpha,\beta}[f]$ and $\lambda_{\alpha,\beta}[f]$, we get the definitions of the order $\varrho[f] = \varlimsup\limits_{r\to+\infty} \dfrac{\ln \ln M(r, f))}{\ln r}$ and the lower order $\lambda[f] = \varliminf\limits_{r\to+\infty} \dfrac{\ln \ln M(r, f))}{\ln r}$ for entire function (1.1). The condition $\alpha(e^x) \in L_{si}$ is used only in Lemma 4 for the proof of the equalities $\varrho_{\alpha,\beta}[\ln \mu, f] = \varrho_{\alpha,\beta}[\nu, f]$ and $\lambda_{\alpha,\beta}[\ln \mu, f] = \lambda_{\alpha,\beta}[\nu, f]$. The function $\alpha(x) = \ln^+ x$ does not satisfy this condition, but it is easy to obtain the equalities $\varrho[\ln \mu, f] = \varrho[\nu, f]$ and $\lambda[\ln \mu, f] = \lambda[\nu, f]$ from estimations (2.8). The condition $\beta \in L_{si}$ is used only in Proposition 6 for the proof of the inequalities (2.9) and (2.10). The function $\beta(x) = x^+$ does not satisfy this condition, and, in this case, we obtain slightly different estimates than (2.9) and (2.10). It is known (see, e.g., [12–13]) that, for the entire function (1.1), $\varrho[f] = \varlimsup\limits_{k\to\infty} \dfrac{k \ln k}{-\ln |f_k|}$. Moreover, if $|f_k|/|f_{k+1}| \nearrow R[f]$ as $k_0 \le k \to \infty$, then $\lambda[f] = \varliminf\limits_{k\to\infty} \dfrac{k \ln k}{-\ln |f_k|}$. Therefore, if $\lambda[f] > 0$ and $\lambda[g] > 0$, then $-\ln |f_k| \le (k \ln k)/\lambda_1$ and $-\ln |g_k| \le (k \ln k)/\lambda_2$ for every $\lambda_1 \in (0, \lambda[f])$, $\lambda_2 \in (0, \lambda[g])$ and all $k \ge k_0$. From whence,

$$\lambda[f * g] = \varliminf_{k\to\infty} \frac{k \ln k}{-\ln |f_k| - \ln |g_k|}$$

$$\ge \varliminf_{k\to\infty} \frac{k \ln k}{(k \ln k)/\lambda_1 + (k \ln k)/\lambda_2} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}.$$

Thus, in view of Proposition 5 and the arbitrariness of λ_1 and λ_2, we have $\lambda[f]\lambda[g]/(\lambda[f] + \lambda[g]) \le \lambda[f * g] \le \min\{\lambda[f], \lambda_[g]\}$. If $\varrho[f] > 0$ and $\lambda[g] > 0$, then $-\ln |f_{k_j}| \le (k_j \ln k_j)/\varrho$ for every $\varrho \in (0, \varrho[f])$ and some sequence $(k_j) \uparrow +\infty$. Moreover, $-\ln |g_k| \le (k \ln k)/\lambda$ for every $\lambda \in (0, \lambda[g])$ and all $k \ge k_0$. From whence, we have

$$\varrho[f * g] \ge \varliminf_{j\to\infty} \frac{k_j \ln k_j}{-\ln |f_{k_j}| - \ln |g_{k_j}|}$$

$$\ge \varliminf_{j\to\infty} \frac{k_j \ln k_j}{(k_j \ln k_j)/\varrho + (k_j \ln k_j)/\lambda} = \frac{\varrho\lambda}{\varrho + \lambda}.$$

With regard for the arbitrariness of ϱ and λ_2, we get $\varrho[f*g] \geq \varrho[f]\lambda[g]/(\varrho[f]+\lambda[g])$. Similarly, we have $\varrho[f*g] \geq \varrho[g]\lambda[f]/(\varrho[g]+\lambda[f])$. Thus, in view of Proposition 5, we get $\max\{\varrho[g]\lambda[f]/(\varrho[g]+\lambda[f]), \varrho[g]\lambda[f]/(\varrho[g]+\lambda[f])\} \leq \varrho[f*g] \leq \min\{\varrho[f], \varrho_[g]\}$, and the following statement is proved. □

Proposition 8. *Let* $|f_k/f_{k+1}| \nearrow +\infty$ *and* $|g_k/g_{k+1}| \nearrow +\infty$ *as* $k_0 \leq k \to \infty$. *Then:*

1) *if the sequences* (l_k) *and* (λ_k) *satisfy* (1.3) *with* $q>1$, *then*

$$\max\left\{\frac{\varrho[f]\lambda[g]}{\varrho[f]+\lambda[g]}, \frac{\varrho[f]\lambda[g]}{\varrho[f]+\lambda[g]}\right\}$$

$$\leq \overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln\ln \frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)} \leq \min\{\varrho[f],\ \varrho[g]\},$$

and if, moreover, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$ *and* $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2 \nearrow 1$ *as* $k_0 \leq k \to \infty$, *then*

$$\frac{\lambda[f]\lambda[g]}{\lambda[f]+\lambda[g]} \leq \underline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln\ln \frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}$$

$$\leq \min\{\lambda[f],\ \lambda[g]\};$$

2) *if* (2.14) *holds, then*

$$\max\left\{\frac{\varrho[f]\lambda[g]}{\varrho[f]+\lambda[g]}, \frac{\varrho[f]\lambda[g]}{\varrho[f]+\lambda[g]}\right\}$$

$$\leq \overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)} \leq \min\{\varrho[f],\ \varrho[g]\},$$

and if, moreover, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$ *and* $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2 \nearrow 1$ *as* $k_0 \leq k \to \infty$, *then*

$$\frac{\lambda[f]\lambda[g]}{\lambda[f]+\lambda[g]} \leq \underline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \frac{r^j\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}$$

$$\leq \min\{\lambda[f],\ \lambda[g]\}.$$

3. Analytic functions in a unit disk

For $\alpha \in L$, $\beta \in L$, and the analytic transcendental function (1.1) with $R[f] = 1$, the quantities

$$\varrho^{(1)}_{\alpha,\beta}[f] := \varrho^{(1)}_{\alpha,\beta}[\ln M, f] = \varlimsup_{r\uparrow 1} \frac{\alpha(\ln M(r,f))}{\beta(1/(1-r))}$$

and

$$\lambda^{(1)}_{\alpha,\beta}[f] := \lambda^{(1)}_{\alpha,\beta}[\ln M, f] = \varliminf_{r\uparrow 1} \frac{\alpha(\ln M(r,f))}{\beta(1/(1-r))}$$

are called the generalized order and the lower generalized order, respectively.

Lemma 5. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, and let, for every $c \in (0, +\infty)$,*

$$\varlimsup_{x\to+\infty} \frac{d \ln \beta^{-1}(c\alpha(x))}{d \ln x} < 1, \quad \lim_{x\to+\infty} \frac{\alpha(x/\beta^{-1}(c\alpha(x)))}{\alpha(x)} = 1. \qquad (3.26)$$

Then

$$\varrho^{(1)}_{\alpha,\beta}[f] = \varlimsup_{k\to\infty} \frac{\alpha(k)}{\beta(k/\ln|f_k|)}. \qquad (3.27)$$

If, moreover, $|f_k/f_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$, then

$$\lambda^{(1)}_{\alpha,\beta}[f] = \varliminf_{k\to\infty} \frac{\alpha(k)}{\beta(k/\ln|f_k|)}. \qquad (3.28)$$

Formula (3.27) was proved in [14], and formula (3.28) follows from the corresponding formula for the Dirichlet series proved in [15–16].

The condition $|f_k/f_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$ implies that $|f_{k+1}| = \prod_{j=k_0}^{k} q_j$ for $k \ge k_0$, where $q_j \searrow 1$. Therefore, $|f_{k+1}| \ge 1$ for all $k \ge k_0$. We assume that $|f_k| \ge 1$ and $|g_k| \ge 1$ for all $k \ge k_0$. Then $R[f * g] \le 1$ and, thus, $R[f * g] = 1$.

Unlike the entire functions, the maximal term for functions (1.1) with $R[f] = 1$ can be a bounded function. In order that $\mu(r, f) \uparrow +\infty$ as $r \uparrow 1$, it is necessary and sufficient that $\varlimsup_{k\to\infty} |f_k| = +\infty$. Indeed, if $|f_k| \le K < +\infty$ for all k, then $\mu(r, f) \le \max\{Kr^k : k \ge 0\} = K$. On the other hand, if $\mu(r, f) \le K$, then $|f_k|r^k \le K$ for all $k \ge 0$ and $r \in [0, 1)$. Directing $r \to 1$, we get $|f_k| \le K$ for all $k \ge 0$.

Lemma 6. *If $\alpha \in L_{si}$, $\beta \in L_{si}$, and $\alpha(\ln x) = o(\beta(x))$ as $x \to +\infty$, then $\varrho^{(1)}_{\alpha,\beta}[f] = \varrho^{(1)}_{\alpha,\beta}[\ln \mu, f]$ and $\lambda^{(1)}_{\alpha,\beta}[f] = \lambda^{(1)}_{\alpha,\beta}[\ln \mu, f]$. If, moreover, $\alpha(e^x) \in$*

L_{si} and $\alpha(x)=o(\beta(x))$ as $x\to+\infty$, then $\varrho^{(1)}_{\alpha,\beta}[\ln\,\mu,f]=\varrho^{(1)}_{\alpha,\beta}[\nu,f]$ and $\lambda^{(1)}_{\alpha,\beta}[\ln\,\mu,f]=\lambda^{(1)}_{\alpha,\beta}[\nu,f]$.

Proof. In view of the conditions $\alpha\in L_{si}$ and $\beta\in L_{si}$, the equalities $\varrho^{(1)}_{\alpha,\beta}[f]=\varrho^{(1)}_{\alpha,\beta}[\ln\,\mu,f]$ and $\lambda^{(1)}_{\alpha,\beta}[f]=\lambda^{(1)}_{\alpha,\beta}[\ln\,\mu,f]$ follow from the estimates

$$\mu(r,f)\le M(r,f)\le\sum_{k=0}^{\infty}|f_k|\left(\frac{1+r}{2}\right)^k\left(\frac{2r}{1+r}\right)^k\le\frac{1+r}{1-r}\mu\left(\frac{1+r}{2},f\right).$$

Indeed, in view of the condition $\alpha\in L_{si}$, we have

$$\alpha(\ln\,\mu(r,f))\le\alpha(\ln\,M(r,f))\le\alpha\left(\ln\,\mu\left(\frac{1+r}{2},f\right)+\ln\,\frac{2}{1-r}\right)$$

$$\le\alpha\left(2\max\left\{\ln\,\mu\left(\frac{1+r}{2},f\right),\,\ln\,\frac{2}{1-r}\right\}\right)$$

$$=(1+o(1))\max\left\{\alpha(\ln\,\mu\left(\frac{1+r}{2},f\right),\,\alpha\left(\ln\,\frac{2}{1-r}\right)\right\}$$

$$=(1+o(1))\left\{\alpha(\ln\,\mu\left(\frac{1+r}{2},f\right)+\alpha\left(\ln\,\frac{2}{1-r}\right)\right\},\quad r\uparrow1,$$

and, thus,

$$\frac{\alpha(\ln\,\mu(r,f))}{\beta(1/(1-r))}\le\frac{\alpha(\ln\,M(r,f))}{\beta(1/(1-r)}$$

$$\le(1+o(1))\left(\frac{\alpha(\ln\,\mu(1+r)/2,f)}{\beta(1/(1-(1+r)/2))}\frac{\beta(2/(1-r))}{\beta(1/(1-r))}+\frac{\alpha(\ln(2/(1-r)))}{\beta(1/(1-r))}\right)$$

as $r\uparrow1$. Since $\beta\in L_{si}$ and $\alpha(\ln x)=o(\beta(x)$ as $x\to+\infty$, we have $\varrho^{(1)}_{\alpha,\beta}[f]=\varrho^{(1)}_{\alpha,\beta}[\ln\,\mu,f]$ and $\lambda^{(1)}_{\alpha,\beta}[f]=\lambda^{(1)}_{\alpha,\beta}[\ln\,\mu,f]$.

Using the equality $\ln\,\mu(r,f)-\ln\,\mu(r_0,f)=\int\limits_{r_0}^{r}\frac{\nu(t,f)}{t}dt$, $r_0\le r<1$, we get

$$\ln\,\mu(r,f)-\ln\,\mu(r_0,f)\le\nu(r,f)\ln(r/r_0)\le\nu(r,f)\ln(1/r_0).\qquad(3.29)$$

On the other hand, for $r\ge r_0$,

$$\ln\,\mu\left(r+\frac{1-r}{2},f\right)-\ln\,\mu(r_0,f)\ge\int\limits_{r}^{r+(1-r)/2}\frac{\nu(t,f)}{t}dt$$

$$\ge\nu(r,f)\ln(1+(1-r)/2r)=(1+o(1))\nu(r,f)(1-r)/2,\quad r\uparrow1,$$

i.e.,

$$\nu(r,f)\le \frac{4}{1-r}\ln\mu\left(r+\frac{1-r}{2},f\right),\quad r\ge r_0^*. \tag{3.30}$$

Since $\alpha(e^x)\in L_{si}$ and $\alpha(x)=o(\beta(x)$ as $x\to+\infty$, relations (3.29) and (3.30) yield

$$(1+o(1))\frac{\alpha(\ln\mu(r,f))}{\beta(1/(1-r)}\le(1+o(1))\frac{\alpha(\nu(r,f))}{\beta(1/(1-r)}$$

$$\le(1+o(1))\frac{\max\{\alpha(\ln\mu(r+(1-r)/2,f)),+\alpha(4/(1-r))\}}{\beta(1/(1-r)}$$

$$=(1+o(1))\frac{\alpha(\ln\mu(r+(1-r)/2,f))}{\beta(1/(1-r-(1-r)/2)}\frac{\beta(2/(1-r)}{\beta(1/(R-r)}$$

$$=(1+o(1))\frac{\alpha(\ln\mu(r+(1-r)/2,f))}{\beta(1/(1-r-(1-r)/2)},\quad r\uparrow 1.$$

This implies that $\varrho^{(1)}_{\alpha,\beta}[\ln\mu,f]=\varrho^{(1)}_{\alpha,\beta}[\nu,f]$ and $\lambda^{(1)}_{\alpha,\beta}[\ln\mu,f]=\lambda^{(1)}_{\alpha,\beta}[\nu,f]$. Lemma 6 is proved. □

Using Lemmas 5 and 6, we prove the following statements.

Proposition 9. *If $\alpha\in L_{si}$, $\beta\in L_{si}$ and $\alpha(\ln x)=o(\beta(x))$ as $x\to+\infty$, then*

$$\varrho^{(1)}_{\alpha,\beta}[f*g]\ge\max\{\varrho^{(1)}_{\alpha,\beta}[f],\ \varrho^{(1)}_{\alpha,\beta}[g]\}$$

and

$$\lambda^{(1)}_{\alpha,\beta}[f*g]\ge\max\{\lambda^{(1)}_{\alpha,\beta}[f],\ \lambda^{(1)}_{\alpha,\beta}[g]\}.$$

Proof. Since $|g_k|>1$ for $k\ge k_0$, we have $\mu(r,f*g)\ge\max\{|f_kg_k|r^k:k\ge k_0\}\}\ge\max\{|f_k|r^k:k\ge k_0\}\}=\mu(r,f)+O(1)$ as $r\uparrow 1$. From whence, we get $\varrho^{(1)}_{\alpha,\beta}[\ln\mu,f*g]\ge\varrho^{(1)}_{\alpha,\beta}[\ln\mu,f]$ and $\lambda^{(1)}_{\alpha,\beta}[\ln\mu,f*g]\ge\lambda^{(1)}_{\alpha,\beta}[\ln\mu,f]$ and, by Lemma 6, $\varrho^{(1)}_{\alpha,\beta}[f*g]\ge\varrho_{\alpha,\beta}[f]$ and $\lambda^{(1)}_{\alpha,\beta}[f*g]\ge\lambda^{(1)}_{\alpha,\beta}[f]$. Similarly, $\varrho^{(1)}_{\alpha,\beta}[f*g]\ge\varrho^{(1)}_{\alpha,\beta}[g]$ and $\lambda^{(1)}_{\alpha,\beta}[f*g]\ge\lambda^{(1)}_{\alpha,\beta}[g]$. □

Proposition 10. *If the functions α and β satisfy the conditions of Lemma 5, then*

$$\varrho^{(1)}_{\alpha,\beta}[f*g]\le\max\{\varrho^{(1)}_{\alpha,\beta}[f],\ \varrho^{(1)}_{\alpha,\beta}[g]\}. \tag{3.31}$$

If, moreover, $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$, $l_kl_{k+2}/l^2_{k+1}\nearrow 1$ and $\lambda_k\lambda_{k+2}/\lambda^2_{k+1}\nearrow 1$ as $k_0\le k\to\infty$, then

$$\lambda^{(1)}_{\alpha,\beta}[f*g]\le\min\{\max\{\lambda^{(1)}_{\alpha,\beta}[f],\ \varrho^{(1)}_{\alpha,\beta}[g]\},\ \max\{\lambda^{(1)}_{\alpha,\beta}[g],\ \varrho^{(1)}_{\alpha,\beta}[f]\}\}. \tag{3.32}$$

Proof. We can consider that $\varrho^{(1)}_{\alpha,\beta}[f] < +\infty$ and $\varrho^{(1)}_{\alpha,\beta}[g] < +\infty$. Then, by Lemma 5, $\ln|f_k| \le \dfrac{k}{\beta^{-1}(\alpha(k)/\varrho_1)}$ and $\ln|g_k| \le \dfrac{k}{\beta^{-1}(\alpha(k)/\varrho_2)}$ for every $\varrho_1 > \varrho^{(1)}_{\alpha,\beta}[f]$, $\varrho_2 > \varrho^{(1)}_{\alpha,\beta}[g]$ and all $k \ge k_0$. Therefore,

$$\ln|f_k g_k| \le \frac{k}{\beta^{-1}(\alpha(k)/\varrho_1)} + \frac{k}{\beta^{-1}(\alpha(k)/\varrho_2)} \le 2\frac{k}{\beta^{-1}(\alpha(k)/\max\{\varrho_1,\varrho_2\})}.$$

This implies that $\varrho^{(1)}_{\alpha,\beta}[f*g] \le \max\{\varrho_1,\, \varrho_2\}$, i.e., in view of the arbitrariness of ϱ_1 and ϱ_2, we get (3.31).

We have $\ln|f_{k_j}| \le \dfrac{k_j}{\beta^{-1}(\alpha(k_j)/\lambda)}$ for every $\lambda \in (\lambda^{(1)}_{\alpha,\beta}[f],\, +\infty)$ and some sequence $(k_j) \uparrow +\infty$. Therefore, as above,

$$\lambda^{(1)}_{\alpha,\beta}[f*g] \le \varliminf_{j\to\infty} \frac{\alpha(k_j)}{\beta(k_j/\ln|f_{k_j} g_{k_j}|} \le \max\{\lambda,\, \varrho_2\}.$$

In view of the arbitrariness of λ and ϱ_2, we get $\lambda^{(1)}_{\alpha,\beta}[f*g] \le \max\{\lambda^{(1)}_{\alpha,\beta}[f],\, \varrho^{(1)}_{\alpha,\beta}[g]\}$. Similarly, $\lambda^{(1)}_{\alpha,\beta}[f*g] \le \max\{\lambda^{(1)}_{\alpha,\beta}[g],\, \varrho^{(1)}_{\alpha,\beta}[f]\}$. Thus, (3.32) is true. □

We prove also the following statement.

Proposition 11. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, and $\alpha(\ln x) = o(\beta(x))$ as $x \to +\infty$. If condition* (2.14) *holds, then $\lambda^{(1)}_{\alpha,\beta}[D^n_l f] = \lambda^{(1)}_{\alpha,\beta}[f]$ and $\varrho^{(1)}_{\alpha,\beta}[D^n_l f] = \varrho^{(1)}_{\alpha,\beta}[f]$.*

Proof. It is sufficient to consider the case where $n = 1$. Condition (2.14) yields the existence of numbers $0 < h_1 \le h_2 < +\infty$ such that $h_1.(k+1) \le l_k/l_{k+1} \le h_2(k+1)$ for all $k \ge 0$. Therefore, $\mu(r, D^1_l f) \ge \max\left\{h_1(k+1)|f_{k+1}|r^{k+1} : k \ge 0\right\} = h_1\mu(r, f')$, and $\mu(r, D^1_l f) \le h_2\mu(r, f')$. Therefore, in view of Lemma 6, $\lambda^{(1)}_{\alpha,\beta}[D^1_l f] = \lambda^{(1)}_{\alpha,\beta}[\ln\mu, D^1_l f] = \lambda^{(1)}_{\alpha,\beta}[\ln\mu, f'] = \lambda^{(1)}_{\alpha,\beta}[f']$ and $\varrho^{(1)}_{\alpha,\beta}[D^1_l f] = = \varrho^{(1)}_{\alpha,\beta}[\ln\mu, D^1_l f] = \varrho^{(1)}_{\alpha,\beta}[\ln\mu, f'] = \varrho^{(1)}_{\alpha,\beta}[f']$. On the other hand (see [8], Lemma 6), $M(r, f') \le \dfrac{2}{1-r} M\left(\dfrac{1-r}{2}, f\right)$ and $M(r,f) \le M(r,f') + |f(0)|$. From whence as in the proof of Lemma 6, we get $\lambda^{(1)}_{\alpha,\beta}[f'] = \lambda^{(1)}_{\alpha,\beta}[f]$ and $\varrho^{(1)}_{\alpha,\beta}[f'] = \varrho^{(1)}_{\alpha,\beta}[f]$. Proposition 11 is proved. □

The following analog of Theorem 1 is true.

Theorem 2. *Let $\alpha(e^x) \in L_{si}$, $\beta \in L_{si}$, and let (3.26) hold. If (2.14) holds, then*

$$\varlimsup_{r\uparrow 1} \frac{1}{\beta(1/(1-r))}\alpha\left(\frac{\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}\right) = \max\{\varrho^{(1)}_{\alpha,\beta}[f],\, \varrho^{(1)}_{\alpha,\beta}[g]\}. \tag{3.33}$$

If, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$, and $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2 \nearrow 1$ as $k_0 \le k \to \infty$, then

$$\max\{\lambda^{(1)}_{\alpha,\beta}[f],\, \lambda^{(1)}_{\alpha,\beta}[g]\} \le \varlimsup_{r\uparrow 1} \frac{1}{\beta(1/(1-r))}\alpha\left(\frac{\mu(r, D_l^{(n+j)}f * D_\lambda^{(m+j)}g)}{\mu(r, D_l^{(n)}f * D_\lambda^{(m)}g)}\right)$$

$$\le \min\{\max\{\lambda^{(1)}_{\alpha,\beta}[f],\, \varrho^{(1)}_{\alpha,\beta}[g]\},\, \max\{\lambda^{(1)}_{\alpha,\beta}[g],\, \varrho^{(1)}_{\alpha,\beta}[f]\}\}. \tag{3.34}$$

Proof. As above, we have (2.21), and condition (3.26) implies that $\alpha(\ln x) \le \alpha(x) = o(\beta(x)$ as $x \to +\infty$. In view of Lemma 6 and the relation $\alpha(e^x) \in L_{si}$, we have

$$\varlimsup_{r\uparrow 1} \frac{\alpha(\nu(r, D_l^{(n)}f * D_\lambda^{(m)}g)^{2j})}{\beta(1/(1-r))} = \varlimsup_{r\uparrow 1} \frac{\alpha(\nu(r, D_l^{(n)}f * D_\lambda^{(m)}g))}{\beta(1/(1-r))}$$

$$= \varrho^{(1)}_{\alpha,\beta}[\nu, D_l^{(n)}f * D_\lambda^{(m)}g)] = \varrho^{(1)}_{\alpha,\beta}[D_l^{(n)}f * D_\lambda^{(m)}]$$

and

$$\varliminf_{r\uparrow 1} \frac{\alpha(\nu(r, D_l^{(n)}f * D_\lambda^{(m)}g)^{2j})}{\beta(1/(1-r))} = \lambda^{(1)}_{\alpha,\beta}[\nu, D_l^{(n)}f * D_\lambda^{(m)}g)]$$

$$= \lambda^{(1)}_{\alpha,\beta}[D_l^{(n)}f * D_\lambda^{(m)}].$$

Therefore, using Propositions 10 and 11 from (2.21), we obtain (3.33) and (3.34). □

Remark 2. Choosing $\alpha(x) = \beta(x) = \ln^+ x$ from the definitions of $\varrho^{(1)}_{\alpha,\beta}[f]$ and $\lambda^{(1)}_{\alpha,\beta}[f]$, we get the definitions of the order $\varrho^{(1)}[f] = \varlimsup_{r\uparrow 1} \dfrac{\ln^+ \ln M(r,f))}{\ln(1/(1-r))}$ and the lower order $\lambda^{(1)}[f] = \varliminf_{r\uparrow 1} \dfrac{\ln^+ \ln M(r,f))}{\ln(1/(1-r))}$ for function (1.1) with $R[f] = 1$. The functions $\alpha(x) = \beta(x) = \ln^+ x$ do not satisfy the conditions $\alpha(e^x) \in L_{si}$ and (3.26). The condition $\alpha(e^x) \in L_{si}$ is used only in Lemma 6 for the proof of the equalities $\varrho^{(1)}_{\alpha,\beta}[\nu, f] = \varrho^{(1)}_{\alpha,\beta}[f]$ and $\lambda^{(1)}_{\alpha,\beta}[\nu, f] = \lambda^{(1)}_{\alpha,\beta}[f]$. Now, we have (see [18]) $\lambda^{(1)}[f] \le \lambda^{(1)}[\nu, f] \le \lambda^{(1)}[f] + 1$ and $\varrho^{(1)}[f] \le \varrho^{(1)}[\nu, f] \le \varrho^{(1)}[f] + 1$. We note (see [18–19]) that $\varrho^{(1)}[f]=\dfrac{\alpha^*[f]}{1-\alpha^*[f]}$, $\alpha^*[f]:=\varlimsup\limits_{k\to\infty}\dfrac{\ln^+\ln|f_k|}{\ln k}$. If $|f_k|/|f_{k+1}|\nearrow 1$ as $k_0\le k\to\infty$, then $\lambda^{(1)}[f]=\dfrac{\alpha_*[f]}{1-\alpha_*[f]}$, $\alpha_*[f]:=\varliminf\limits_{k\to\infty}\dfrac{\ln^+\ln|f_k|}{\ln k}$.

Since $|f_k|>1$ and $|g_k|>1$, we have $\alpha^*[f*g]\ge\max\{\alpha^*[f],\,\alpha^*[g]\}$ and $\alpha_*[f*g]\ge\max\{\alpha_*[f],\,\alpha_*[g]\}$. From whence, we get

$$\varrho^{(1)}[f*g]\ge\frac{\max\{\alpha^*[f],\,\alpha^*[g]\}}{1-\max\{\alpha^*[f],\,\alpha^*[g]\}}$$

$$=\frac{\max\{\varrho^{(1)}[f]/(1-\varrho^{(1)}[f]),\,\varrho^{(1)}[g]/(1-\varrho^{(1)}[g])\}}{1-\max\{\varrho^{(1)}[f]/(1-\varrho^{(1)}[f]),\,\varrho^{(1)}[g]/(1-\varrho^{(1)}[g])\}}$$

$$=\max\{\varrho^{(1)}[f],\,\varrho^{(1)}[g])\}.$$

On the other hand, $\ln|f_k|\le k^{\alpha_1}$ and $\ln|g_k|\le k^{\alpha_2}$ for every $\alpha_1\in(\alpha^*[f],1)$, $\alpha_2\in(\alpha^*[g],1)$ and all $k\ge k_0$. Therefore, $\alpha^*[f*g]\le\varlimsup\limits_{k\to\infty}\dfrac{\ln^+(k^{\alpha_1}+k^{\alpha_2})}{\ln k}\le\max\{\alpha_1,\alpha_2\}$. In view of the arbitrariness of α_1 and α_2, we get $\alpha^*[f*g]\le\max\{\alpha^*[f],\alpha^*[g]\}$ and, as above, $\varrho^{(1)}[f*g]\le\max\{\varrho^{(1)}[f],\,\varrho^{(1)}[g]\}$.

If $|f_k/f_{k+1}|\nearrow+\infty$, $|g_k/g_{k+1}|\nearrow+\infty$ as $k_0\le k\to\infty$, then we obtain $\lambda^{(1)}[f*g]\ge\max\{\lambda^{(1)}[f],\lambda^{(1)}[g])\}$ and $\lambda^{(1)}[f*g]\le\min\{\max\{\lambda^{(1)}[f],\varrho^{(1)}[g]\},\max\{\lambda^{(1)}[g],\,\varrho^{(1)}[f]\}\}$. Finally, (2.21) implies that

$$\ln\frac{\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}=2j\ln\nu(r,D_l^{(n)}f*D_\lambda^{(m)}g),\quad r\to+\infty.$$

Therefore, the following statement is true.

Proposition 12. *If* (2.14) *holds, then*

$$2j\max\{\varrho^{(1)}[f],\,\varrho^{(1)}[g]\}\le\varlimsup_{r\uparrow1}\frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}$$

$$\le 2j\max\{\varrho^{(1)}[f]+1,\,\varrho^{(1)}[g]+1\}.$$

If, moreover, $|f_k/f_{k+1}|\nearrow1$, $|g_k/g_{k+1}|\nearrow1$, $l_kl_{k+2}/l_{k+1}^2\nearrow1$ *and* $\lambda_k\lambda_{k+2}/\lambda_{k+1}^2\nearrow1$ *as* $k_0\le k\to\infty$, *then*

$$2j\max\{\lambda^{(1)}[f],\,\lambda^{(1)}[g]\}\le\varlimsup_{r\uparrow1}\frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r,D_l^{(n+j)}f*D_\lambda^{(m+j)}g)}{\mu(r,D_l^{(n)}f*D_\lambda^{(m)}g)}$$

$$\le 2j\min\{\max\{\lambda^{(1)}[f],\,\varrho^{(1)}[g]\}+1,\max\{\lambda^{(1)}[g],\,\varrho^{(1)}[f]\}+1\}.$$

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